Goal: Head toward Fenchel-Nielsen coordinates

Lecture 3: Hyperbolic Geometry

Conclusion: All hyperbolic geodesic surfaces of $(\Sigma, ds^2)$ are invariant of the Riemann surface $\Sigma / \Gamma$. In particular, the length of the shortest geodesics are locally constant, i.e., $\ell_\Gamma : \mathcal{L}(\Sigma / \Gamma) \to \mathbb{R}$ is locally constant.

Definition: A geodesic $\gamma : [a, b] \to (\Sigma, ds^2)$ is locally constant, i.e., $\ell_\Gamma : \mathcal{L}(\Sigma / \Gamma) \to \mathbb{R}$ is locally constant.

Example: The hyperbolic annulus $X_\epsilon = \mathbb{H} / \Gamma \times \mathbb{R}$, where $\Gamma = \{ z \in \mathbb{H} : \text{Re}(z) < \epsilon \}$, has length $\log(\log(\pi/\epsilon))$.

What is the length of the shortest geodesic $\gamma_{\text{min}}$ in $\Sigma / \Gamma$?

Solution:
What are the closed geodesics in $X_\epsilon$?

Indeed, if $\gamma$ is a closed geodesic in $X_\epsilon$, then $\gamma \in \mathcal{H}$, i.e., $\gamma$ is a geodesic in the hyperbolic plane $\mathbb{H}$ invariant under $\mathbb{H}/\Gamma$.

\[ \ell_\Gamma(\gamma) = \int_{x_0}^{x_1} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

Now, $i \sim i' \Rightarrow \text{distance log}_2 \Rightarrow \text{result.}$

Conclusion: If $X_\epsilon$ is biholomorphic to $X_{\epsilon'}$, $\epsilon, \epsilon' > 0 \Rightarrow \log(\log(\pi/\epsilon')) = \log(\log(\pi/\epsilon)) = 1 = \lambda$

Therefore, $\mu < d < 2 \Rightarrow \mu < d < 2 \Rightarrow \mu' < d' < 2 \Rightarrow \mu = \mu'$

\[ \text{Mod}(1 < d < 2) \cong \mathbb{H} \cup \mathbb{H}, \text{ hyperbolic space.} \]

Goal: Understand Moduli space

Homework: Use Schwarz lemma to show that $f : \Sigma_1 \to \Sigma_2$ is analytic between two hyperbolic surfaces. Then $f$ decreases hyperbolic distances.

Next goal: understand $\text{Mod}(\Sigma)$, Teichmüller space, use the use of Poincaré metric.

HW 1: Compute the length of a circle of radius $1$ in hyperbolic space.

2. Show that a circle in $(\mathbb{H}, ds^2)$ is a Euclidean circle in $\mathbb{H}$.

3. Show Gauss-Bonnet theorem

4. Show that the area of a hyperbolic triangle is less than the length of its edge.

(Gromov)

\[ |\log(\text{Sigma}(\pi))| = \text{length of geodesic corresponding to } \pi. \]
Lecture 9. Basic geometry of geodesics

\((M, g)\) Riemannian manifold. \(\forall p \in M, \text{ the ball } B_r(p) = \{ x \in M \mid d(x, p) < r \}\)

Geodesic: \([a, b] \to M: \text{ locally the shortest path}\)

Key Fact: \(\forall p \in M, \exists 0 < r \to B_r(p) \text{ is convex and } \forall x_1, x_2 \in B_r(p), \exists! \text{ shortest geodesic } \gamma : [x_1, x_2] \to B_r(p) \text{ from } x_1 \text{ to } x_2\).

Furthermore: \(\exists a, b \in B_r(p), \lim_{x \to a} \gamma(x, y) = g(y, y) \quad \text{length}(\gamma(x, y)) \to \text{length}(\gamma(a, b))\).

Equi. True w.r.t \((H, ds^2)\) or \((\mathbb{D}, ds^2)\). \(\text{geodesics} \quad \text{are in}\)

**Basic Thm.** \((M, g)\) closed Riemannian manifold \(M \ni S^1 \to M\) so that \(d \not\equiv p\).

(\(d\) continuous & cannot be extended to \(\mathbb{D}^2\)). Then \(\exists \alpha \text{ a geodesic closed } \beta : S^1 \to M\) so that \(\beta \equiv \alpha\). \(\forall t \in \text{ continuous } H : S^1 \times [0, 1] \to M\)

\(s.t. H(t, 0) = \alpha(t), H(t, 1) = \beta(t) \quad \forall t\).

**Pf.** Note if \(\alpha \equiv pt\), \(\beta = \text{ constant geodesic}\).

**Pf.**

**Lemma 1.** \(\exists \delta > 0 \text{ s.t. } \forall p \in M, \exists \text{ geodesics convex and } \text{ in } \mathbb{R}^2\).

**Pf.** Lebesgue lemma from point set topology \(G = \{ Br_p(p) \mid p \in M, p\}

the radius \(\ln \text{ key fact}\).

**Lemma 2.** If \(d_1, d_2 : S^1 \to M \text{ s.t } d(x_1(t), x_2(t)) \leq \delta/2 \forall t, \text{ then } d_1 \leq d_2\).

**Pf.** \(d_2(t) \in B_{\delta/2} (x_1(t))\). \(H(x, t) : \text{ the geodesic path, parameterized proportional to arc length } \left(\| \frac{d}{dt} H(x,t) \| = \text{ const}\right) \text{ from } x_1(t) \text{ to } x_2(t)\).

Continuity follows from the solution of ODE, depends on initial point. \(\square\)

Now the proof: Let \(L = \inf \frac{1}{2} \text{ length}(r) \mid r \in M \}

Then \(L \geq \delta/2\). Since \(L < \delta/2 \Rightarrow r \in B_{\delta/2}(p) \Rightarrow r \not\equiv pt \Rightarrow d(x, p)\)
Lecture 7. Basic geometry of geodesics

Let \( r_n \) be a seq. of piecewise smooth loops \( r_n \equiv a \), \( \lim_{n \to \infty} \text{length}(r_n) = L \) and \( \text{length}(r_n) \leq 2L \).

Modify \( r_n \) s.t.

1. \( \| r_n'(t) \| = \frac{\text{length}(r_n)}{2\pi} \) (reparametrization) (Each path \( y \to y(t) \| = 1 \))

   \[ \Rightarrow \text{for } t, t' \in S^1 \]
   \[ d(r_n(t), r_n(t')) \leq \frac{\text{length}(r_n[S^1])}{2\pi} \leq \frac{2L}{2\pi} = d(t, t') \]

2. Take \( N \gg 1 \) s.t. \( \frac{2L}{N} \leq \frac{\pi}{2} \), and \( t_i = \text{equal distance points of } S^1 \) w/ \( d(t_i, t_{i+1}) = \frac{2\pi}{N} \).

\[ \Rightarrow d(r_n(t_i), r_n(t_{i+1})) \leq \frac{\pi}{2} \quad r_n[t_i, t_{i+1}] \subset B_{\frac{\pi}{2}}(r_n(t_i)) \]

3. Replace \( r_n \), making \( r_n[t_i, t_{i+1}] \) the shortest geodesic joining \( t_i \) to \( B_{\frac{\pi}{2}}(r_n(t_i)) \).

Now use key fact to define \( \beta: [t_i, t_{i+1}] \) to be \( \lim_{n \to \infty} r_n(t_i, t_{i+1}) \).

\[ \Rightarrow \beta = \lim_{n \to \infty} r_n \text{ uniformly s.t. } \text{length}(\beta) = \lim_{n \to \infty} \text{length}(r_n) = L \]

We claim \( \beta \) is a closed geodesic s.t. \( \beta \equiv a \).

\[ \exists N \gg 1 \text{ s.t. } d(\beta(t_i), r_n(t)) \leq \frac{\pi}{2} \quad \forall t \Rightarrow \beta \equiv r_n \equiv a. \]

Next \( \beta \) is geodesic except possibly at \( \beta(t_i)'s \)

Since otherwise \( \Rightarrow \beta = \beta_i \text{ s.t. } \text{length}(\beta_i) < L \).

\[ \square \]

Homework: Write down carefully the proof of: for any complete Riemannian \( (M, g) \) and \( d:(C^0, \o, \cdot, a) \to (M, R^2), \exists \) a geodesic path \( (\beta, C^0, \o, a) \to (M, R^2) \) s.t. \( d \equiv \beta \text{ relative to } \o. \)
lecture 9. Basic Geodesics

Where do you use the completeness?

RM1 The geodesic $\beta$ may not be unique: $S^1 \times S^1$ flat torus $\mathbb{R}^2 / \mathbb{Z} \times \mathbb{Z}$.

RM2 If $M$ is not compact, even $(M, g)$ is complete, $\beta$ may not exist.

$$\Sigma = \mathbb{H} / \langle z \rightarrow z + 1 \rangle \cong \mathbb{D} \text{-fol} \text{ hyperbolic} \times \text{the quasiflats } \mathcal{Q}(t) = e^{it}, 0 \leq t \leq 1.$$

First, $l = 0$ now since $d(M_i, M_i + 1) = \frac{1}{n} \to 0$.

Next, $\Sigma$ contains NO closed geodesics at all. Indeed if $q: S^1 \to \Sigma$ is a closed geodesic then $\tilde{q}(t) = q$ is a geodesic if $H$ invariant under $Z \to Z + 1$. But there is no such. (Not shown)

Proposition. If $(\Sigma, g)$ is hyperbolic and $\alpha \neq \beta$ are two homotopic closed geodesics, then $\alpha = \beta$ (contradiction of $\neq$). (True also for neg curv)

Proof. We need the basic information. Easy proof (left to do).

$\neg$ Theorem. If $\beta$ be the positive geodesic in $H$. Then for any $r > 0$

$$N_r(\beta) = \{ x \in H / d(x, \beta) \leq r \} = \{ x \in H / r - 0 \leq \text{Arg}(z) \leq 2 \}$$

where $\text{Arg}(z) = \theta$,

$$d(x, \beta) = \sqrt{\text{atan}^2 (r) + \text{atan}^2 (\theta - r)}$$

Proof The isometry $z \mapsto z$ leaves $\beta$ invariant $\forall \lambda > 0 \Rightarrow$

$$N_r(\beta) \text{ is invariant under } z \mapsto \lambda z \Rightarrow N_r(\beta) = \{ \frac{1}{\lambda} z : \text{Arg}(z) \leq \theta \}$$

The actual calculation is homework. $x(t) = (\text{cost}, \text{sipt})$

$$\gamma = \sqrt{\frac{1}{2} - \frac{\text{sipt}^2}{\text{cost}^2}} \Rightarrow \ln \text{tangent} \frac{\theta}{2} = \ln \text{tangent} \frac{\pi}{2} - \theta \Rightarrow$$
Lecture 9. Basic Geodesics

Corollary: (1) For any geodesic \( \gamma \in H \cup D^2 \), for any \( y \). \( N_y(\gamma) \) is

\[ \text{tg}(y) = 5 \sinh(r) \]

\( y \) s.t. \( \exists N_y(\gamma) \) circles, union of two circular arcs

(2) if \( \gamma_1, \gamma_2 \) are two geodesics s.t. \( \gamma_1 \subset N_y(\gamma_2) \) for some \( y \Rightarrow \gamma_1 = \gamma_2 \)

Now the proof of uniqueness \( H/\Gamma \)

Suppose \( \alpha, \beta : S^1 \to \Sigma \) two homotopic geodesics s.t. \( H : S^1 \times I \to \Sigma \)

is the smooth homotopy. Define \( F : \mathbb{R} \times [0, 1] \to \Sigma \) by

\[ F(x, t) = H(e^{2\pi i x}, t) \quad s.t \ F(0, 0), F(0, 1) \text{ are geodesics} \]

Lifting Thm: If \( \pi : X \to Y \) is a covering map (i.e. \( \pi, H \to H/\Gamma \)) and \( f : A \to Y \) is a continuous map from a simply connected

manifold, then \( \exists \) a continuous \( \tilde{f} : A \to X \) s.t. \( \pi \circ \tilde{f} = f \)

(\( \tilde{f} \) is called a lift of \( f \)).

Let \( \tilde{F} : \mathbb{R} \times [0, 1] \to H \) be a lifting of \( F \), s.t \( \tilde{F}(0, 0), \tilde{F}(0, 1) \)

are two geodesics \( \alpha, \beta \) in \( H \).

Projecting down to \( \alpha, \beta \). \[ \text{max} \{ \text{length}(H(x, t)) \} = \text{compactness} \]

Now let \( \gamma = \max \{ \text{length}(H(x, t)) \mid x \in S^1 \} \) 

\[ d(\tilde{\alpha}(s), \tilde{\beta}(s)) \leq \text{length}(H(e^{2\pi i s}, t)) \leq \gamma \]

\[ \Rightarrow \gamma \leq 1 \leq N_y(\gamma) \Rightarrow \tilde{a} = \tilde{\beta} \] by the corollary

\[ \Rightarrow d(\tilde{a}, \tilde{\beta}) = 0 \]

\[ \Rightarrow d = \beta. \]
Corollary: Suppose $\alpha$ is an essential loop in a closed hyperbolic surface $\Sigma$ and homotopic to the closed geodesic $\beta$. If $\tilde{\beta}$ is a lift of $\beta$ to $\tilde{\Sigma}$, then there exists a lift $\tilde{\alpha}$ of $\alpha$ such that $\tilde{\alpha} \cap N_r(\tilde{\beta})$ for some $r > 0$.

**Thm:** If $\alpha$ is a simple essential loop homotopic to a closed geodesic $\beta \in \pi_1(\Sigma)$, then $\beta$ is simple.

**Proof:** If $\beta$ is not simple, then $\exists$ two lifts $\tilde{\beta}_1, \tilde{\beta}_2$ of $\beta$ such that $\tilde{\beta}_1, \tilde{\beta}_2$ are not distinct. Let $\tilde{\alpha}_i$ be the lift of $\alpha$ such that $\tilde{\alpha}_i \subseteq N_r(\tilde{\beta}_i)$. Then $\tilde{\alpha}_1 \cap \tilde{\alpha}_2 \neq \emptyset$, contradicting simplicity. Thus, $\alpha$ is not simple.

**RM:** The same argument shows that if $\alpha_1, \alpha_2$ are two disjoint simple essential loops homotopic to geodesics $\beta_1, \beta_2$, then $\beta_1 \cap \beta_2 = \emptyset$.

Let us produce F-N coordinates. Now

**Mention Priyan's Thm**

**Open Question:** Does $\exists$ constant $c > 0$ such that, if a closed geodesic $\alpha \in (\Sigma, g) = \pi_1(\Sigma)$ has a lift to a simple closed geodesic $\tilde{\alpha}$ at most $c$π1 that covers $\Sigma$?

**May be some fundamental domain.**
Lemma 10. The Fenchel Nielsen Coordinates of Teichmüller Space.

All surfaces are assumed to be orientable.

Thm Suppose \((\Sigma, g)\) closed orientable hyperbolic surface and \((s_1, \ldots, s_{g_s-3})\) as a topological decomposition of \(\Sigma\) into 3-holed sphere \(s_i\) essential loops. Then their geodesic representative \(\gamma_{s_1, \ldots, s_{g_s-3}}\) is also a 3-holed sphere decomposition.

Why \(3g_s-3 = 3g-2k+1\)?

\[\begin{array}{ccc}
E \\
| s_i | s_i \quad \vdash \quad s_i \quad \vdash \quad s_i \\
\end{array} \]

Proof Each \(s_i^*\) is simple since \(s_i\) is. Also \(s_0^* \cap s_j^* = \emptyset\) since \(s_i \cap s_j = \emptyset\). Furthermore \(s_i^* \neq s_j^*\) \(\Rightarrow\) topological reason that each component \(X\) of \(\Sigma - U s_i^* \approx \Sigma_{a, 3}\) (Homework why?)

Hint: Euler characteristic + classification. \(X(X) = -1\)

Def A hyperbolic pants, hyperbolic merini on \(\Sigma_{a,3}\) with geodesic boundary

How to understand them?

Key lemma \(\forall b_1, b_2, b_3 > 0, \exists!\) right-angled hyperbolic hexagon \(P\) whose three non-pairwise adjacent edges have lengths \(b_1, b_2, b_3\)

Proof Existence

Uniqueness: from the above proof.
Corollary: \( \forall \ a_1, a_2, a_3 > 0 \ \exists \ \text{a hyperbolic part where boundary } \gamma_{a_i} \text{ lengths } a_1, a_2, a_3. \)

**Proof:** Existence, two two copies of hexagons + double \( \Delta \)

Uniqueness: Suppose \( X \) is a hyperbolic part of geodesic boundary components \( B_1, B_2, B_3 \) let \( c_i \) be the shortest path (geodesic) from \( B_{i+1} \) to \( B_{i+2} \) \( (B_1=B_4, B_2=B_3) \). Then

1. \( c_i \perp B_j \) \( j \neq i \) (Shortest)
2. \( c_i \cap c_j = \emptyset \) \( i \neq j \)

Indeed if so, there will exist a hyperbolic triangle of inner angle \( \frac{\pi}{3} \frac{a_i}{a_j} \).

Thus (Gauss-Bonnet) If \( \Delta \) is a hyperbolic triangle \( \Delta \) \( \Gamma \) of inner angles \( a_1, a_2, a_3 \), then area(\( \Delta \)) = \( \pi - a_1 - a_2 - a_3 \). In particular \( a_1 + a_2 + a_3 < \pi \).

**Proof:** Let

\[
\text{Area}(\Delta) = \int_{\Delta} \frac{\partial x \partial y}{y^2} = \int_{\Delta} \left( \int_{\Delta} \frac{d(\partial x)}{y} \right)
\]

Stacks this

\[
\int_{\Delta} \frac{dx}{y} = \sum_{i=1}^{3} \int_{B_i} \frac{dx}{y}
\]

Now \( B_i \) part of a circle \( \perp x \)-axis:

Equation \( \begin{cases} x = r \cos t + a \\ y = r \sin t \end{cases} \) \( \frac{dx}{y} = -\frac{r \sin t dt}{rt} = -dt \)

So \( \int_{B_i} \frac{dx}{y} = \theta - \varphi \). Now you homewrk to finish \( \square \).
Lecture 10: Fenchel-Nielsen

RM. After a PSL(2,R), always may assume:
It is better to calculate.

Corollary. Suppose \((\Sigma, P)\) is a closed topological surface with a pants decomposition. Then all hyperbolic metrics on \((\Sigma, P)\) are obtained by isometric gluing of hyperbolic pants along their boundaries.

Proof (1). For a hyperbolic metric \(g\) on \(\Sigma\)

\[ \Rightarrow \text{Maximizing all } \partial P \text{ geodesic} \Rightarrow \text{done} \]

(2) If each pants hyperbolic, gluing isometry \(\Rightarrow\) hyperbolic metric.

The Fenchel-Nielsen coordinate for Teichmüller space.

By the uniformization theorem, for a closed surface \(\Sigma\) by \(X(\Sigma) \cong \mathbb{H}^2\) the Teichmüller space \(T(\Sigma) = \{ [g(\Sigma, g)] \} \) of hyperbolic metrics \((\Sigma, g), \sim (\Sigma, g') \) if there exists an isometry \(h : (\Sigma, g) \to (\Sigma, g')\) such that \(h \circ id \) is homotopic to \(id\).

F.N. coordinate. Fix a pants decomposition by \(3g-3\) loops, then

\[ T(\Sigma) \cong \mathbb{R}^{6g-6} = (\mathbb{R}^3 \times \mathbb{R})^{3g-3} \]