2.11. **Error estimates by scaling.** A more general technique for error estimates, which requires more mathematical background, is to use the Bramble-Hilbert lemma and scaling. We present here only the main ideas, most without proof. We again let T denote a triangle in the mesh $\mathcal{T}_h$ and $\hat{T}$ the reference triangle.

Lemma 6. *(Bramble-Hilbert) Suppose k and m are integers satisfying $0 \le m \le k+1$. Suppose $u \in H^{k+1}(\hat{T})$ and $\hat{\Pi}$ is an interpolation operator satisfying:*

$$\hat{\Pi}\hat{p} = \hat{p}, \quad \hat{p} \in P_k, \qquad \|\hat{\Pi}\hat{u}\|_{m,\hat{T}} \le \hat{C}_1 \|\hat{u}\|_{k+1,\hat{T}}$$

for some constant C_1 independent of $\hat{u}$. Then there exists a constant $\hat{C}_2$, depending only on k, $\hat{m}$, $\hat{T}$, and $\hat{\Pi}$, but independent of $\hat{u}$, such that

$$\|\hat{u} - \hat{\Pi}\hat{u}\|_{m,\hat{T}} \le \hat{C}_2 |\hat{u}|_{k+1,\hat{T}}, \qquad \text{where} \quad |\hat{u}|^2_{k+1,\hat{T}} = \sum_{|\alpha|=k+1} \|D^\alpha \hat{u}\|^2_{L^2(\hat{T})}.$$

Of course the lemma would be obvious if we used the full norm $\|\hat{u}\|_{k+1,\hat{T}}$ on the right hand side. The result is that we only need the highest order derivatives. Note that we have seen this result for $k = 1$, when $\hat{\Pi}\hat{u}$ is the linear polynomial interpolating $\hat{u}$ at the vertices. The technical part of the lemma is to show that $\|\hat{\Pi}\hat{u}\|_{m,\hat{T}} \le \hat{C}_1 \|\hat{u}\|_{k+1,\hat{T}}$. When $\hat{\Pi}\hat{u}(\hat{\boldsymbol{x}}) = \sum_{i=1}^N \hat{u}(\hat{\boldsymbol{x}}_i)\hat{p}_i(\hat{\boldsymbol{x}})$, i.e., the degrees of freedom are point values of $\hat{u}$ (and $\hat{p}_i$ is the corresponding dual basis function in P_k), this is done by applying the Sobolev imbedding theorem, which says that for all $\hat{\boldsymbol{x}}$, $|\hat{u}(\hat{\boldsymbol{x}})| \le C\|\hat{u}\|_r$, where $r > n/2$, n the number of space dimensions. Then

$$\|\hat{\Pi}\hat{u}\|_{m,\hat{T}} \le \sum_{i=1}^N |\hat{u}(\hat{\boldsymbol{x}}_i)| \, \|\hat{p}_i\|_{m,\hat{T}} \le C\|\hat{u}\|_r \sum_{i=1}^N \|\hat{p}_i\|_{m,\hat{T}} \le C\|\hat{u}\|_r.$$

So we have the desired estimate as long as $k+1 \ge n/2$. In particular, when $n = 2$ or 3, the result holds for $k \ge 1$ and $m = 0, 1$, the cases we need to get estimates for the finite element method for Poisson's equation.

Let $x = F_T(\hat{x}) = B_T\hat{x} + b_T$ be the affine mapping from the reference triangle $\hat{T}$ to the triangle T which maps vertices of $\hat{T}$ to vertices of T and let $v(x) = \hat{v}(\hat{x})$, i.e., $v(F_T(\hat{x})) = \hat{v}(\hat{x})$. Then we have the following estimates:

Lemma 7. *For each integer $m \ge 0$ with $v \in H^m(T)$, $\hat{v} \in H^m(\hat{T})$,*

$$|\hat{v}|_{m,\hat{T}} \le C_1 \|B_T\|^m |\det B_T|^{-1/2} |v|_{m,T}, \qquad |v|_{m,T} \le C_2 \|B_T^{-1}\|^m |\det B_T|^{1/2} |\hat{v}|_{m,\hat{T}}.$$

This result estimates the norms when we change variables between the triangles T and $\hat{T}$. Recall that in the case $m = 1$, we have derived the formula:

$$\int_T \nabla u \cdot \nabla v \, dx \, dy = |\det B| \int_{\hat{T}} (\hat{\nabla}\hat{u})^T B^{-1} (B^{-1})^T \hat{\nabla}\hat{v} \, d\hat{x} d\hat{y}.$$

Hence,

$$\int_T |\nabla v|^2\,dx\,dy = |\det B| \int_{\hat T} (|B^{-1})^T \hat\nabla \hat v|^2\,d\hat x d\hat y$$
$$\le |\det B| \int_{\hat T} \|(B^{-1})^T\|^2 |\hat\nabla \hat v|^2\,d\hat x d\hat y \le |\det B| \|B^{-1}\|^2 \int_{\hat T} |\hat\nabla \hat v|^2\,d\hat x d\hat y.$$

The next item is to estimate the quantities $|\det B_T|$, $\|B_T\|$, and $\|B_T^{-1}\|$ is terms of geometric quantities related to the triangles.

Lemma 8. *Let h_T denote the diameter of T and ρ_T the diameter of the largest ball contained in T. Then*

$$|\det B_T| = \textit{area } (T)/\textit{area } \hat T, \qquad \|B_T\| \le h_T/\rho_{\hat T}, \qquad \|B_T^{-1}\| \le h_{\hat T}/\rho_T.$$

We can combine these results to obtain error estimates for polynomial interpolation. Let $\sigma_T = h_T/\rho_T$ denote the shape regularity constant for the triangle T. Define an interpolation operator Π on the triangle T by $(\Pi v)(x) = (\hat\Pi \hat v)(\hat x)$. Then we get the following error estimate.

Lemma 9. *Suppose $u \in H^{k+1}(T)$ and $\hat\Pi$ is an interpolation operator satisfying:*

$$\hat\Pi \hat p = \hat p, \quad \hat p \in P_k, \qquad \|\hat\Pi \hat u\|_{m,\hat T} \le \hat C_1 \|\hat u\|_{k+1,\hat T}$$

for some constant C_1 independent of $\hat u$. Then there exists a constant C_3, depending only on k, $\hat m$, $\hat T$, and $\hat\Pi$, but independent of u, such that

$$|u - \Pi u|_{m,T} \le C_3 \sigma_T^m h_T^{k+1-m} |u|_{k+1,T}.$$

Proof.

$$|u - \Pi u|_{m,T} \le C_2 \|B_T^{-1}\|^m |\det B_T|^{1/2} |\hat u - \hat\Pi \hat u|_{m,\hat T} \le C_2 \|B_T^{-1}\|^m |\det B_T|^{1/2} \hat C_2 |\hat u|_{k+1,\hat T}$$
$$\le C_2 \|B_T^{-1}\|^m |\det B_T|^{1/2} \hat C_2 C_1 \|B_T\|^{k+1} |\det B_T|^{-1/2} |u|_{k+1,T}$$
$$\le C_3 \|B_T^{-1}\|^m \|B_T\|^{k+1} |u|_{k+1,T} \le C_3 \left(\frac{h_{\hat T}}{\rho_T}\right)^m \left(\frac{h_T}{\rho_{\hat T}}\right)^{k+1} |u|_{k+1,T} \le C_3 \sigma_T^m h_T^{k+1-m} |u|_{k+1,T}.$$

□

Recall that a family of triangulations $\mathcal{T}_h$ is shape regular if there exists a constant $\sigma > 0$ independent of h and T such that $\sigma_T \le \sigma$ for all $T \in \mathcal{T}_h$. Under this condition, we then have the following global approximation result, where Πu denotes a piecewise polynomial interpolant of u of degree $\le k$ that preserves polynomials on the reference triangle.

Lemma 10. *If in addition to the hypotheses of the previous lemma, we assume the mesh is shape-regular, then setting $h = \max_T h_T$, we have*

$$|u - \Pi u|_{m,\Omega} \le C_3 \sigma^m h^{k+1-m} |u|_{k+1,\Omega}.$$

Proof.

$$|u-\Pi u|^2_{m,\Omega} = \sum_T |u-\Pi_T u|^2_{m,T} \le \sum_T (C_3\sigma_T^m h_T^{k+1-m}|u|_{k+1,T})^2$$
$$\le (C_3\sigma^m h^{k+1-m})^2 \sum_T |u|^2_{k+1,T} = (C_3\sigma^m h^{k+1-m})^2|u|^2_{k+1,\Omega}.$$

□

2.12. **Order of convergence estimates for Ritz-Galerkin approximation schemes.** We now return to the approximation of the variational problem: Find $u \in V$ such that

$$a(u,v) = F(v), \quad \text{for all } v \in V,$$

by the standard Ritz-Galerkin approximation scheme:

Find $u_h \in V_h$ such that $\qquad a(u_h, v_h) = F(v_h), \quad$ for all $v_h \in V_h$.

We have previously established that under the conditions

$$a(v,v) \ge \alpha\|v\|_1^2, \qquad |a(u,v)| \le M\|u\|_1\|v\|_1, \quad |F(v)| \le K\|v\|_1,$$

we have the quasi-optimal error estimate

$$\|u-u_h\|_1 \le \frac{M}{\alpha}\|u-v_h\|_1, \quad \text{for all } v_h \in V_h.$$

The interpolation error estimate derived above then shows that if we choose the space V_h to consist of continuous piecewise polynomials of degree $\le k$ and the solution $u \in H^{r+1}(\Omega)$ with $1 \le r \le k$, then

$$\|u-u_h\|_1 \le Ch^r|u|_{r+1}.$$

We observe, however, that the error $\|u-u_I\|_{L^2} \le Ch^{r+1}|u|_{r+1}$. We next show that the same improved order of convergence in $L^2(\Omega)$ also holds for the error $u-u_h$.

Lemma 11. *Suppose Ω is a convex polygon. Then, under the hypotheses stated above,*

$$\|u-u_h\|_{L^2(\Omega)} \le Ch\|u-u_h\|_1 \le Ch^{r+1}|u|_{r+1}.$$

Proof. The proof uses a combination of elliptic regularity and duality. For simplicity, we consider only the variational formulation corresponding to the boundary value problem:

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

i.e., $V = \mathring{H}^1(\Omega)$, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx$, $F(v) = \int_\Omega fv\,dx$. More general problems can be done in a similar way. For such problems, the following regularity result for the solution u is known: Given $f \in L^2(\Omega)$, there exists a constant C independent of u and f, such that $\|u\|_2 \le C\|f\|_{L^2(\Omega)}$. Note that the equation says that the combination $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 \in L^2(\Omega)$. The regularity result says that each of the second derivatives $\in L^2(\Omega)$ and satisfies the indicated bound. To establish the lemma, we introduce the "dual problem": Find $w \in V = \mathring{H}^1(\Omega)$ such that

$$a(v,w) = (u-u_h, v), \quad \text{for all } v \in V,$$

i.e., w is the solution of the boundary value problem:

$$-\Delta w = u - u_h \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega.$$

The dual problem is chosen to have the same form as the original boundary value problem, but where f is replaced by the error $u - u_h$. From the elliptic regularity result, we know that w satisfies $\|w\|_2 \le C\|u - u_h\|_{L^2}$. Then, using Galerkin orthogonality,

$$\begin{aligned}\|u - u_h\|_{L^2}^2 &= (u - u_h, u - u_h) = a(u - u_h, w) = a(u - u_h, w - w_I)\\ &\le M\|u - u_h\|_1\|w - w_I\|_1 \le C\|u - u_h\|_1 h\|w\|_2 \le Ch\|u - u_h\|_1\|u - u_h\|_{L^2}.\end{aligned}$$

Hence,

$$\|u - u_h\|_{L^2} \le Ch\|u - u_h\|_1 \le Ch^{r+1}|u|_{r+1}.$$

□