Math 244 – Solutions for Sample Exam 1 – Professor Feehan
(100 points – 1 hour and 20 minutes)

1 (20 points). Consider the initial value problem \( ty′ + 2y = 4t^2, \ y(1) = 2. \)
(a) What type of first-order differential equation is this?
(b) What is the interval on which the solution exists?
(c) Find the solution.

Solution: (a) Linear. (b) Rewrite equation as \( y′ + 2t^{-1}y = 4t. \) The functions \( p(t) = 2t^{-1} \)
and \( g(t) = 4t \) are continuous on the intervals \((-\infty, 0)\) and \( (0, \infty)\). The interval \( (0, \infty) \)
contains the point \( t = 1 \) where the initial condition is specified, so the interval where
the solution to the initial value problem exists is \( (0, \infty) \). (c) The integrating factor is
\( \mu(t) = \exp(\int p(t) \, dt) = \exp(\int 2t^{-1} \, dt) = \exp(2 \ln |t|) = t^2. \) The general solution is
\( y = \mu^{-1} [\int \mu g \, dt + c] = t^{-2} [\int 4t^3 \, dt + c] = t^2 + ct^{-2}. \) Using \( y(1) = 2 \) gives \( c = 1 \) and so the
solution to the initial value problem is \( y(t) = t^2 + t^{-2}, \) for \( t > 0. \)

2 (10 points). Consider the equation \( y′ + y^2 \sin x = 0. \)
(a) What type of first-order differential equation is this?
(b) Find a solution other than \( y = 0. \)

Solution: (a) Non-linear, separable. (b) For \( y \neq 0, \) we can rewrite the equation as
\( y^2 \, dy = -\sin x \, dx. \) Integrating both sides gives \( -y^{-1} = \cos x + c \) and so \( y = -(c + \cos x)^{-1} \)
is the general solution.

3 (15 points). Consider the equation \( t^2 y′ + 2ty - y^3 = 0, t > 0. \)
(a) What type of first-order differential equation is this?
(b) Find a solution other than \( y = 0. \) [Hint: Use the substitution \( v = y^{-2} \) and first
solve the resulting equation for \( v. \)]

Solution: (a) For \( t \neq 0, \) we can rewrite the equation as \( y′ + 2t^{-1}y = -t^{-2}y^3. \) This is a
Bernouilli equation with \( p(t) = 2t^{-1}, q(t) = -t^{-2}, \) and \( n = 3: \) see Problem # 2.4.27 on
page 73 of Boyce-Diprima, an example we discussed in class. (b) Setting \( v(t) = y(t)^{-2} \)
as suggested, we have \( v′ = -2y^{-3}y′ \) and \( y′ = -\frac{1}{2}y^3v′. \) Substitute \( y′ = -\frac{1}{2}y^3v \) in the given
ordinary differential equation, divide the resulting equation by \( y^3 \) (assuming \( y \neq 0 \), and
use \( y^{-2} = v, \) to give \( -\frac{1}{2}t^2v′ + 2tv - 1 = 0. \) Divide across by \( -\frac{1}{2}t^2 \) to give \( v′ - 4t^{-1}v =
-2t^{-2}. \) This is a first-order, linear equation with \( p(t) = -4t^{-1} \) and \( q(t) = -2t^{-2}. \) The
integrating factor is \( \mu(t) = \exp(\int p(t) \, dt) = \exp(-\int 4t^{-1} \, dt) = \exp(-4 \ln |t|) = t^{-4}. \) Then
\( v = \mu^{-1} [\int \mu g \, dt + c] = t^4 [\int t^{-4} (-2t^{-2}) \, dt + c] = t^4 [\frac{2}{3} t^{-5} + c] = \frac{2}{3} t^{-1} + ct^4. \) But \( y = \pm v^{-1/2} \)
and so \( y = \pm (\frac{2}{3} t^{-1} + ct^4)^{-1/2} = \pm \sqrt{5t/(2 + 5ct^5)}^{1/2}. \)

4 (20 points). Consider the equation \( y′ = \frac{y}{y \sin y - x}. \)
(a) When is an equation of the form \( M(x, y) + N(x, y)y′ = 0 \) exact?
(b) Is the given equation exact? Explain using your answer to (a).
(c) Solve the given equation. [Hint: If it is not exact, use an integrating factor of the
form \( \mu(y) \) to solve a related exact equation.]

Solution: (a) The equation \( M + Ny′ \) is exact when \( M_y = N_x. \) (b) To check exactness,
rewrite equation as \( ydx + (x - y \sin y) \, dy = 0, \) so \( M(x, y) = y \) and \( N(x, y) = x - y \sin y. \)
Since \( M_y = 1 = N_x, \) the equation is exact when written in this form. [If one assumed
\( y \neq 0 \) and rewrote the equation in the form \( dx + (\frac{x}{y} - \sin y) \, dy = 0, \) one instead has
$M = 1$, $N = \frac{x}{y} - \sin y$, and so $M_y = 0 \neq N_x = 1/y$, so this equation is not exact, but could be made exact by solving for an integrating factor $\mu(y)$ such that $(\mu M)_y = (\mu N)_x$: one finds that $\mu = y$ works. (c) We seek a function $\psi(x, y)$ such that $\psi_x = M = y$ and $\psi_y = N = x - y \sin y$. Integration gives $\psi = xy + f(y)$ and so $\psi_y = x + f'(y) = N = x - y \sin y$. Hence, $f'(y) = y \sin y$ and integration by parts gives $f(y) = y \cos y - \sin y + c$. Therefore, $\psi(x, y) = xy + y \cos y - \sin y + c$ and the general solution $y$ is given implicitly by the equation $xy + y \cos y - \sin y + c = 0$.

5 (15 points). Euler’s method can be used to approximate the solution to first-order differential equations of the form $y' = f(t, y)$ at times $t_n$ by a sequence of solutions $y_n$ to the first-order difference equation

$$y_{n+1} = y_n + f(t_n, y_n)h, \quad y_0 = y(0), \quad n = 0, 1, 2, \ldots,$$

where $h = t_{n+1} - t_n$ is the step size.

Use Euler’s method to find approximate values to the initial value problem $y' = 5(t + 1) - 3 \sqrt{y}$, $y(0) = 1$ at times $t = 1, 2$ using $h = 1$. It is not necessary to simplify your answer for $y_2$.

**Solution:** We wish to solve $y' = f(t, y) = 5(t + 1) - 3 \sqrt{y}$, $y(0) = 1$, using Euler’s method [note correction to $f(t, y)$]. From the given information, we have $t_0 = 0$, $y_0 = 1$, $h = 1$, $t_1 = t_0 + h = 1$, and so $y_1 = y_0 + f(t_0, y_0)h = 1 + f(0, 1) = 1 + 5 - 3 = 3$. Then $y_2 = y_1 + f(t_1, y_1)h = 3 + f(1, 3) = 3 + 10 - 3 \sqrt{3} = 13 - 3 \sqrt{3} \approx 7.8038$.

6 (20 points). Consider the initial value problem $y'' + y' - 2y = 0$, $y(0) = 1$, $y'(0) = 1$.

(a) Find two solutions, $y_1, y_2$, to the equation $y'' + y' - 2y = 0$.

(b) Compute the Wronskian determinant $W(y_1, y_2)$ of the solutions you found in (a).

What property must $W(y_1, y_2)$ satisfy in order that $y_1, y_2$ form a fundamental set of solutions?

(c) Find the solution to the given initial value problem.

**Solution:** (a) The characteristic equation is $r^2 + r - 2 = (r - 1)(r + 2) = 0$, so $r_1 = 1$, $r_2 = -2$ and we can take $y_1 = e^t$, $y_2 = e^{-2t}$. (b) The Wronskian determinant is $W(y_1, y_2) = y_1y_2' - y_1'y_2 = -3e^{-t}$. For $\{y_1, y_2\}$ to be a fundamental set of solutions, we must have $W(y_1, y_2)(t) \neq 0$ for at least one $t$ on the interval where the solutions are defined: this property holds here since $W(y_1, y_2) = -3e^{-t}$ is never zero, for any $t$ in $(-\infty, \infty)$. (c) The general solution is $y = c_1e^t + c_2e^{-2t}$, so we solve for $c_1, c_2$ satisfying $y(0) = 1, c_1 + c_2 = 1, y'(0) = c_1 - 2c_2 = 1$. This gives $c_1 = 1, c_2 = 0$ and so the solution to the initial value problem is $y = e^t$. 