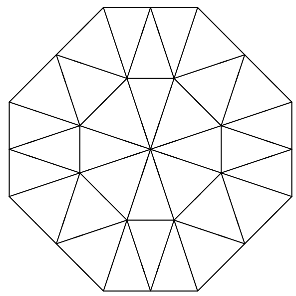
2.6. **Definition of finite elements.** We first define triangular finite element spaces, considering the case when Ω is a convex polygon. For each $0 < h < 1$, we let $\mathcal{T}_h$ be a triangulation of $\bar{\Omega}$ with the following properties:

i: $\bar{\Omega} = \cup_i T_i$,
ii: If T_i and T_j are distinct, then exactly one of the following holds: (a) $T_i \cap T_j = \emptyset$, (b) $T_i \cap T_j$ is a common vertex, (c) $T_i \cap T_j$ is a common side.
iii: each $T_i \in \mathcal{T}_h$ has diameter $\leq h$.

Triangulation of Ω



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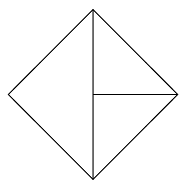

Given a triangulation $\mathcal{T}_h$ of Ω, a finite element space is a space of piecewise polynomials with respect to $\mathcal{T}_h$ that is constructed by specifying the following things for each $T \in \mathcal{T}_h$:

Shape functions: a finite dimensional space $V(T)$ of polynomial functions on T.

Degrees of freedom (DOF): a finite set of linear functionals $\phi_i : V(T) \to \mathbb{R}$ $(i = 1, \ldots, N)$ which are unisolvent on $V(T)$, i.e., given any numbers α_i, $i = 1, \ldots, N$, there exists a unique function $p \in V(T)$, such that $\phi_i(p) = \alpha_i$, $i = 1, \ldots, N$. Since these equations amount to a square linear system of equations, to show unisolvence, we show that the only solution of the system with all $\alpha_i = 0$ is $p = 0$.

We further assume that each degree of freedom on T is associated to a subsimplex of T, i.e., to a vertex, an edge, or T itself. Moreover, if a subsimplex is shared by two different triangles T_1 and T_2 in $\mathcal{T}_h$, the DOFs for T_1 and T_2 associated to the subsimplex are the same.

Once we define the shape functions and DOFs for each triangle, we then define the *assembled finite element space* as all functions $v \in L^2(\Omega)$ such that

(i) $v|_T \in V(T)$ for all $T \in \mathcal{T}_h$ and (ii) The DOFs are single-valued in the sense that whenever q is a subsimplex shared by T_1 and T_2, then the corresponding DOFs associated to q applied to $v|_{T_1}$ and $v|_{T_2}$ take on the same value. Note that we do not specify the interelement continuity explicitly. It is determined by the fact that the shared DOFs are single-valued.

We first consider the construction of finite element subspaces of $H^1(\Omega)$. In general, one can show that a function v which is smooth on each triangle $T \in \mathcal{T}_h$ (e.g., v is a polynomial on each $T \in \mathcal{T}_h$) will belong to the space $H^1(\Omega)$, if and only if $v \in C^0(\Omega)$.

Hence, we construct $V_h = \{v_h \in C^0(\bar{\Omega}) : v_h|_T \in P_k, T \in \mathcal{T}_h\}$, where P_k denotes the space of polynomials of degree $\leq k$. This is the space of shape functions. How do we do this?

$k = 0$. On each triangle T_i, $v_h = c_i$, a constant. But since $v_h \in C^0(\bar{\Omega})$, $c_i = c$ for all i, and so the space consists of a single constant. This is not a useful space, since there is only one degree of freedom.

$k = 1$. On each triangle, $v_h|_T = c_0 + c_1x_1 + c_2x_2$. The issue is how to find degrees of freedom so that the global function we construct is continuous. We shall show that the correct degrees of freedom are the values of v_h at the vertices of the triangulation. This involves establishing two facts. The first is to show that a linear function on a triangle is uniquely determined by its values at the vertices. The second is to show that a piecewise linear function defined on two triangles sharing a common edge will be continuous everywhere on that edge if it is continuous at the vertices of that edge.

To see how this can be done, first consider the case of an interval $[a_1, a_2]$ in one dimension. One way of writing a linear polynomial is $P_1(x) = c_0 + c_1x$. In this form, the degrees of freedom c_0 and c_1 are the values $P_1(0)$ and $P_1'(0)$, respectively, i.e., $c_0 = P_1(0)$ and $c_1 = P_1'(0)$. However, we may also write any linear polynomial in the form:

$$P_1(x) = b_1 \frac{x - a_2}{a_1 - a_2} + b_2 \frac{x - a_1}{a_2 - a_1}.$$

In this form, the degrees of freedom b_1 and b_2 are the values $P_1(a_1)$ and $P_1(a_2)$, respectively, i.e., $b_1 = P_1(a_1)$ and $b_2 = P_1(a_2)$. This is because, instead of taking 1 and x as the basis functions for linear polynomials, we are using the functions

$$\phi_1(x) = \frac{x - a_2}{a_1 - a_2}, \qquad \phi_2(x) = \frac{x - a_1}{a_2 - a_1}.$$

The key property is that $\phi_1(a_1) = 1, \phi_1(a_2) = 0$ and $\phi_2(a_1) = 0, \phi_2(a_2) = 1$.

We shall show that the corresponding functions in higher dimensions are the barycentric coordinates of a point $\boldsymbol{x}$. To define the barycentric coordinates, we let $\boldsymbol{a}_j = (a_{1j}, a_{2j})$ be the vertices of a triangle T. Then

$$T = \{\boldsymbol{x} = \sum_{j=1}^{3} \lambda_j \boldsymbol{a}_j, \quad 0 \leq \lambda_j \leq 1, \quad 1 \leq j \leq 3, \quad \sum_{j=1}^{3} \lambda_j = 1\}.$$

The barycentric coordinates $\lambda_j = \lambda_j(\boldsymbol{x})$, $1 \leq j \leq 3$ of any point $\boldsymbol{x} \in \mathbb{R}^2$ with respect to the the vertices $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are the unique solutions of the linear system:

$$\sum_{j=1}^{3} a_{ij} \lambda_j = x_i, \ i = 1, 2, \qquad \sum_{j=1}^{3} \lambda_j = 1,$$

i.e., we solve

$$A \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 1 & 1 & 1 \end{pmatrix}.$$

One can show that the matrix is nonsingular if the triangle is nondegenerate.

Observe that when $\boldsymbol{x} = \boldsymbol{a}_k$, i.e., $x_i = a_{ik}$, then $\lambda_1, \lambda_2, \lambda_3$ is the unique solution of

$$\sum_{j=1}^{3} a_{ij}\lambda_j = a_{ik}, \qquad \sum_{j=1}^{3} \lambda_j = 1.$$

The solution is given by $\lambda_j = \delta_{jk} = 1$, for $j = k$ and $= 0$ for $j \neq k$.

If we write the general system in the form $A\lambda = \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix}$, then $\lambda(\boldsymbol{x}) = A^{-1} \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix}$. Define $A^{-1} = B = (b_{ij})$. Then $\lambda_i(\boldsymbol{x}) = \sum_{j=1}^{2} b_{ij}x_j + b_{i3}, \quad 1 \leq i \leq 3$, so the λ_i are affine functions of x_1, x_2. Since when $x_i = a_{ik}$, $\lambda_j = \delta_{jk}$, we have $\lambda_j(\boldsymbol{a}_k) = \delta_{jk}$. So λ_j is precisely the function we need, since it is a linear function equal to 1 at the vertex $\boldsymbol{a}_j$ and equal to zero at the other two vertices.

Example: Consider the triangle with vertices $\boldsymbol{a}_1 = (1, 0)$, $\boldsymbol{a}_2 = (0, 1)$, and $\boldsymbol{a}_3 = (0, 0)$. Then $\lambda_1(\boldsymbol{x}) = x_1$, $\lambda_2(\boldsymbol{x}) = x_2$, and $\lambda_3(\boldsymbol{x}) = 1 - x_1 - x_2$.

Also observe that if $\boldsymbol{x} = (1 - \theta)\boldsymbol{y} + \theta\boldsymbol{z}$, $0 \leq \theta \leq 1$, i.e., $\boldsymbol{x}$ lies on the line segment between $\boldsymbol{y}$ and $\boldsymbol{z}$, then

$$\begin{aligned} \lambda(\boldsymbol{x}) = A^{-1} \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} &= A^{-1} \begin{pmatrix} (1-\theta)\boldsymbol{y} + \theta\boldsymbol{z} \\ (1-\theta)1 + \theta 1 \end{pmatrix} \\ &= (1-\theta)A^{-1} \begin{pmatrix} \boldsymbol{y} \\ 1 \end{pmatrix} + \theta A^{-1} \begin{pmatrix} \boldsymbol{z} \\ 1 \end{pmatrix} = (1-\theta)\lambda(\boldsymbol{y}) + \theta\lambda(\boldsymbol{z}). \end{aligned}$$

Hence, on the edge joining $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, $\lambda_3(\boldsymbol{x}) = 0$. At the point midway between $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$, $\lambda_3(\boldsymbol{x}) = 1/2$.

We note that the *barycenter* of a triangle is the point of T for which all the $\lambda_i = 1/3$.

Using this notation, we now show that any linear polynomial $P(\boldsymbol{x}) = c_0 + c_1x_1 + c_2x_2$ defined on T is uniquely determined by its values at the three vertices $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ of T. The idea is the same as in one dimension. Instead of writing a linear polynomial as a linear combination of the basis functions $1, x_1, x_2$, we write it as a linear combination of the λ_j, $j = 1, 2, 3$, i.e.,

$$P_1(\boldsymbol{x}) = b_1\lambda_1(\boldsymbol{x}) + b_2\lambda_2(\boldsymbol{x}) + b_3\lambda_3(\boldsymbol{x}).$$

From the properties of the λ_j, we immediately get that

$$P_1(\boldsymbol{x}) = P_1(\boldsymbol{a}_1)\lambda_1(\boldsymbol{x}) + P_1(\boldsymbol{a}_2)\lambda_2(\boldsymbol{x}) + P_1(\boldsymbol{a}_3)\lambda_3(\boldsymbol{x}).$$

Next consider the degrees of freedom for higher order polynomials: Define $\boldsymbol{a}_{ij}$ to be the midpoint of the edge joining $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$. We can show that we can write every polynomial of degree ≤ 2 defined on T in the form:

$$P_2(\boldsymbol{x}) = \sum_{i=1}^{3} \lambda_i(2\lambda_i - 1)P(\boldsymbol{a}_i) + \sum_{i<j} 4\lambda_i\lambda_j P(\boldsymbol{a}_{ij}).$$

Note that the term $\lambda_3(2\lambda_3 - 1) = 0$ everywhere along the line segment joining $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ and when $\lambda_3 = 1/2$, which includes the midpoint of the edges joining $\boldsymbol{a}_1$ to $\boldsymbol{a}_3$ and joining $\boldsymbol{a}_2$ to $\boldsymbol{a}_3$. Furthermore $\lambda_3(2\lambda_3 - 1) = 1$ at $\boldsymbol{a}_3$. In addition $4\lambda_1\lambda_2 = 0$ at all vertices and at $\boldsymbol{a}_{13}$ and $\boldsymbol{a}_{23}$ and equals 1 at the midpoint $\boldsymbol{a}_{12}$.

To represent cubic polynomials, we define for $i \neq j$, $\boldsymbol{a}_{iij} = (2/3)\boldsymbol{a}_i + (1/3)\boldsymbol{a}_j$ and $\boldsymbol{a}_{123} = (1/3)(\boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3)$ Then we can write every cubic polynomial in the form

$$P_3(\boldsymbol{x}) = \sum_{i=1}^{3} \lambda_i(3\lambda_i - 1)(3\lambda_i - 2)/2P(\boldsymbol{a}_i) + \sum_{i\neq j} 9\lambda_i\lambda_j(3\lambda_i - 1)/2P(\boldsymbol{a}_{iij}) + 27\lambda_1\lambda_2\lambda_3 P(\boldsymbol{a}_{123}).$$

Note that in each of these cases, we have degrees of freedom for the space of shape functions $V(T)$ of the form $P(\boldsymbol{b}_i)$, where the $\boldsymbol{b}_i$ are points in T. Corresponding to each of these degrees of freedom, we can find a basis function $\phi_i \in V(T)$ with the property that $\phi_i(\boldsymbol{b}_j) = 1$ if $i = j$ and $\phi_i(\boldsymbol{b}_j) = 0$ if $i \neq j$. We then can write an arbitrary $P \in V(T)$ in the form $P(\boldsymbol{x}) = \sum_{i=1}^{M} P(\boldsymbol{b}_i)\phi_i(\boldsymbol{x})$. Here M is the dimension of the space $V(T)$. If the degrees of freedom are unisolvent, then the number of degrees of freedom must also equal M.

In general, if T is an n-simplex with vertices $\boldsymbol{a}_j$, $1 \leq j \leq n+1$, then for a given integer $k \geq 1$, any polynomial $p \in P_k$, the set of polynomials of degree $\leq k$, is uniquely determined by its values on the set:

$$L_k(T) = \{x = \sum_{j=1}^{n+1} \lambda_j \boldsymbol{a}_j, \quad \sum_{j=1}^{n+1} \lambda_j = 1, \quad \lambda_j \in \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}, 1 \leq j \leq n+1\}.$$

Example: $n = 2$, $k = 1$. Linear polynomials on a triangle are uniquely determined by their values at the 3 vertices.

Example: $n = 2$, $k = 2$. Quadratric polynomials on a triangle are uniquely determined by their values at the 3 vertices and the midpoints of the three edges.

Example: $n = 2$, $k = 3$. Cubic polynomials on a triangle are uniquely determined by their values at the 3 vertices, two points on each edge, and at the centroid of the triangle.

Example: $n = 3$, $k = 1$. Linear polynomials on a tetrahedron are uniquely determined by their values at the 4 vertices.

Example: $n = 3$, $k = 2$. Quadratric polynomials on a tetrahedron are uniquely determined by their values at the 4 vertices and the midpoints of the 6 edges. Note this means the dimension of this space is 10.