2.7. **Affine families of finite elements.** To describe a finite element on an arbitrary triangle in $\mathbb{R}^2$ in a way that will simplify computations, we use the following idea. Let $\hat{T}$ denote the reference triangle with vertices $\hat{\boldsymbol{a}}_1 = (1,0)$, $\hat{\boldsymbol{a}}_2 = (0,1)$, and $\hat{\boldsymbol{a}}_3 = (0,0)$. Given any triangle T with vertices $\boldsymbol{a}_1 = (a_{11}, a_{21})$, $\boldsymbol{a}_2 = (a_{12}, a_{22})$, $\boldsymbol{a}_3 = (a_{13}, a_{23})$, we can find a unique invertible affine mapping F_T mapping the triangle $\hat{T}$ to the triangle T given by $\boldsymbol{x} = F_T(\hat{\boldsymbol{x}}) = B_T\hat{\boldsymbol{x}} + b_T$, where B_T is an invertible 2×2 matrix and b_T a two-dimensional vector, satisfying: $F_T(\hat{\boldsymbol{a}}_i) = \boldsymbol{a}_i, \quad i = 1, \dots 3$. In fact, we can easily show that

$$B_T = \begin{pmatrix} a_{11} - a_{13} & a_{12} - a_{13} \\ a_{21} - a_{23} & a_{22} - a_{23} \end{pmatrix}, \qquad b_T = \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}.$$

Hence, we have the relations:

$$x_1 = (a_{11} - a_{13})\hat{x}_1 + (a_{12} - a_{13})\hat{x}_2 + a_{13}, \qquad x_2 = (a_{21} - a_{23})\hat{x}_1 + (a_{22} - a_{23})\hat{x}_2 + a_{23}.$$

If we invert this relationship, we find that

$$\begin{aligned} \hat{x}_1 &= [(a_{22} - a_{23})(x_1 - a_{13}) - (a_{12} - a_{13})(x_2 - a_{23})]/\det(B_T), \\ \hat{x}_2 &= [-(a_{21} - a_{23})(x_1 - a_{13}) + (a_{11} - a_{13})(x_2 - a_{23})]/\det(B_T). \end{aligned}$$

Given a set of shape functions $\hat{p} \in V(\hat{T})$ on the reference triangle $\hat{T}$, we then define the set of shape functions $V(T)$ on the triangle $T = F_T(\hat{T})$ by:

$$V(T) = \{p : p(\boldsymbol{x}) = \hat{p} \circ F_T^{-1}(\boldsymbol{x}) = \hat{p}(\hat{\boldsymbol{x}}), \quad \hat{p} \in V(\hat{T})\}.$$

With these definitions, we can translate the degrees of freedom and basis functions on the reference element to degrees of freedom and basis functions on an arbitrary triangle. For example, if we take as degrees of freedom on the reference triangle the values of a polynomial $\hat{p}$ at the points $\hat{\boldsymbol{x}}_i$, i.e., $\hat{p}(\hat{\boldsymbol{x}}_i)$, then the degrees of freedom on the triangle T will be the values $p(\boldsymbol{x})$ satisfying $p(\boldsymbol{x}) = \hat{p}(\hat{\boldsymbol{x}})$. In the case of linear functions, this means the values at the vertices of the triangle, since the map F_T is constructed to map vertices to vertices. We have already shown that the basis functions for linear functions on a triangle corresponding to vertex degrees of freedom are given by the barycentric coordinate functions $\lambda_i(\boldsymbol{x})$. By the presentation above, this means that if we denote by $\hat{\lambda}_i(\hat{\boldsymbol{x}})$, the barycentric coordinates on the reference triangle, then we should have $\hat{\lambda}_i(\hat{\boldsymbol{x}}) = \lambda_i(\boldsymbol{x})$. To get an understanding about how these mappings work, let us verify this fact. By definition,

$$A \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 1 & 1 & 1 \end{pmatrix}.$$

and

$$\hat{A} \begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \hat{\lambda}_3 \end{pmatrix} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ 1 \end{pmatrix}, \quad \text{where} \quad \hat{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

An easy calculation shows that

$$\begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \hat{\lambda}_3 \end{pmatrix} = \hat{A}^{-1} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ 1 - \hat{x}_1 - \hat{x}_2 \end{pmatrix}.$$

Recall that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} - a_{13} & a_{12} - a_{13} \\ a_{21} - a_{23} & a_{22} - a_{23} \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} + \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ 1 - \hat{x}_1 - \hat{x}_2 \end{pmatrix} = A \begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \hat{\lambda}_3 \end{pmatrix}.$$

So

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = A^{-1} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = A^{-1} A \begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \hat{\lambda}_3 \end{pmatrix} = \begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \hat{\lambda}_3 \end{pmatrix}.$$

Thus, we can construct basis functions in terms of the barycentric coordinates on the reference triangle and then map them to obtain the basis functions on the general triangle.

We can use these ideas to calculate the integrals in the finite element method. For example, to calculate an integral of the form

$$\int_T \nabla u \cdot \nabla v \, dx\, dy = \int_T \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) dx\, dy,$$

we transform the integral to the reference triangle by making the change of variable $\boldsymbol{x} = B\hat{\boldsymbol{x}} + b$. Since $\hat{v}(\hat{x}, \hat{y}) = v(x, y)$, we get by the chain rule that

$$\frac{\partial v}{\partial x} = \frac{\partial \hat{v}}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{v}}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial x}, \qquad \frac{\partial v}{\partial y} = \frac{\partial \hat{v}}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial y} + \frac{\partial \hat{v}}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial y}$$

Now $\hat{\boldsymbol{x}} = B^{-1}(\boldsymbol{x} - b)$, so

$$\frac{\partial \hat{x}}{\partial x} = B^{-1}_{11}, \quad \frac{\partial \hat{x}}{\partial y} = B^{-1}_{12}, \quad \frac{\partial \hat{y}}{\partial x} = B^{-1}_{21}, \quad \frac{\partial \hat{y}}{\partial y} = B^{-1}_{22}.$$

Hence,

$$\begin{pmatrix} \partial v/\partial x \\ \partial v/\partial y \end{pmatrix} = (B^{-1})^T \begin{pmatrix} \partial \hat{v}/\partial \hat{x} \\ \partial \hat{v}/\partial \hat{y} \end{pmatrix}, \quad \text{i.e.,} \quad \nabla v = (B^{-1})^T \hat{\nabla} \hat{v}.$$

Since $(B^{-1})^T = (B^T)^{-1}$, we also have $\hat{\nabla}\hat{v} = B^T \nabla v$. By the change of variable formula, we get

$$\begin{aligned} \int_T \nabla u \cdot \nabla v \, dx\, dy &= \int_{\hat{T}} (B^{-1})^T \hat{\nabla} \hat{u} \cdot (B^{-1})^T \hat{\nabla} \hat{v} |\det B| \, d\hat{x} d\hat{y} \\ &= |\det B| \int_{\hat{T}} (\hat{\nabla} \hat{u})^T B^{-1} (B^{-1})^T \hat{\nabla} \hat{v} \, d\hat{x} d\hat{y}. \end{aligned}$$

In the case of piecewise linear functions,

$$\frac{\partial \hat{v}}{\partial \hat{x}} = \hat{v}(\hat{\boldsymbol{a}}_1) - \hat{v}(\hat{\boldsymbol{a}}_3) = v(\boldsymbol{a}_1) - v(\boldsymbol{a}_3), \qquad \frac{\partial \hat{v}}{\partial \hat{y}} = \hat{v}(\hat{\boldsymbol{a}}_2) - \hat{v}(\hat{\boldsymbol{a}}_3) = v(\boldsymbol{a}_2) - v(\boldsymbol{a}_3).$$

We then compute

$$\int_\Omega \nabla u \cdot \nabla v \, dx\, dy = \sum_T \int_T \nabla u \cdot \nabla v \, dx\, dy.$$

2.8. **Tensor product finite elements.** We define the space Q_k as the space of polynomials which are of degree $\leq k$ in each of the variables $x_1, \dots x_n$ separately. The dimension of the space Q_k is $(k+1)^n$.

Example: $k = 1$:

$$Q_1(x, y) = c_0 + c_1 x + c_2 y + c_3 xy \quad (n = 2),$$
$$Q_1(x, y, z) = c_0 + c_1 x + c_2 y + c_3 z + c_4 xy + c_5 xz + c_6 yz + c_7 xyz \quad (n = 3).$$

Lemma 5. *A polynomial $q \in Q_k$ is uniquely determined by its values on the set*

$$M_k = \{\boldsymbol{x} = (i_1/k, i_2/k, \dots, i_n/k) \in \mathbb{R}^n : i_j \in \{0, 1, \dots, k\}, \quad 1 \leq j \leq n\}.$$

Example: $k = 1$: $n = 2$: $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$

$$q(x, y) = q(0, 0)(1 - x)(1 - y) + q(1, 0)x(1 - y) + q(0, 1)y(1 - x) + q(1, 1)xy.$$

Example: $k = 2$: $n = 2$: $(0, 0)$, $(1/2, 0)$, $(1, 0)$, $(0, 1/2)$, $(1/2, 1/2)$, $(1, 1/2)$, $(0, 1)$, $(1/2, 1)$, $(1, 1)$.

Quadrilateral elements: These are defined by using bilinear mappings of a rectangle, i.e., we define as the reference element the unit square $\hat{K}$. Then we can define the mapping

$$\boldsymbol{F}(\hat{x}, \hat{y}) = \boldsymbol{a}_1(1 - \hat{x})(1 - \hat{y}) + \boldsymbol{a}_2\hat{x}(1 - \hat{y}) + \boldsymbol{a}_3\hat{y}(1 - \hat{x}) + \boldsymbol{a}_4\hat{x}\hat{y}.$$

that maps $\hat{K}$ to the quadrilateral K with vertices $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$. To define the shape functions on the element K, we begin with a set $\hat{V}$ of polynomial shape functions defined on the reference element (e.g., the space $Q_1(\hat{x}, \hat{y})$). We then define a space of functions on the quadrilateral K by

$$V_F(K) = \{u : K \mapsto \mathbb{R} : \hat{u}_{K,F} \in \hat{V}\},$$

where $\hat{u}_{K,F}(\hat{x}, \hat{y}) = u \circ \boldsymbol{F}(\hat{x}, \hat{y}) = u(x, y)$. The complication here is that the elements in $V_F(K)$ need not be polynomials if the mapping $\boldsymbol{F}$ is not affine, i.e., if K is not a parallelogram.

2.9. C^1 **finite elements.** The Argyris element is a C^1 finite element defined on each triangle by a quintic polynomial with 21 degrees of freedom:

$$p(\boldsymbol{a}_i), (\partial p/\partial x)(\boldsymbol{a}_i), (\partial p/\partial y)(\boldsymbol{a}_i), (\partial^2 p/\partial x^2)(\boldsymbol{a}_i), (\partial^2 p/\partial x \partial y)(\boldsymbol{a}_i), (\partial^2 p/\partial y^2)(\boldsymbol{a}_i), \quad 1 \leq i \leq 3,$$
$$(\partial p/\partial n)(\boldsymbol{a}_{12}), (\partial p/\partial n)(\boldsymbol{a}_{13}), (\partial p/\partial n)(\boldsymbol{a}_{23}).$$

There are also other choice of degrees of freedom.