A list of problems in geometry and topology, part I

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Here is a list of problems that I have been working on.

Call a closed set \( X \subset \mathbb{C} \) of circle type if each connected component of \( X \) is either a point or a closed round circle. Consider the Riemann sphere \( \mathbb{C} \cup \{\infty\} \) as the infinity of the (upper-half-space model of) hyperbolic 3-space \( \mathbb{H}^3 \).

**Problem 1.** For any genus zero connected complete hyperbolic surface \( \Omega \), there exists a circle type closed set \( X \subset \mathbb{C} \) such that \( \Omega \) is isometric to the boundary of the convex hull of \( X \) in \( \mathbb{H}^3 \). Furthermore, \( X \) is unique up to Möbius transformation.

Comments: The work of Fillastre and Rivin show that problem 1 has affirmative solution if \( \Omega \) is of finite topological type. If \( \Omega \) conformal to \( \mathbb{C} - V \) or \( \{z \in \mathbb{C} | |z| < 1\} - V \) where \( V \) is a discrete set, then Problem 1 is equivalent to the conjectural discrete uniformization theorem for simply connected polyhedral surfaces. Furthermore, Problem 1 can be considered as a geometric counterpart of the Köbe’s circle domain conjecture which states that any genus zero Riemann surface is biholomorphic to the complement of a circle type closed set in \( \mathbb{C} \).

This is a conjecture by F. Luo, Jian Sun and Tianqi Wu. With David Gu, we introduced a notion of discrete conformality for compact polyhedral surfaces and proved a discrete uniformization theorem in http://arxiv.org/abs/1309.4175. Problem 1 can be considered as a version of discrete uniformization for non-compact simply connected polyhedral surfaces.

**Problem 2** (Differential Geometry). Suppose \( f : A \rightarrow B \) is a diffeomorphism between two strictly convex smooth closed surfaces in \( \mathbb{R}^3 \) so that \( f \) preserves the second fundamental form. Show that \( f \) is an isometry.

Comments: This is a smooth version of Stoker’s conjecture on convex polytopes. If the answer is affirmative, then by the rigidity theorem of convex surfaces, \( f \) is induced by a rigid motion of \( \mathbb{R}^3 \).

**Problem 3** (Square tiling of the plane) Suppose \( \{S_i| i \in J\} \) is a square tiling of the plane so that each square intersects exactly six others. Show that all squares have the same size.

Comments: This is a counterpart of Thurston’s conjecture on rigidity of circle packings, proved by Rodin-Sullivan, that the hexagonal circle packing of the plane is unique up to scaling and rigid motion. In the case of square tiling, the uniqueness is no longer true.

**Problem 4** (Topology). For any connected 3-manifold \( M^3 \) and any non-trivial element \( \alpha \in \pi_1(M^3) \), show that there exist a finite commutative ring \( K \) with identity and a group homomorphism \( \rho : \pi_1(M) \rightarrow PGL(2, K) \) so that \( \rho(\alpha) \neq id \).
Comments: By the solution of the geometrization conjecture and a theorem of J. Hempel, it is known that 3-manifold groups are residually finite. This problem asks for the specific list of finite groups which detect non-triviality. This problem is motivated by solving Thurston’s equation over a commutative ring.

**Problem 5 (Casson conjecture).** Suppose $M^3$ is a non-compact hyperbolic 3-manifold of finite volume and $T$ is an ideal triangulation of $M$. If one realizes (abstractly) each tetrahedron in $T$ by an ideal hyperbolic tetrahedron so that the sum of the dihedral angles of these tetrahedra around each edge (in $T$) is $2\pi$, then the sum of the volume of these tetrahedra is at most the volume of the complete hyperbolic metric on $M$.

Comments: This conjecture is usually stated in terms of angle structures.

**Problem 6 (Geometric triangulations).** Is there any geometric triangulation of the hyperbolic plane so that
(a) each vertex is adjacent to exactly 6 triangles and
(b) the diameter of all triangles are uniformly bounded?

Comments: It is easy to construct a geometric triangulation of the hyperbolic plane satisfying (a) but not (b). It is likely that such a triangulation does not exist.