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**MATH 421. ADVANCED CALCULUS FOR  
 ENGINEERING. SPRING 2014. QUIZ 6**

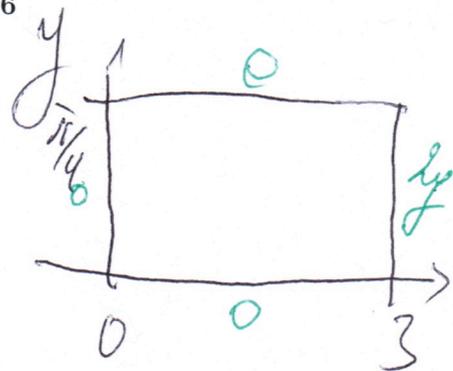
1. (100 points) Solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < 3, 0 < y < \pi/4,$$

subject to the given conditions

$$u(x, 0) = u(x, \pi/4) = 0,$$

$$u(0, y) = 0, u(3, y) = 2y.$$



$$Y'' + \lambda Y = 0, \quad Y(0) = Y(\pi/4) = 0$$

$$X'' - \lambda X = 0$$

$$Y_n(y) = \sin(4ny), \quad n = 1, 2, \dots$$

$$\lambda_n = 16n^2 \quad X'' - 16n^2 X = 0$$

$$X_n(x) = a_n \cosh 4nx + b_n \sinh 4nx,$$

$$u(x, y) = \sum_{n=1}^{\infty} \sin(4ny) (a_n \cosh 4nx + b_n \sinh 4nx)$$

$$u(0, y) = 0 \Rightarrow a_n = 0, \quad n = 1, 2, \dots$$

$$u(3, y) = \sum_{n=1}^{\infty} b_n \sinh(12n) \sin(4ny) = 2y =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(4ny)$$

$$b_n = \frac{(-1)^{n+1}}{n \sinh(12n)}$$

$$u(x, y) = \sum_{n=1}^{\infty} \sin(4ny) \frac{(-1)^{n+1}}{n \sinh(12n)} \sinh(4nx)$$

$$0 < x < 3, \quad 0 < y < \frac{\pi}{4}$$

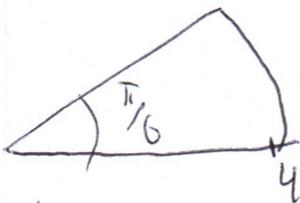
2. (100 points) Solve Laplace's equation in the sector of the disk (in radial coordinates)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, 0 < \theta < \pi/6, 0 < r < 4$$

subject to the given conditions

$$u(4, \theta) = \theta - 1, 0 < \theta < \pi/6,$$

$$\frac{\partial u}{\partial \theta}(r, 0) = 0, \frac{\partial u}{\partial \theta}(r, \pi/6) = 0.$$



$$\textcircled{H} \frac{d^2 H}{d\theta^2} + \lambda \textcircled{H} = 0$$

$$r^2 R'' + r R' - \lambda R = 0$$

$$\textcircled{H}'(0) = \textcircled{H}'\left(\frac{\pi}{6}\right) = 0$$

$$\textcircled{H}_n(\theta) = \cos(6n\theta), n=0, 1, 2, \dots$$

$$\lambda_n = 36n^2,$$

$$R_n(r) = c_n r^{6n} + d_n r^{-6n}, n \geq 1$$

$$R_0 = c_0 + d_0 \ln r$$

Regularity at  $r=0 \Rightarrow d_n = 0, n=0, 1, 2, \dots$

$$u(r, \theta) = c_0 + \sum_{n=1}^{\infty} c_n \cos(6n\theta) r^{6n}$$

$$u(4, \theta) = c_0 + \sum_{n=1}^{\infty} c_n 4^{6n} \cos(6n\theta) = \theta - 1$$

$$= \frac{\pi}{12} - 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos(6n\theta)$$

$$c_0 = \frac{\pi}{12} - 1, c_n = \frac{(-1)^{n-1}}{n^2} 4^{-6n}$$

$$u(r, \theta) = \frac{\pi}{12} - 1 + \frac{1}{3\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \left(\frac{r}{4}\right)^{6n} \cos(6n\theta).$$

$$0 < \theta < \frac{\pi}{6}$$

$$0 < r < 4$$

MATH 421.03. ADVANCED CALCULUS FOR  
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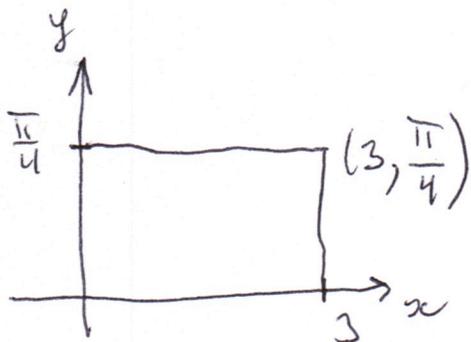
1. (100 points) Solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, 0 < x < 3, 0 < y < \pi/4,$$

subject to the given conditions

$$\frac{\partial u}{\partial x}(0, y) = \frac{\partial u}{\partial x}(3, y) = 0,$$

$$u(x, 0) = 1, u(x, \pi/4) = 2x.$$



$$X'' + \lambda X = 0, \quad X'(0) = X'(3) = 0$$

$$Y'' - \lambda Y = 0$$

$$X_n(x) = \cos\left(\frac{n\pi}{3}x\right), \quad n=0, 1, 2, \dots; \quad \lambda_n = \frac{k^2}{9}$$

$$Y_n(y) = a_n \cosh\left(\frac{n\pi}{3}y\right) + b_n \sinh\left(\frac{n\pi}{3}y\right), \quad n=1, 2, \dots$$

$$Y_0(y) = a_0 + b_0 y.$$

$$u(x, y) = a_0 + b_0 y + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{3}x\right) \left( a_n \cosh\left(\frac{n\pi}{3}y\right) + b_n \sinh\left(\frac{n\pi}{3}y\right) \right)$$

$$u(x, 0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{3}x\right) \equiv 1; \quad a_0 = 1, \quad a_n = 0, \quad n \geq 1.$$

$$u(x, \frac{\pi}{4}) = 1 + \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi^2}{12}\right) \cos\left(\frac{n\pi}{3}x\right) = 2x, \quad 0 < x < 3.$$

$$= 3 - \frac{12}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\left(\frac{n\pi}{3}x\right)$$

$$1 + b_0 \frac{\pi}{4} = 3, \quad b_0 = \frac{8}{\pi},$$

$$b_n \sinh\left(\frac{n\pi^2}{12}\right) = -\frac{12}{\pi^2} \frac{(-1)^n - 1}{n^2}$$

$$b_n = -\frac{12}{\pi^2} \frac{(-1)^n - 1}{n^2 \sinh\left(\frac{n\pi^2}{12}\right)},$$

$$u(x, y) = 1 + \frac{8}{\pi}y - \frac{12}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2 \sinh\left(\frac{n\pi^2}{12}\right)} \cos\left(\frac{n\pi}{3}x\right) \sinh\left(\frac{n\pi}{3}y\right)$$

$$0 < x < 3, \quad 0 < y < \frac{\pi}{4}.$$

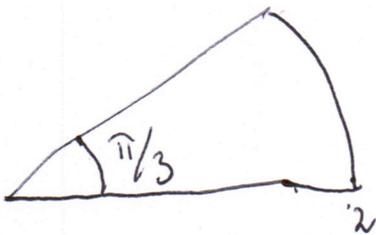
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subject to the given conditions

$$u(2, \theta) = \theta, 0 < \theta < \pi/3,$$

$$u(r, 0) = 0, u(r, \pi/3) = 0.$$



$$\textcircled{H}'' + \lambda \textcircled{H} = 0$$

$$r^2 R'' + r R' - \lambda R = 0$$

$$\textcircled{H}(0) = \textcircled{H}\left(\frac{\pi}{3}\right) = 0$$

$$\textcircled{H}(\theta) = \sin(3n\theta), \lambda_n = 9n^2, n=1, 2, \dots$$

$$R_n(r) = c_n r^{3n} + d_n r^{-3n},$$

$$r^2 R'' + r R' - 9n^2 R = 0.$$

Regularity at  $r=0$ :  $d_n = 0, n=1, 2, \dots$

$$u(r, \theta) = \sum_{n=1}^{\infty} c_n r^{3n} \sin(3n\theta) = \theta$$

$$\begin{aligned} u(2, \theta) &= \sum_{n=1}^{\infty} c_n 2^{3n} \sin(3n\theta) = \theta = \\ &= \frac{2}{3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(3n\theta); \end{aligned}$$

$$c_n = \frac{2}{3} \frac{(-1)^{n+1}}{n} 8^{-n}.$$

$$u(r, \theta) = \frac{2}{3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{r}{2}\right)^{3n} \sin(3n\theta),$$

$$0 < \theta < \frac{\pi}{3}, \quad 0 < r < 2,$$

# Lecture 28.

28.

## Fourier integral.

We take Fourier series on  $[-p, p]$  and take limit as  $p \rightarrow \infty$ , considering it as an integral sum.

$$f(x) = \frac{1}{\pi} \int_0^{\infty} [A(\lambda) \cos(\lambda x) + B(\lambda) \sin(\lambda x)] d\lambda$$

$$A(\lambda) = \int_{-\infty}^{\infty} f(x) \cos(\lambda x) dx,$$

$$B(\lambda) = \int_{-\infty}^{\infty} f(x) \sin(\lambda x) dx.$$

More symmetry.

The are Cosine and Sine integral  
(analogues of Cosine and Sine series).

Complex Fourier integral  $\Rightarrow$  Fourier transform:

$$\mathcal{F}f(\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx = F(\omega)$$

$$(\mathcal{F}^{-1})F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega = f(x).$$

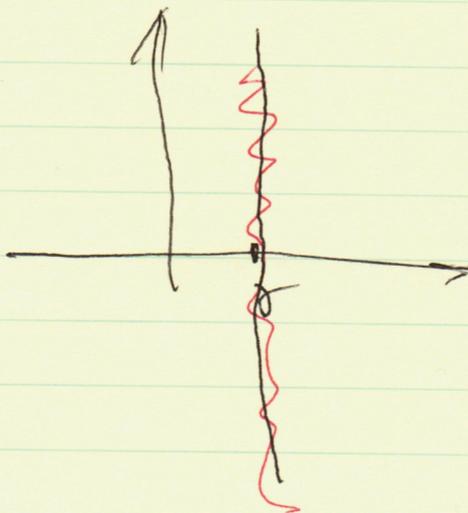
$$\mathcal{F}^{-1} \circ \mathcal{F} = \text{Id.}$$

Application to Laplace transform:

Let  $f(\omega) = 0, \omega < 0$ .

The 
$$\mathcal{L}f(s) = \mathcal{F}f(is) = \int_0^{\infty} f(t) e^{-st} dt$$

$$(\mathcal{L}^{-1} \mathcal{L}F)(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) ds$$



The final exam is  
cumulative.

Pay the attention:

- Laplace transform and its  
inversion

- Applications to differential  
equations

- Convolution; integral and  
integrodifferential equations

- All types of Fourier transforms

- Sturm-Liouville problems

- Separation of variables for  
boundary problems of partial  
differential equations.

Refer on all quizzes, midterms and  
lecture notes on webpage.