Key lemma  \( \forall b_1, b_2, b_3 > 0, \exists! \) right-angled hyperbolic hexagon \( P \) whose non-parallel adjacent edges have lengths \( b_1, b_2, b_3 \).

**Proof**

The existence of \( P \) is guaranteed by the uniqueness of the geodesic \( A_i \) of distance \( b_1, b_2, b_3 \).

**Uniqueness**

If \( l_1, l_2 \) are two disjoint geodesics of positive distance apart, then there exists exactly one geodesic \( A \) such that \( d(A, l_i) = b_i \) for \( i = 1, 2, 3 \).

**Proof**

Transfer common perpendicular symmetric.

Now add Gauss-Bonnet here.
Lecture 11  Fenchel-Nielsen Coordinate

Corollary. (a) The sum of the inner angles of a hyperbolic $n$-gon $< (n-2)\pi$.
In fact its area $= (n-2)\pi - \sum_{i=1}^{n} \alpha_i$.

(b) If $E_g$ is a closed genus $g$ hyperbolic surface, then its area is

$$-2\pi \chi(E_g) = \int_{E_g} -dA$$

(Gauss-Bonnet)

Pf. (a) Decompose the polygon into triangles, total of $(n-2)$ of them,

so

$$\text{Area} = \sum_{\Delta_i} \left( \pi - \alpha_i - \beta_i - \gamma_i \right) = (n-2)\pi - \sum \text{inner angles}$$

(b). \(\square\) Translated the surface $(\Sigma, d)$ into hyperbolic triangles, say there are $V, E, F$ many vertices, edges and faces.

Let $\alpha_i$ be the set of all inner angles.

$$\alpha_i, \beta_i, \gamma_i, \Delta_1, \ldots, \Delta_F$$

$$\text{Area}(\Sigma) = \sum_{\Delta_i} \text{Area}(\Delta_i) = \sum_{\Delta_i} \left( \pi - \alpha_i - \beta_i - \gamma_i \right)$$

$$= F \cdot \pi - \sum \alpha_i$$

$$= F \cdot \pi - \sum_{\text{vertices}} \left( \sum_{\text{adjacent}} \alpha_i \right)$$

$$= F \cdot \pi - 2\pi \cdot V$$

$$= -2\pi \cdot \chi(\Sigma)$$.

\(\square\)

E.g. Between any two geodesics $L_1, L_2$ there exists at most one geodesic $g$ perpendicular to both.

$$\Rightarrow$$ if not $\exists$ right angled quadrilateral, or triangles of two min. 90 $\frac{\pi}{2}, \frac{\pi}{2}$. 

2-9-2013
Thus \( l_1, l_2, l_3 > 0 \) \( \exists \) hyperbolic 3-holed sphere w/ geodesic boundary of lengths \( l_1, l_2, l_3 \).

\( \text{Proof } \) Existence, take two right-angled hexagons of edge lengths \( l_1/2, l_2/2, l_3/2 \) isometrically glue them along the other three edges to obtain:

(Mehri double)

Uniqueness: suppose \( P \) is such a hyperbolic 3-holed sphere let \( a_1, a_2, a_3 \) be the shortest paths between pairs of boundary edges.

Claim: (1) \( a_i \cap l_j \) is (shortest)

(2) \( a_i \) is simple, no self-intersection

(3) \( a_i \cap a_j = \emptyset \) if \( a_i \cap a_j \neq \emptyset \)

\( \Rightarrow \exists \) a hyperbolic triangle \( \Delta \) of inners \( a_1, a_2, a_3 \)

\( \frac{l_1}{2}, \frac{l_2}{2}, \frac{l_3}{2} \)

\( \Rightarrow \) but \( P \) open along \( a_1, a_2, a_3 \) we obtain two right-angled hexagons \( H_+, H_- \). \( H_+ \) is isometric to \( H_- \); they have the same edges lengths at three non-adjacent edges

\( \Rightarrow P \) is a Mehri double of \( H_+ \) \( \Box \)

BM: This is very similar to \( \forall l_1, l_2, l_3 \geq 0 \) \( l_1 + l_2 > l_3 \) \( \Rightarrow \exists \) tri of edge lengths \( l_1, l_2, l_3 \).
hecht & Ber's Constant.

Recall \( \text{Area}(B_r(p)) = 4\pi \sinh^2 \left( \frac{r}{2} \right) \)

Thm (Bers) There exists a constant \( C_g \) s.t. \( \forall \) hyperbolic metric \( d \) on \( \Sigma_g \) surface of genus \( g \) has a pants decomposition so that each decomp piece curve has length \( \leq C_g \).

Notation Given an essential loop \( \alpha \), \( d^* \) is the closed geodesic \( d^* \sim \alpha \).

If \( \alpha \) is not known the best constant \( C_g \) \( (C_g \leq \text{g} \cdot \text{Ln} g) \)

Def \((M, d)\) Riemannian metric \( p \in M \), \( r > 0 \) is called a geodesic radius \( r \).

The exponential map \( \text{Exp}: T_p M \to M \) is 1-1 on \( B_r(o) \)
\[ \Rightarrow B_r(p) \text{ is diffeomorphic (homeomorphic) to } B_r(o) \]

Ex. Cylinder \( S^1 \times [0, 1] \)

E.g.: Cylinder

\[ y = \text{const.} \text{ max} \]

Example 1: If \( d: S^1 \to H/\pi = \Sigma_g \) is a geodesic \( d \neq \pi \) \( d \neq pt \)

E.g.: \( d(a) \) and \( d(b) \) may have tubular neighbors.

If \( d \neq pt \) \( \Rightarrow \exists \) an extension \( F: D \to H/\pi \) of \( d \) \( \cong \pi \cdot pt \)

By the lifting thm, \( F \) lifts to \( \widetilde{F}: D \to H \) s.t. \( \widetilde{F}(0) \neq \text{pt} \)

If two geodesics in \( \Sigma_g \) intersect at two points

But that is impossible.

A geodesic \( d \circ [0, 1] \to \Sigma_g \), \( d(0) = x(c) + \alpha(0) \), geodesic

(May not be a closed geodesic) \( \Rightarrow d \neq \text{pt} \).
Lemma 2. For \((\Sigma, d)\), let \(l = \) the length of the shortest geodesic in \(\Sigma\). \(\delta = \frac{\pi}{2}\).
(Why does it exist?) Then \(\delta\) is an injectively regular for \(p \in \Sigma\).

Proof. If not, \(\exists \ p \in \Sigma\) and \(\delta' \leq \delta\) s.t. \(\exp : T_p \Sigma \to \Sigma\) is not injective in \(B_\delta'(p)\).

Thus, the nice loop \(a \to a\) of length \(2\delta\) is essential.
\(\Delta = a \to a^*\) closed geodesic s.t.
\(\ell(a^*) < 2\delta\), \(1 < 2\delta' \leq 2\delta = L\).
Contradicting the choice of \(L\).

Corollary 3. The length of the shortest geodesics in \((\Sigma, d)\) \(\leq 2\sinh(\sqrt{g-1})\).

Proof. If otherwise, \(L \geq 2\ln(4g-2)\) then
\[ \text{Area}(\Sigma, d) = 2\pi (4g-2) \geq \text{Area}(B_{\frac{L}{2}}(p)) \]
\[ 1 = \frac{4\pi \sinh^2(\frac{L}{2})}{4g-2} \]
\[ = \frac{4\pi}{4g-2} (e^L - e^{-L} + 2) = (e^L - e^{-L}) + 2 \]
So
\[ \frac{e^L - e^{-L} + 2}{4} \leq 1 \]
\[ \sinh^2(\frac{L}{2}) \leq g-1 \]
So \(e^L < 4g-2\).

In general, \(p\) is injectively regular if \(p \in \) an opt. surface \(X\) with geod. \(x\).

Note: If \(r > 0\) is injectively regular of \(p\) in a opt. surface \(X\) with geod. \(x\) \(+ Br(p) \subset X\), \(-2\pi x(x) \geq 4\pi \sinh(\frac{\pi}{2})\) \(\frac{e^L - e^{-L} + 2}{4} \leq - \frac{\pi L}{4g-2}\).
Lemma (X, d): opt hyperbolic, \( \partial X \) geodesic, \( X \equiv \Sigma_0 \) s.t. each boundary component of \( X \) has length \( \leq c \). Then \( \exists \) a closed geodesic \( \alpha \) in \( X \) such that:

1. \( L(\alpha) \leq 2c + 2\ln(-2\chi(X) + 2) = 2c + 2R \)
2. \( \alpha \) not in \( \partial X \)

pf. If not, \( \exists \) closed geodesic \( \beta \subset X-\partial X \) has length \( > 2c + 2R \).

Claim: If \( \alpha \) is a geodesic arc from \( \partial X \) to \( \partial X \) \( \Rightarrow \) \( \text{length}(\alpha) \leq 2R \)

pf. If not \( \exists \) such \( \alpha \) of length \( > 2R \)

(i) \( \alpha \) joins different boundary components: produce a new curve \( a \) as shown:

\[
\text{length}(a) \leq 2c + 2\text{length}(\alpha) \leq 2c + 2R
\]

\[\Rightarrow \text{length}(a) < 2c + 2R \]

\[\Rightarrow a \subset \partial X \Rightarrow X = \Sigma_0 \text{ closed} \]

(ii) \( \alpha \) joins the same boundary:

One of \( a_1 \) or \( a_2 \) in \( X \) not boundary component of \( X \)

s.t. \( \text{length}(a_i) \leq c + \text{length}(\alpha) \). \( \Rightarrow \) contradiction.

Now, let \( \alpha \) be the shortest arc from \( \partial X \) to \( \partial X \) (perpendicular to \( \partial X \)) + \( p \) be the mid point of \( \alpha \)

We claim the ball \( B_{\frac{R}{2}}(p) \) is embedded:

\[
\text{Area}(X) \geq \text{Area}(B_{\frac{R}{2}}(p)) \Rightarrow \text{Area}(B_{\frac{R}{2}}(p)) \leq \frac{R}{2} \ln(-2\chi(X) + 2) \]

\[K < \ln(-2\chi(X) + 2) \text{ embeded} \]

To see this:

(i) \( B_{\frac{R}{2}}(p) \cap \partial X = \emptyset \)

\[\exists \text{ a path } s \text{ joining } \partial X \text{ to } \partial X \text{ of length } < K \]

(ii) \( \Rightarrow \text{ area against } \Rightarrow \text{ Genus } T : \mathbb{R} \rightarrow \mathbb{R} \text{ again } \]

\[\text{Pt of Bor's the } \Rightarrow \text{ Easy} \]
Lecture 11. Basic Facts about Hyperbolic Surfaces

Recall right-angled hexagon, Gauss-Bonnet, existence of hyperbolic pants.

This gives a way to describe hyperbolic metrics on surface $S_g$ of genus $g \geq 2$.

Corollary. Each hyperbolic metric on $S_g$ is an isometric gluing of hyperbolic pants.

The color lemma and Boas' constant

Color lemma. Suppose $\alpha$ is a simple closed geodesic on a hyperbolic surface $\Sigma = \mathbb{H}/\Gamma$ of length $\ell$. Let $\delta$ be the neighborhood

$$N_{\delta}(\alpha) = \{ z \in \mathbb{H} / \Gamma | d(z, \alpha) \leq \delta \}$$

is an embedded annulus.

Topologically

$$N_{\delta}(\alpha) \sim \text{small, embedded annulus}$$

Proof. $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ quotient map $\pi'(\alpha)$ disjoint union of geodesics.

Since $\alpha$ is simple:

$\Gamma$: action

The condition $\Rightarrow$ for any two $\alpha_1 \neq \alpha_2$ conjugates of $\pi'(\alpha)$ $\alpha_1 \cap \alpha_2 = \emptyset$.

Then $N_{\delta}(\alpha_1) \cap N_{\delta}(\alpha_2) = \emptyset$.

Let $\hat{\delta}$ be the limit $\delta \to \infty$. $N_{\delta}(\alpha)$ embedded annulus.
Lemma 1.3 The collar lemma

\[ H/\mathbb{R} \ni \delta \mapsto \frac{\pi}{\lambda} \delta \]

The collar of a \( \gamma \) is
\[ N_\delta (\gamma) = \{ z \in H/\mathbb{R} \mid d(\gamma, z) < \delta \} / \mathbb{R} \]

Note the moduli of \( N_\delta (\gamma) + H/\mathbb{R} \) are related by

\[ \frac{\sec \phi}{\sec \delta} = \sinh \delta \]

Thus
\[ N_\delta (\gamma) \sim \frac{H/\mathbb{R}}{\text{biholo}} \]

Thus the length of \( \gamma \)

\[ \text{Length}(\gamma) = \frac{\pi}{2 \phi} \lambda \]

Collar Thm Suppose \( \gamma \) is a simple closed geodesic in \( H/\mathbb{R} \), complete hyperbolic surface of length \( \lambda \). Let \( \delta \) be: \( \sinh(2\delta) \sinh(\frac{\delta}{\lambda}) = 1 \). Then the neighborhood \( N_{\delta} (\gamma) = \{ z \in H/\mathbb{R} \mid d(\gamma, z) < \delta \} / \mathbb{R} \) is isometric to

\[ \left\{ \frac{\pi}{2 \phi} \lambda \phi \right\} \]

(Margulis tube)

Homework 3.20 The moduli of \( \text{mod } N_\delta (\gamma) \geq \epsilon \) for all \( \delta, \epsilon > 0 \).

Now for \( \delta = \frac{\pi}{2} \sinh^{-1} \left( \frac{\epsilon \phi}{\sinh(\frac{\delta}{\lambda})} \right) \)

\[ \sinh(\delta) = \sinh \left( \frac{\pi}{2} \sinh^{-1} \left( \frac{\epsilon \phi}{\sinh(\frac{\delta}{\lambda})} \right) \right) \]

For \( \delta > 1 \),

\[ \sinh^{-1} \left( \frac{\epsilon \phi}{\delta} \right) = \sinh^{-1} (e^{\frac{\epsilon \phi}{\delta}}) \approx e^{\frac{\epsilon \phi}{\delta}} \]

\[ \sinh(\delta) \approx e^{\frac{\epsilon \phi}{\delta}} \]
Collar lemma

What happens when \( \lambda \to \infty \): \( \text{length}(N_\lambda(a)) > \text{length}(a) = \lambda \to \infty \). So no way.

Let \( S \) be the largest number at \( n_x \in \mathbb{N} \) embedded.

Proof. let \( \pi: \mathbb{H} \to \mathbb{H}/\Gamma \) be the quotient map. Since \( \Gamma \) is discrete, \( \pi^{-1}(a) \) consists of disjoint geodesics in \( \mathbb{H} \).

The condition \( \theta: \forall \bar{a}_1 \neq \bar{a}_2 \text{ good in } \pi^{-1}(a) \quad N_\lambda(\bar{a}_1) \setminus N_\lambda(\bar{a}_2) = \emptyset \).

Notation: \([a, b] \in \mathbb{H} \setminus \Gamma \), \([a, b] \) geodesic from \( a \to b \).

After a conjugation, we may assume:

(1) \([0, 1] \in \pi^{-1}(a) \) and \( f(z) = \lambda z \in \Gamma \), \( \lambda = e^k \). \([0, 1] \in \pi^{-1}(a) \) and \( f(z) = \lambda z \in \Gamma \), \( \lambda = e^k \).

(2) \( f([a, b]) \subset \pi^{-1}(a) \) \( a < b \Rightarrow \delta([a, b]) \cap [a, b] = \emptyset \).

Note: \( \delta([a, b]) \cap [a, b] \).

Now by the assumption of tangency

\[ 2 \delta = \delta([0, 1], [a, b]) \]

so \( \text{Sinh}(2 \delta) = \text{tg}(\varphi) = \sqrt{\left(\frac{a+b}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2} \]

(distance arc length)

\[ = \frac{2 \sqrt{b-a}}{b-a} \geq 2 \frac{\sqrt{\lambda}}{\lambda} = 2 \frac{2}{\sqrt{\lambda-1}} = 2 \frac{2}{\text{Sinh}(\frac{\delta}{2})} \]

\[ \text{Sinh}(2 \delta) \text{Sinh}(\frac{\delta}{2}) \geq 1 \]
Lehre 18. Collar lemma

**Def.** The module of the rig $\mathcal{L} = \{ x_i \in \mathbb{R} \cup \{ \infty \} \} \mod \mathcal{L} = 1g \frac{\mathcal{L}}{\mathcal{L}'}$. By uniqueness theorem, each rig (or Riemann surface homeo to $\mathbb{C}^*$) is biholomorphic to $\mathcal{L}$. We define $\mod(L) = \mod((\mathcal{L} \otimes \mathcal{N})$

**Eq.** $\mathcal{L} = \{ \frac{2}{2+\cos 2\pi t} \}$

**Corollary.** $\forall \alpha > 0 \exists g \in \mathcal{L}$ such that mod($\mathcal{L}$) = $\alpha$. (Hw)

**Proof.** Let $\gamma$ be the shortest geodesic in $\mathcal{L}$ $\delta = \frac{1}{2} \sinh^{-1}(\frac{1}{\delta})$. $\mathcal{L} = \mathbb{R} \cup \{ \infty \}$

**Known.** $l(\alpha) \leq l(g(\alpha)) \Rightarrow \mod(L) \geq g$ a lower bound

$\exists \delta > \delta(\alpha)$ ($\lim \mod(L) \to \infty$). Hw.

There were not too much study of max module rig in Riemann surfaces. This is a very interesting subject.

**Def.** $\Sigma$ Riemann surface, $\text{Maxmod}(\Sigma) = \max \{ \text{mod}(L) \mid L \text{ essential rig in } \Sigma \}$.

($\Theta$ counter part of the shortest geodesic.)

An extremal surface is a Riemann surface $\Sigma$ s.t. $\text{Maxmod}(\Sigma) = \text{Maxmod}(\Sigma')$ for all other Riemann surfaces $\Sigma' \cong \text{homeo } \Sigma$. What can you say about these surfaces?

**Eq.** Maximmum torus $\mathcal{T} \cong \mathbb{R}^2 / 2\pi i \mathbb{Z}$. In maxmod($\mathcal{T}$) = $\alpha$, $2\pi$ Schwarze lemma infinitesimal version. $\mathcal{T}$ hyperboloth surface $\Sigma$, $\Sigma \to \Sigma$, holonomy map. Then $\forall v \in \Gamma_p \mathcal{E}_1 \exists w$ ($\forall \gamma \in \mathcal{L} \forall \gamma$). s.t. equality holds for one $v$ with $\phi$ is an isometry. In particular $\forall$ loop $\gamma \in \Sigma$, $\gamma(x) = \gamma(y)(\gamma(x))$.

**Proof.** Let $\Gamma_p$; $D \to \Sigma$ be the cover map s.t. $\Gamma_p(\gamma) = \gamma$, $\Gamma_p(0) = \gamma(0)$.

By cover map theory $\Sigma$ holonomy lifting $\tilde{\gamma} : (D, 0) \to (\tilde{D}, 0)$ s.t. $\tilde{T}_p \tilde{\gamma} = \gamma(0)$. Now Schwartz lemma applied to $\tilde{\gamma}$: $\|D\tilde{\gamma}(\tilde{D})\| \leq \|D\|$, done.

Corollary
Lecture 13  Collar Lemma

Corollary If \( \varphi : S_1 \to S_2 \) is a holomorphic embedding of a riemann surface \( S_1 \) to \( S_2 \) such that \( \varphi \) is a homotopy equivalence (i.e. \( \varphi \) and \( \varphi^{-1} \) ), then \( \text{mod}(S_1) \geq \text{mod}(S_2) \) and equality holds if \( \varphi \) is biholomorphic.

Proof Let \( d_1, d_2 \) be the shortest geodesic on \( \text{mod}(S_1) \) \((S_1, d_1)\). Then \( l(d_2) \leq l(\varphi(d_1)) \leq l(d_1) \) \( \Rightarrow \) result

Since shortest Schwarzkopf \( (\varphi(d_1), d_2) \)

\[ l(d) = \frac{4\pi^2}{\text{mod}(S)} \Rightarrow \text{done} \]

For a flat torus \( T = \mathbb{C}/\mathbb{Z} + \mathbb{Z} - \mathbb{Z} \) if \( \alpha \) is a simple loop in \( T \)

then \( \text{mod}(\text{Riem}(\alpha)) = \frac{2\pi \text{Area}(T)}{l^2(\alpha)} \).

Thus, we are looking for torus of the (largest) minimal width.

Conclude if must be \( \frac{1}{2\pi} + \mathbb{Z} i \).
Lecture 14: Moduli space and Teichmüller space

Example: \( \text{Teich}(\Delta) = \{ (l_1, l_2, l_3) \in \mathbb{R}^3 \mid l_i + l_j > l_k \} \)

= cone from \( \Delta \) to an equilateral triangle \( T \) (open 1)

\[ \Delta(a, b, c) = \frac{a+b-c}{2} \]

\[ \Delta(a, b, b) \]

\[ \Delta(a, b, a) \]

\[ \Delta(a, a, a) \]

Symmetry

\[ S_3 \text{ action} \]

\[ \mathbb{R}^3 \]

\[ \text{Top} \]

\[ \mathbb{R}_{>0} \times \mathbb{R}^2 \]

\[ \text{Mod}(\Delta) = \text{cone over} \]

\[ \{ \text{isometry classes of triangles} \} \]

\[ \text{Riemann surface} \]

\[ \mathbb{S} \text{ topological surface} \]

\[ \text{Mod}(S) = \{ (\Sigma, \phi) \mid \Sigma \cong S \ (\Sigma, \phi) \text{ biholo } \Sigma' \} \]

\[ \text{homeo} \]

\[ \text{isometry biholomorphism classes of } \mathbb{S} \text{-surf homeo to } S \].

The Teichmüller space of \( S \),

Easy definition: \( \text{Teich}(S) = \{ (S, \phi) \mid h(S, \phi) \cong (S, \phi') \text{ biholo } h \mid h = \text{id} \}

Problem: when the underlying \( S \) changes, how to define \( \text{Teich}(S) \)?

Definition: A marked Riemann surface (of type \( S \)) is \( (\Sigma, \phi, h) \) where

\[ h: S \to \Sigma \text{ homeo, } (0, p) (\Sigma, \phi) \text{ complex structure} \]

\[ (\Sigma, \phi, h_1) \cong (\Sigma, \phi, h_2) \] \( \implies \) biholo \( \phi: (\Sigma, \phi, h_1) \to (\Sigma, \phi, h_2) \)

So that

\[ S \xrightarrow{h_1} \Sigma_1 \]

\[ \phi_0 \circ h_1 \circ \phi_1 \text{ biholo } h_2 \]

\[ \phi_0 \circ h_1 \cong h_2 \text{ homotopic} \]

Example: \( \Sigma = S \) \( h = \text{id} \) \( \implies \phi = \text{id}: S \to S \)

Remark 1: We may replace \( \phi: S \to \Sigma \) by the homotopy class of \( \phi \).

Remark 2: We may even assume \( \phi \) by \( \pi_1(S) \to \pi_1(\Sigma) \text{ isomorphism} \)

replace \( \phi \)
Claim \( \text{Mod} (S' \times S') \cong \mathbb{H} / \text{PSL}(2, \mathbb{Z}) \) + \text{Teich} (S' \times S') \cong \mathbb{H}.

Unification: \( \Rightarrow \) Each complex torus \( \cong \mathbb{C} / \omega \) \( \mapsto \mathbb{C} / (\omega_1 + \omega_2) \) \( \omega_1 / \omega_2 \in \mathbb{R} \)

Note. If \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbb{Z}) \) then \( \omega_1 + b \omega_2, c \omega_1 + d \omega_2 \) are basis

\[
\Gamma = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 = \mathbb{Z} (\omega_1 + b \omega_2) + \mathbb{Z} (c \omega_1 + d \omega_2)
\]

(1) The isomorphism \( \psi(\tau) = \lambda \tau + \lambda \in \mathbb{C} \Rightarrow \mathbb{R} \)

conjugate \( \Gamma \) to \( \lambda \Gamma = \mathbb{Z} (\lambda \omega_1) + \mathbb{Z} (\lambda \omega_2) \)

\[
\begin{array}{c}
\mathbb{C} \xrightarrow{\psi} \mathbb{C} \\
\mathbb{C} \xrightarrow{\lambda} \mathbb{C} \\
\mathbb{C} \xrightarrow{\lambda \psi} \mathbb{C}
\end{array}
\]

\[
\Rightarrow \; \mathbb{C} / \mathbb{Z} \cong \mathbb{C} / \mathbb{Z} + \mathbb{Z} (\lambda \omega)
\]

In particular: \( \mathbb{C} / \mathbb{Z} \cong \mathbb{C} / \mathbb{Z} + \mathbb{Z} (\omega)
\]

\[
\therefore \quad \mathbb{C} / \mathbb{Z} (\omega_1 + \omega_2) \cong \mathbb{C} / \mathbb{Z} + \mathbb{Z} (\omega_2)
\]

\[
\therefore \quad \mathbb{C} / \mathbb{Z} (\omega_1 + b \omega_2) + \mathbb{Z} (c \omega_1 + d \omega_2) \cong \mathbb{C} / \mathbb{Z} + \mathbb{Z} \left( \frac{a \omega_1 + b \omega_2}{c \omega_1 + d \omega_2} \right)
\]

Conclusion. If \( \tau \in \mathbb{H} \) and \( \tau = \frac{a \tau + b}{c \tau + d} \Rightarrow \mathbb{C} / \mathbb{Z} + \mathbb{Z} \)

On the other hand, if \( \psi: \mathbb{C} / \mathbb{Z} + \mathbb{Z} \rightarrow \mathbb{C} / \mathbb{Z} + \mathbb{Z} \) \( \tau, \tilde{\tau} \leq \mathbb{R} \)

\( \psi(\tau) = \tilde{\tau}\)

\( \psi \) biholomorphic \( \tilde{\tau} = \text{bilocally} \psi(\tau)

Therefore, we claim \( \tau = \frac{a \tau}{c \tau + d} \) \( \leq \mathbb{R} \).

Proof. First. \( \forall \alpha \in \mathbb{C} \) \( \leq \mathbb{C} + \mathbb{Z} \) \( \psi(\tau) = \frac{\tau}{\mathbb{Z} + \mathbb{Z}} \)

So we may assume that \( \psi(\alpha) = \alpha \)

Now by covering lifting theorem \( \Rightarrow \) biholomorphic \( \psi: \mathbb{C} \rightarrow \mathbb{C} \) \( \psi(\tau) = \alpha \)

\( \therefore \quad \psi(\tau) = \lambda \tau \quad \lambda \in \mathbb{C} \)

\( \Rightarrow \quad \psi(\tau) = \lambda \tau \quad \lambda \in \mathbb{C} \)
Furthermore \( \hat{\varphi}(\hat{r}) = \hat{r}' \) for \( \hat{r}' = \lambda \hat{r} \) and \( \hat{\varphi}(\hat{S}) = \lambda \hat{S} \).

\[ \Rightarrow \quad \lambda z = a\hat{z} + c \]
\[ \lambda z' = c\hat{z}' + d \]
\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}(2,\mathbb{R}) \quad (\det A = \lambda) \]

But \( z, z' \in \mathbb{H} \Rightarrow ad - bc = 1 \) \( \det(A) = 1 \) \( \text{Done} \)

Prop The Teichmüller space \( T(5\times5) = \mathbb{H} \).

Key topological fact \( h_1, h_2 : S^5 \times S^1 \rightarrow S^5 \times S^1 \) homeom \( f \) \( (h_1)_* = (h_2)_* : \pi_1(S^5 \times S^1) \rightarrow \overline{h}_1(S^5 \times S^1) \)

Thus a marked complex torus \( (X, \tilde{g}, \tilde{p}) \) \( \cong (X, \tilde{g}, \tilde{p}, \tilde{n}) \) \( \tilde{n} \) two germs of \( \pi_1(X) \). (oriented germs) \( \omega \) two detached orientations \( \omega \) \( \omega \) \( \omega \)

\[ \Rightarrow \text{Each marked torus } = \left( \frac{\omega_1}{\omega_2}, (\omega_1, \omega_2) \right) \quad \frac{\omega_1^2}{\omega_2^2} \in \mathbb{H} \]

Now \( \left( \frac{\omega_1}{\omega_2}, (\omega_1, \omega_2) \right) \overset{\text{Teich}}{\sim} \left( \frac{\omega_1'}{\omega_2'}, (\omega_1', \omega_2') \right) \) \( \frac{\omega_1'}{\omega_2'} = \frac{\omega_1}{\omega_2} \)

The same proof as above \( \Rightarrow \) done.

Questions What about \( \frac{\omega_1}{\omega_2} \in -\mathbb{H} \) \( \Rightarrow \) orientation issue \( \Rightarrow \) ?
S a closed surface of genus $g \geq 2$

$$\text{Teich}(S) = \{ [X, d, \varphi] \mid d \text{ hyperbolic metric} \varphi : S \to X, \text{ "homotopy clan" of orientation preserving homeos } \}$$

Now, fix a 3-holed sphere decay $P = \{ \gamma_i \} \ldots , \gamma_{3g-3} \}$ of $S$, let $\Gamma'$ be a set of disjoint SCC called "seams" s.t. for each 3-holed sphere $\Sigma_{3g-3} \subset S - \Gamma$

$\Gamma' \cap \Sigma_{3g-3}$ consists of 3 arcs joining different comp.

$\Gamma'$ with $(S, P, \Gamma')$ we can associate the FN coordinate,

$$\text{FN} : \text{Teich}(S) \to (R_{>0} \times R)^{3g-3} : [X, d, \varphi] \mapsto (l_1, \ldots )$$

$(l_1, \ldots, l_{3g-3}) \in R_{>0}$ length cond $(x_1, \ldots, x_{3g-3})$ twist-cond

The lengths... if $i$, let $\varphi_i = \varphi_{(\gamma_i)}$ be the good $\varphi(X, d, \varphi) \in (X, d, \varphi) \in (X, d, \varphi)$.

We may assume, for $(X, d)$ that $\varphi_i : \varphi_{(\gamma_i)}$ after a homotopy

For each $\gamma_i$, let $Q_i = \text{ union of two hyperbolic 3-holed sphere } \cup X$

adjoint to $\varphi_i$

$$P_e \cup P_o$$

Now, $\varphi(\gamma') \cap Q_i \subset \gamma_1 \times \gamma_2 \times \gamma_3 \text{ rel } (\partial \Sigma_i) \quad \partial \Sigma_i - \text{ invariant.}$

where $\gamma_1, \gamma_3$ are the shortest geodes $\in P_e + P_o$, $\gamma_2 \in \varphi(\gamma';\gamma_i)$

Define $x_i = \text{ the signed distance of } d_2 < \partial R.$

Signed distance Fix any orientation on $\varphi_{(\gamma_i)}$, use manifolds of $X \Rightarrow \text{ left + right}$.
Theorem (Fenchel-Nielsen) Fix $(S, \gamma, \gamma')$ the map
\[ \text{FN: } \text{Teschl}(S) \rightarrow \mathbb{R}^{3g-3} \times \mathbb{R}^{3g-3} \]

Prop.: Well defined on $\mathcal{N}$ (it is independent of the choice of two $\gamma'(\gamma')$).

Onto: clearly from the construction.

Given $(x, \tau) \in \text{Teschl}(S)$ produce hyperbolic pairs of given lengths

Use $t$ to glue them isometrically. $\Rightarrow (x, d) \in \mathcal{N}$, use $t_i \rightarrow \text{map}$

$\phi_i : S \rightarrow X$, sending $\phi_i(\gamma')$ to a curve homotopic to $\gamma_i \ast \gamma_i \ast \gamma_i$.

1-1: If $(x, d, \phi), (x', d', \phi')$ two marked hyperbolic surfaces of

the same FN coordinates $\Rightarrow (x, d, \phi) \cong (x', d', \phi')$.

By the construction $x, x' \in \exists$ an isometry $h : X \rightarrow X'$

sending geodesic $\phi_i(\gamma')$ to $\phi_i'(\gamma')$. SOME twist $\Rightarrow$ glued nicely

The homotopy $\gamma_i \\
S \\
\phi \longrightarrow \phi_i \ \text{s.t. } h \circ \phi \cong \phi_i$.

$\phi_i(\gamma') \Rightarrow h \circ \phi_i' \phi_i \Rightarrow A$ topological fact
Thus (Wolpert) If $d_0$ is a family of hyperbolic metrics obtained by $t$-twisting along a simple closed geodesic $\phi$ in $d_0$, and $\alpha(t)$ is the closed geodesic $\phi$ at homotopy to geodesic $\alpha(0)$,

$$\phi \Rightarrow \frac{d|}{dt} \mid_{t=0} \phi(\alpha(t)) = \sum_{p \in d_0 \cap \phi_0} \cos \theta_p$$

$\theta_p$ from $\phi \Rightarrow \alpha$.

Proof: It is basically the cosine law for triangles.

Lemma: 1. Cosine law

$$\cosh(q_3) = \cosh(q_1) \cosh(q_2) - \sinh(q_1) \sinh(q_2) \cos \theta_3$$

2. Standard, I take it as homework.

In $\triangle ABC$

$$\cosh(l(t)) = \cosh(t \cosh(l(t)) - \sinh(t \sinh(l(t)) \cos(\theta_1 - \tau(t)), \tau(0) = 0$$

$$\Rightarrow \frac{d}{dt} \mid_{t=0} \cosh(l(t)) = \cosh(l(0)) \frac{d}{dt} \left( \sinh(l(0)) a'(0) - \sinh(a(0)) \cos(\theta_1) \right)$$

$$\Rightarrow \cosh(l(0)) \frac{d}{dt} \mid_{t=0} \sinh(l(0)) a'(0) = \cosh(l(0)) a'(0) - \sinh(l(0)) \cos(\theta_1)$$

Now, $a(t) \cdot \cosh(a(t)) = \cosh(t \cosh(l(t)) - \sinh(t \sinh(l(t)) \cos \theta_2$

$\Rightarrow \left( \sinh(l(0)) a'(0) = \cosh(l(0)) a'(0) - \sinh(l(0)) \cos \theta_2 \right)$

$$\Rightarrow a'(0) = - \cos \theta_2$$
Wolpert's formula

Proof: lift everything to the universal cover and write that's down in $\widetilde{d}_0$ metric.

$\Rightarrow$ result follows from the formula we just derived.

In reality it should be

that is the actual situation.

In this case:

\[
\cosh l(t) = \cosh t \cosh a(t-1) - \sinh(t) \sinh a(t) \cos (a_2 \cdot \pi - a_1 + c(t))
\]

\[
= \cosh a(t) - t \cdot \sinh(a(t)) \cos(\pi - a_1) + o(t^3)
\]

\[
\Rightarrow l'(t) = a'(t) + \sinh(l(t)) \cos(a_1)
\]

Next:

\[
\cosh a(t) = \cosh \cosh l(t) - \sinh \sinh l(t) \cos(\pi - a_1)
\]

\[
= \cosh l(t) + t \sinh l(t) \cos(\pi)
\]

\[
\Rightarrow a'(t) = \boxed{\cos a_1}.
\]

(Wolpert)

Corollary: The symplectic form $\sum d\lambda_i \wedge d\lambda_i$ on $T(\Sigma)$ is invariant under the action of the MCG.