The general solution is $y = c_1 y_1 + c_2 y_2$, of the equation $y'' - 2y' + y = 0$. 

**Solution:** Characteristic equation is $r^2 - 2r + 1 = (r - 1)^2 = 0$, so $r_1 = r_2 = 0$ are the characteristic roots. Hence, the general solution is $y(t) = c_1 e^t + c_2 t e^t$.

2 (20 points). Consider the equation $t^2 y'' + 2ty' - 2y = 0, t > 0$.

(a) Verify that $y_1(t) = t$ is a solution to the equation.

(b) Use the method of reduction of order to find a second solution $y_2(t)$ of the given equation. [Hint: Look for a solution of the form $y(t) = v(t)y_1(t)$ and solve the resulting first order equation for $v'$.]

**Solution:** (a) $y' = 1$, $y'' = 0$, and substituting for $y$ into the given equation gives $t^2(0) + 2t(1) - 2t = 0$, as required.

(b) Set $y = tv$, so $y' = tv' + v$, and $y'' = tv'' + 2v'$. Substituting for $y$ into the given equation and simplifying gives $t^2 v'' + 4tv' = 0$. Set $u = v'$ and, for $t \neq 0$, this gives $\frac{du}{dt} = -\frac{4}{t} u$ or $\frac{du}{u} = -\frac{4}{t} dt$. Integrating both sides gives $\ln |u| = \ln(t^{-4}) + C$, so $u = t^{-4}$ is one solution. Now integrating $v' = t^{-4}$ gives $v = -\frac{1}{4} t^{-3}$ and so $y_2 = t^{-2}$ is a second solution. The general solution is $y = c_1 y_1 + c_2 y_2 = c_1 t^2 + c_2 t^{-2}$. (Credit only given for this method.)

3 (20 points). Consider the equation $y'' - 2y' - 3y = 3e^{2t}$.

(a) Find the general solution, $y_h = c_1 y_1 + c_2 y_2$, of the associated homogeneous equation.

(b) Find a particular solution, $y_p$, to the given inhomogeneous equation.

(c) Find the solution to the initial value problem, if $y(0) = 0$ and $y'(0) = 1$.

**Solution:** (a) Characteristic equation is $r^2 - 2r - 3 = (r - 3)(r + 1) = 0$, so $r_1 = 3, r_2 = -1$. The general solution of the homogeneous equation is $y_h(t) = c_1 e^{3t} + c_2 e^{-t}$.

(b) From Table 3.6.1, we seek a particular solution of the form $y = Ae^{2t}$, so $y' = 2Ae^{2t}$, $y'' = 4Ae^{2t}$. Substituting into the given inhomogeneous equation yields $-3Ae^{2t} = 3e^{2t}$, so $A = -1$ and we can take $y_p = -e^{2t}$.

(c) The general solution to the inhomogeneous equation is $y(t) = c_1 e^{3t} + c_2 e^{-t} - e^{2t}$. Applying the initial conditions gives the equations $c_1 + c_2 - 1 = 0, 3c_1 - c_2 - 2 = 1$. Solving this pair of equations yields $c_1 = 1$ and $c_2 = 0$, so the solution to the initial value problem is $y(t) = e^{3t} - e^{2t}$.

4 (10 points). Consider the equation $y'' + y = g(t)$, where $g(t) = \tan t$.

(a) Find the general solution, $y_h = c_1 y_1 + c_2 y_2$, of the associated homogeneous equation.

(b) Compute the Wronskian determinant $W(y_1, y_2)(t)$ of the two solutions you found in (a).

(c) Find a particular solution $y_p$ to the given inhomogeneous equation. If you use the following formula for a particular solution, it is not necessary to evaluate the integrals:

$$y_p = -y_1(t) \int \frac{y_2(t) g(t)}{W(y_1, y_2)(t)} \, dt + y_2(t) \int \frac{y_1(t) g(t)}{W(y_1, y_2)(t)} \, dt.$$
Solution: (a) The characteristic equation is $r^2 + 1 = 0$, so $r_1 = i$, $r_2 = -i$. The solution to the homogeneous equation is $y_h(t) = c_1 \cos t + c_2 \sin t$. (b) For $y_1 = \cos t$, $y_2 = \sin t$, their Wronskian determinant is

$$
\det \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} = \det \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \cos^2 t + \sin^2 t = 1.
$$

(c) From the given formula,

$$
y_p(t) = -\cos t \int \frac{\sin^2 t \cos t}{\cos t} dt + \sin t \int \sin t dt.
$$

Although not required, one finds that $y_p(t) = -(\cos t) \ln(\tan t + \sec t)$.

5 (20 points). Consider the matrix

$$
A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}.
$$

(a) (2 points) Explain whether the matrix is self-adjoint or not. (2 points) What can one say in general about the eigenvalues of self-adjoint matrices?

(b) (8 points) Find all the eigenvalues of $A$, given that $\lambda = -1$ is one eigenvalue. You may use a calculator to verify your work, but you must do the computation by hand and show all your work to receive full credit.

(c) (8 points) Compute all eigenvectors of $A$ corresponding to $\lambda = -1$, bearing in mind that there may be more than one linearly independent eigenvector. You may use a calculator to verify your work, but you must do the computation by hand and show all your work to receive full credit.

Solution: (a) The matrix is self-adjoint: Since $A$ is real, $\overline{A} = A$ and so $A^* = A^T = A$ (symmetric). Self-adjoint matrices have real eigenvalues.

(b) The determinant of $A - \lambda I$ is

$$
\det \begin{pmatrix} 3 - \lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3 - \lambda \end{pmatrix} = -\lambda^3 + 6\lambda^2 + 15\lambda + 8.
$$

We are told $\lambda + 1$ is a factor of this polynomial. Synthetic division gives $\lambda^3 - 6\lambda^2 - 15\lambda - 8 = (\lambda + 1)(\lambda^2 - 7\lambda - 8)$ and the quadratic factor has roots $\lambda = -1$, $\lambda = 8$.

(c) Setting $\lambda = -1$, we must solve the system $(A + I)x = 0$, where $x = (x_1, x_2, x_3)^T$ is an eigenvector. Hence, we solve

$$
\begin{align*}
4x_1 + 2x_2 + 4x_3 &= 0, \\
2x_1 + x_2 + 2x_3 &= 0, \\
4x_1 + 2x_2 + 4x_3 &= 0.
\end{align*}
$$

This reduces to $2x_1 + x_2 + 2x_3 = 0$ and so $x_2 = -2x_1 - 2x_3$. Hence, a typical eigenvector is a linear combination

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.
$$

and two linearly independent eigenvectors are $(1, -2, 0)^T$, $(0, -2, 1)^T$. (They are not unique.)
6 (20 points). Consider the following linear, second-order differential equation with variable coefficients:

\[ x^2 y'' - 3xy' + 4y = 0. \]

(a) (3 points) What type of equation is this?

(b) (14 points) Determine the general solution to the differential equation.

(c) (3 points) On which interval(s) is your solution valid?

Solution: (a) Euler’s equation. (b) Looking for a solution of the form \( y = x^r \) yields the characteristic equation \( r(r - 1) - 3r + 4 = 0 \), which factors as \( r^2 - 4r + 4 = (r - 2)^2 \). Hence, \( r_1 = r_2 = 2 \) and the general solution is \( y(x) = c_1 x^2 + c_2 x^2 \ln |x| \). (c) The solution is valid for all \( x \) except \( x \neq 0 \).