2.10. **Error estimates for piecewise linear interpolation.** The error estimate we derived previously related the error in the finite element method to the error in the approximation of the solution by the best approximation in the finite element subspace. We will now obtain bounds for this error by defining and analyzing the properties of the finite element interpolant. We consider the case of piecewise linear interpolation in detail. The derivation presented below is by multipoint Taylor expansions.

We begin with the Taylor series expansion with integral remainder in one dimension.

$$\begin{aligned} F(s) - F(s_0) &= \int_{s_0}^{s} F'(t)\,dt = \int_{s_0}^{s} F'(t)\frac{d}{dt}(t-s)\,dt \\ &= F'(t)(t-s)\,\Big|_{t=s_0}^{t=s} - \int_{s_0}^{s} F''(t)(t-s)\,dt = F'(s_0)(s-s_0) - \int_{s_0}^{s} F''(t)(t-s)\,dt. \end{aligned}$$

Hence,

$$F(s) = F(s_0) + F'(s_0)(s-s_0) + \int_{s_0}^{s} F''(t)(s-t)\,dt.$$

We next give a simple derivation of the error in 1-dimension. Let $a < b$ be given points. Then the linear function interpolating u at the points a and b is given by:

$$u_I(x) = \frac{b-x}{b-a}u(a) + \frac{x-a}{b-a}u(b).$$

To estimate the error $u(x) - u_I(x)$, we expand $u(a)$ and $u(b)$ in a Taylor series about the point x, using the integral remainder formula given above.

$$u(a) = u(x)+u'(x)(a-x)+\int_{x}^{a} u''(t)(a-t)\,dt, \qquad u(b) = u(x)+u'(x)(b-x)+\int_{x}^{b} u''(t)(b-t)\,dt.$$

Then

$$\begin{aligned} u(x) - u_I(x) &= u(x) - \frac{b-x}{b-a}u(a) - \frac{x-a}{b-a}u(b) \\ &= u(x)\left[1 - \frac{b-x}{b-a} - \frac{x-a}{b-a}\right] - u'(x)\left[\frac{b-x}{b-a}(a-x) + \frac{x-a}{b-a}(b-x)\right] \\ &\quad - \frac{b-x}{b-a}\int_{x}^{a} u''(t)(a-t)\,dt - \frac{x-a}{b-a}\int_{x}^{b} u''(t)(b-t)\,dt \\ &= -\frac{1}{b-a}\int_{a}^{b} u''(t)G(t,x)\,dt, \end{aligned}$$

where

$$G(t,x) = \begin{cases} (b-x)(t-a), & a < t < x \\ (x-a)(b-t), & x < t < b \end{cases}.$$

Hence,

$$|u(x) - u_I(x)|^2 = \frac{1}{(b-a)^2}\left|\int_{a}^{b} u''(t)G(t,x)\,dt\right|^2 \le \frac{1}{(b-a)^2}\int_{a}^{b} |u''(t)|^2\,dt \int_{a}^{b} |G(t,x)|^2\,dt,$$

and so

$$\int_a^b |u(x)-u_I(x)|^2\,dx \le \frac{1}{(b-a)^2}\int_a^b |u''(t)|^2\,dt \int_a^b\int_a^b |G(t,x)|^2\,dt\,dx$$
$$= \frac{(b-a)^4}{90}\int_a^b |u''(t)|^2\,dt.$$

Suppose now that we have a set of points $x_0, x_1, \ldots, x_N$ with $x_i - x_{i-1} = h_i$, $i = 1, \ldots, N$ and we denote by u_I the continuous piecewise linear interpolant of u, i.e., a continuous function that is linear on each subinterval $[x_{i-1}, x_i]$ and satisfies $u_I(x_i) = u(x_i)$, $i = 0, \ldots, N$. Then, applying the above formula with $a = x_{i-1}$ and $b = x_i$, and assuming that $h_i \le h$, we get

$$\int_{x_0}^{x_N} |u(x)-u_I(x)|^2\,dx = \sum_{i=1}^N \int_{x_{i-1}}^{x_i} |u(x)-u_I(x)|^2\,dx$$
$$= \frac{1}{90}\sum_{i=1}^N h_i^4 \int_{x_{i-1}}^{x_i} |u''(t)|^2\,dt \le \frac{h^4}{90}\int_{x_0}^{x_N} |u''(t)|^2\,dt.$$

Hence,

$$\|u-u_I\|_{L^2[x_0,x_N]} \le \frac{h^2}{\sqrt{90}}\|u''\|_{L^2[x_0,x_N]}.$$

We can derive an error estimate for the derivative in a similar way.

$$u_I'(x) = \frac{u(b)-u(a)}{b-a} = u'(x)\frac{(b-x)-(a-x)}{b-a}$$
$$+ \frac{1}{b-a}\int_x^b u''(t)(b-t)\,dt - \frac{1}{b-a}\int_x^a u''(t)(a-t)\,dt.$$

Hence,

$$u'(x) - u_I'(x) = -\frac{1}{b-a}\int_a^b u''(t)H(t,x)\,dt,$$

where

$$H(t,x) = \begin{cases} a-t, & a<t<x \\ b-t, & x<t<b \end{cases}.$$

Then

$$|u'(x)-u_I'(x)|^2 \le \frac{1}{(b-a)^2}\int_a^b |u''(t)|^2\,dt \int_a^b H^2(t,x)\,dt,$$

and so

$$\int_a^b |u'(x)-u_I'(x)|^2\,dx \le \frac{1}{(b-a)^2}\int_a^b |u''(t)|^2\,dt \int_a^b H^2(t,x)\,dt\,dx = \frac{(b-a)^2}{6}\int_a^b |u''(t)|^2\,dt.$$

Translating these results to each subinterval and summing, we get

$$\|u'-u_I'\|_{L^2[x_0,x_N]} \le \frac{h}{\sqrt{6}}\|u''\|_{L^2[x_0,x_N]}.$$

One can derive similar estimates in higher dimensions, but the proofs are more complicated.

In the case of piecewise linear interpolation, one first shows that on a triangle T,

$$\|u - u_I\|_{L^2(T)} \le C_T h_T^2 |u|_{2,T}, \qquad |u|_{2,T}^2 = \|u_{xx}\|_{L^2(T)}^2 + \|u_{xy}\|_{L^2(T)}^2 + \|u_{yy}\|_{L^2(T)}^2.$$

If we assume that $h_T \le h$ for all T, then as in the one-dimensional case, we get

$$\|u - u_I\|_{L^2(\Omega)}^2 = \sum_T \int_T (u - u_I)^2 \, dx \, dy \le C \sum_T h_T^4 |u|_{2,T}^2 \le Ch^4 \sum_T |u|_{2,T}^2 = Ch^4 |u|_{2,\Omega}^2.$$

Thus, we finally obtain

$$\|u - u_I\|_{L^2(\Omega)} \le Ch^2 |u|_{2,\Omega}.$$

To get estimates for the derivatives, we usually make the assumption that the mesh is *shape regular*. Let h_T denote the diameter of T and ρ_T the diameter of the largest ball contained in T. Define the *shape constant* $\sigma_T = h_T/\rho_T$. If we consider a family of meshes $\mathcal{T}_h$, $0 < h < 1$, we say that the family is *shape regular* if for all $T \in \mathcal{T}_h$ and all $0 < h < 1$, $\sigma_T \le C$ independent of T and h. One can then show that

$$\|\nabla u - \nabla u_I\|_{L^2(T)} \le C_T h |u|_{2,T},$$

where C_T depends on the shape constant σ_T, and then for shape regular meshes, we have by summing the squares of this inequality that

$$\|\nabla u - \nabla u_I\|_{L^2(\Omega)} \le Ch |u|_{2,\Omega}.$$