1 (10 points). For which values of $x$ does the following series converge absolutely?

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2(x+2)^n}{3^n}.$$ 

2 (15 points). In the following steps, you are asked to solve the differential equation for $y(x)$ by means of a power series about the point $x_0 = 1$.

$$y'' - xy - y = 0.$$ 

(a) Is $x_0 = 1$ an ordinary or singular point for this equation [3 points]? Explain your answer [1 point].

(b) Find the recurrence relation for the coefficients $a_n, n \geq 0$ [8 points].

(c) Find the first three (3) terms in each of the two linearly independent solutions, $y_1(x), y_2(x)$ [4 points].

3 (20 points). Consider the first-order system for $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

$$x' = -(x - y)(1 - x - y), \quad y' = x(2 + y).$$ 

(a) Find all critical points of the system [4 points].

(b) Find the corresponding linear system near the critical point $(0, 0)$ [4 points], find the eigenvalues of the linear system [2 points], and explain whether the linear system is stable, unstable, or asymptotically stable [2 points].

(c) Find the corresponding linear system near the critical point $(-2, -2)$ [6 points], find the eigenvalues of the linear system [1 point], and explain whether the linear system is stable, unstable, or asymptotically stable [1 point].

4 (15 points). Consider the following linear, first-order system, where $x(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$:

$$x' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} x.$$ 

(a) Find the eigenvalues and eigenvectors of the given matrix. You must show all your work to receive credit [8 points].

(b) Find the general solution, $x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$ [3 points].

(c) If $x(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, solve the initial value problem [2 points].

(d) Explain whether the linear system is stable, unstable, or asymptotically stable [2 points].

5 (20 points). Consider the following linear, first-order system, where $x(t) = (x(t), y(t))^T$:

$$x' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} x.$$ 

(a) Find the eigenvalues and eigenvectors of the given matrix; if there is only one eigenvalue and one linearly independent eigenvector, find a generalized eigenvector corresponding to that eigenvalue and eigenvector. You must show all your work to receive credit [8 points].

(b) Find the general solution, $x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$ [7 points].

(c) Explain whether the linear system is stable, unstable, or asymptotically stable [5 points].
6 (15 points). Consider the following inhomogeneous differential equation for $y(t)$:

$$y''' - 3y'' + 3y' - y = 12e^t.$$

(a) Find the general solution, $y_h = c_1y_1 + c_2y_2 + c_3y_3$, of the associated homogeneous equation [6 points].

(b) Find a particular solution, $y_p$, to the given inhomogeneous equation [9 points].

7 (15 points). Consider the following inhomogeneous differential equation for $y(t)$:

$$y''' - 4y' = t + 3\cos t + e^{-2t}.$$

(a) Find the general solution, $y_h = c_1y_1 + c_2y_2 + c_3y_3$, of the associated homogeneous equation [4 points].

(b) Find a particular solution, $y_p$, to the given inhomogeneous equation [11 points].

8 (15 points). Consider the following differential equation for $y(t)$:

$$16y''' - 8y' + 145y = 0.$$

(a) Find the general solution, $y(t) = c_1y_1(t) + c_2y_2(t)$, taking care to express your solution in terms of real (not complex) functions $y_1(t), y_2(t)$ [4 points].

(b) If $y_1(t), y_2(t)$ are your two solutions, compute their Wronskian determinant $W(y_1, y_2)$ [4 points].

(c) For which values of $t$ are your solutions $y_1, y_2$ linearly independent [3 points]?

(d) If $y(0) = -2$ and $y'(0) = 1$, solve the initial value problem [4 points].

9 (15 points). Find the general solution, $y(x) = c_1y_1(x) + c_2y_2(x)$, to the following differential equation, taking care to express your solution in terms of real (not complex) functions $y_1(x), y_2(x)$ [11 points]:

$$x^2y'' + 3xy' + 5y = 0, \quad x \neq 0.$$

Is $x_0 = 0$ a regular point or singular point for this equation [1 point]? What about $x_0 = 1$ [1 point]? Explain your answers [2 points].

10 (10 points). Transform the following second-order scalar differential equation for $x(t)$ into a system of first-order differential equations, $\frac{d}{dt}x = Ax + g$, for a suitable $2 \times 2$ matrix $A$ and vector-valued functions $x(t), g(t)$:

$$x'' + 3x' + 7x = e^t + \sin t, \quad x(0) = 1, \quad x'(0) = 2.$$

11 (15 points). Consider the vectors $x^{(1)}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$, $x^{(2)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$.

(a) Compute the Wronskian determinant of $x^{(1)}(t), x^{(2)}(t)$ [5 points].

(b) For which values of $t$ are the vectors $x^{(1)}(t), x^{(2)}(t)$ linearly independent [5 points]?

(c) Find the system, $\frac{d}{dt}x = P(t)x$, for $x(t)$ satisfied by $x^{(1)}(t), x^{(2)}(t)$ [5 points].

12 (10 points). Euler’s method can be used to approximate the solution to first-order differential equations of the form $y' = f(t, y)$ at times $t_n$ by a sequence of solutions $y_n$ to the first-order difference equation

$$y_{n+1} = y_n + f(t_n, y_n)h, \quad y_0 = y(0), \quad n = 0, 1, 2, \ldots,$$

where $h = t_{n+1} - t_n$ is the step size. Use Euler’s method to find approximate values for the solution to the initial value problem

$$y' = 2t + e^{-ty}, \quad y(0) = 1,$$

at times $t = 1, 2$ using $h = 1$. It is not necessary to simplify your answer for $y_2$. 

13 (15 points). Consider the equation $(3xy + y^2) + (x^2 + xy)y' = 0$ for $y(x)$.

(a) When is an equation of the form $M(x, y) + N(x, y)y' = 0$ exact [3 points]?

(b) Is the given equation exact? Explain using your answer to (a) [3 points].

(c) Find the general solution of the form $\psi(x, y) = c$ to the given equation [9 points]. [Hint: If the equation is not exact, use an integrating factor of the form $\mu(x)$ to solve a related exact equation.]

14 (10 points). Consider the first-order scalar differential equation, $y' + \frac{1}{t}y = 3\cos 2t$.

(a) For which values of $t$ will the solution exist? Your explanation should be based on the equation coefficients [2 points].

(b) Find the general solution [8 points].