

FIVE STAR. ★★★★☆ Advanced Engineering

Mathematics

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Part 1 (of 3)

The Laplace Transform

Ch. 4.8, AEM (textbook)

Lecture 1. The definition and examples

Functional transforms (operators)

$$f(t) \Rightarrow \mathcal{L}f(s)$$

$$\underline{F(s)}$$

\mathcal{L} - transform or
operator

{ functions transform at functions }

(compare with geom. transforms)

Why : transform differential equations
at algebraic ones

Definition. Let $f(t)$ be a function for $0 \leq t < \infty$. Then

$$F(s) = (\mathcal{L}f)(s) = \int_0^\infty e^{-st} f(t) dt$$

is called the Laplace transform of f .

Usually, (i) $0 < s < \infty$

(ii) f is - continuous

(seldom piecewise)

- absolutely bounded

$\exists M$ such that
exists

$$|f(t)| < M, \forall t > 0$$

Under these conditions

$\mathcal{L}f$ is well defined

(sufficient conditions)

 Calculus: Improper integrals.

$$\int_0^\infty f(t) dt = \lim_{T \rightarrow \infty} \int_0^T f(t) dt$$

$T > 0$

Examples:

Compute $\mathcal{L}f$.

① $f(t) \equiv 1$.

$$\mathcal{F}(s) = \mathcal{L}f(s) = \lim_{T \rightarrow +\infty} \int_0^T e^{-st} dt =$$

$$= \lim_{T \rightarrow +\infty} \left\{ \frac{1}{s} (-e^{-sT}) \right\} \Big|_0^T$$

$$= \lim_{T \rightarrow +\infty} \frac{1}{s} [-e^{-sT} + 1] = \frac{1}{s}$$

Condition $s > 0$ is essential!

② $f(t) = t$

$$\mathcal{L}f = \int_0^\infty t e^{-st} dt = -\frac{te^{-st}}{s} \Big|_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} dt$$

!! !! Ex. 1

$\approx \frac{1}{s^2}$.

Compare the growth of t , e^{-st} if $t \rightarrow +\infty$



Integration by parts:

$$\int u v' dt = u v - \int u' v dt$$

Calculus

(3) Generalization of Ex. 1, 2:

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}} \quad n \geq 0, s > 0.$$

(4) $f(t) = e^{at}$, $a \leq 0 \Rightarrow$ decreases

$$\mathcal{L}f(s) = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}$$

Essential condition $\& a \leq 0$ (Why?)

(5) $f(t) = \sin t$, $g(t) = \cos t$

Compute simultaneously

$$A = \mathcal{L}f, \quad B = \mathcal{L}g$$

$$A = \int_0^\infty \sin t e^{-st} dt = -\cos t e^{-st} \Big|_0^\infty - \int_0^\infty -\cos t e^{-st} dt$$

Integr. by parts

$$= 1 - sB$$

$$B = \int_0^\infty \cos t e^{-st} dt = \sin t e^{-st} \Big|_0^\infty + s \int_0^\infty \sin t e^{-st} dt$$

$$A = 1 - sB$$

$$= s^2$$

$$B = sA \Rightarrow A = 1 - s^2 A \Rightarrow A = \frac{1}{1+s^2} \Rightarrow B = \frac{s}{1+s^2}$$

Linear property of the Laplace transform:

$$\boxed{\mathcal{L}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{L}f_1 + c_2 \mathcal{L}f_2}$$

Exercises:

Find

1. $f(t) = 2t^2 - 3t + 5$

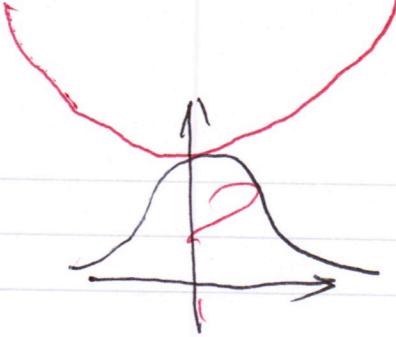
$$\mathcal{L}f(s) = \frac{4}{s^3} - \frac{3}{s^2} + \frac{5}{s}$$

2. $f(t) = -2t^5 + \sin 2t - 3\cos 3t + e^{-2t}$

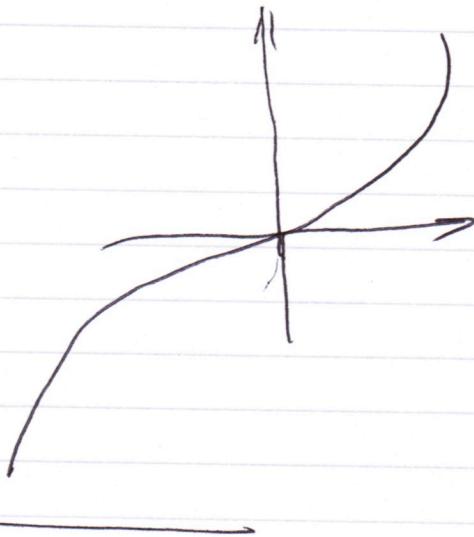
$$\mathcal{L}f(s) = -\frac{205!}{s^6} + \frac{2}{s^2+4} - \frac{3s}{s^2+9} + \frac{1}{s+2}$$

Hyperbolic functions.

$$\cosh t \stackrel{\text{def}}{=} \frac{1}{2} (e^t + e^{-t}),$$



$$\sinh t = \frac{1}{2} (e^t - e^{-t})$$



$$f(t) = \cosh t \Rightarrow \mathcal{L}f = \frac{1}{2} \left(\frac{1}{s-1} + \frac{1}{s+1} \right),$$

$$\underline{\mathcal{L}\left(\frac{1}{2}(e^t + e^{-t})\right)}$$

$$= \frac{s}{s^2 - 1}, \quad s > 1$$

[If $a > 0$ then $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ only for $s > a$]

$$g(t) = \sinh t \Rightarrow \mathcal{L}g = \frac{1}{2} \left(\frac{1}{s-1} - \frac{1}{s+1} \right) = \frac{1}{s^2 - 1}, \quad s > 1$$

Let us to start a table

$f(t)$	$\mathcal{L}f(s)$
$t^n, n=0, 1, \dots$	$\frac{n!}{s^{n+1}}$
$e^{at}, a < 0$	$\frac{1}{s-a}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sinh kt$	$\frac{k}{s^2-k^2}, s \geq k $
$\cosh kt$	$\frac{s}{s^2-k^2}, s \geq k $