CORRECTIONS TO

REPRESENTATIONS AND INVARIANTS OF THE CLASSICAL GROUPS
by Roe Goodman and Nolan R. Wallach
(1998 hard-cover edition)

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Note: Most of the following corrections are incorporated into the 1999 (paperback) printing.

p.15, l.-14 to l.-8 (proof of assertion (2)) REPLACE:

We may assume that ... this proves (2).

BY:

The point evaluations $\{\delta_x\}_{x\in X}$ span V^*. Choose $x_i \in X$ so that $\{\delta_{x_1}, \ldots, \delta_{x_q}\}$ is a basis for V^* and let $\{g_1, \ldots, g_q\}$ be the dual basis for V. Then we can write

$$R(x)g_j = \sum_{i=1}^{q} c_{ij}(x)\, g_i$$

for $x \in X$. Since

$$c_{ij}(x) = \langle R(x)g_j,\, \delta_{x_i}\rangle = g_j(x_i x),$$

we see that $x \mapsto c_{ij}(x)$ is a regular function on X. This proves (2).

p.15, l.-3 REPLACE: $\{f_1, \ldots, f_m\} \subset \rho^*\mathrm{Aff}(G)$ BY: $\{f_1, \ldots, f_n\} \subset \Phi^*\mathrm{Aff}(G)$

p.16, l.1 to l.26 REPLACE :

The following theorem shows that ...

(STATEMENT AND PROOF OF THEOREM 1.1.14)

... so σ^{-1} is regular (see Section A.4.3). □

BY:

Example

Let B be a bilinear form on $\mathbb{C}^n$. We define a multiplication $*_B$ on $\mathbb{C}^{n+1}$ by

$$\begin{bmatrix} x \\ \lambda \end{bmatrix} *_B \begin{bmatrix} y \\ \mu \end{bmatrix} = \begin{bmatrix} x+y \\ \lambda+\mu+B(x,y) \end{bmatrix}$$

for $x, y \in \mathbb{C}^n$ and $\lambda, \mu \in \mathbb{C}$. From the bilinearity of B we calculate easily that this multiplication is associative. Since

$$\begin{bmatrix} x \\ \lambda \end{bmatrix} *_B \begin{bmatrix} -x \\ -\lambda+B(x,x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we conclude that $*_B$ defines a group structure on $\mathbb{C}^{n+1}$ with 0 as the identity element. Multiplication and inversion are regular maps, so by Theorem 1.1.13 there is a linear algebraic group G_B with $\mathrm{Aff}(G_B) \cong \mathrm{Aff}(\mathbb{C}^{n+1})$ as a $\mathbb{C}$-algebra and $G_B \cong (\mathbb{C}^{n+1}, *_B)$ as a group.

We can use the proof of Theorem 1.1.13 to obtain an explicit matrix realization of G_B. Let $f_i(x) = x_i$ for $x \in \mathbb{C}^{n+1}$ and let $g_i \in (\mathbb{C}^n)^*$ for $i = 1, \ldots, n$ be the linear functionals such that

$$B(x, y) = \sum_{i=1}^{n} f_i(x) g_i(y)$$

for $x, y \in \mathbb{C}^n$. Let $f_0(x) = 1$ for all $x \in \mathbb{C}^{n+1}$. For $f \in \mathrm{Aff}(\mathbb{C}^{n+1})$ and $y \in \mathbb{C}^{n+1}$ let $R_(y)f(x) = f(x *_B y)$. From the definition of the multiplication $*_B$ we have $R(y)f_0 = f_0$, $R(y)f_i = f_i + f_i(y)$ for $1 \le i \le n$, and

$$R(y)f_{n+1} = f_{n+1} + f_{n+1}(y) + \sum_{i=1}^{n} g_i(y) f_i$$

(we define $g_i(y) = g_i(\bar{y})$, where $\bar{y}$ is the projection of y onto $\mathbb{C}^n$). Thus the $(n+2)$-dimensional subspace V of $\mathrm{Aff}(\mathbb{C}^{n+1})$ spanned by the functions $f_0, \ldots, f_{n+1}$ is invariant under $R(y)$. Let $\Phi(y)$ be the restriction of $R(y)$ to V. Then $\Phi(y)$ has the matrix

$$\begin{bmatrix} 1 & f_1(y) & \cdots & f_n(y) & f_{n+1}(y) \\ 0 & 1 & \cdots & 0 & g_1(y) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & g_n(y) \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

relative to the ordered basis $\{f_0, f_1, \ldots, f_{n+1}\}$ for V. Since f_i and g_i are linear functions and $\{f_i(y)\}$ are the coordinates of y, it is clear that $G_B = \Phi(\mathbb{C}^{n+1})$ is a closed subgroup of $\mathrm{GL}(n+2, \mathbb{C})$ that is isomorphic to $(\mathbb{C}^{n+1}, *_B)$ as a group and as an affine algebraic set.

p.25, l.10 REPLACE: We denote by s_0

BY: We denote by s_l

p.25, l.11 (display) REPLACE: s_0 BY: s_l

p.25, l.13 REPLACE:

$$J_+ = \begin{bmatrix} 0 & s_0 \\ s_0 & 0 \end{bmatrix}, \qquad J_+ = \begin{bmatrix} 0 & s_0 \\ -s_0 & 0 \end{bmatrix},$$

BY:

$$J_+ = \begin{bmatrix} 0 & s_l \\ s_l & 0 \end{bmatrix}, \qquad J_+ = \begin{bmatrix} 0 & s_l \\ -s_l & 0 \end{bmatrix},$$

p.25, l.-10 REPLACE: $s_0 a^t s_0$ BY: $s_l a^t s_l$

p.25, l.-7 REPLACE:

$$A = \begin{bmatrix} a & b \\ c & -s_0 a^t s_0 \end{bmatrix},$$

BY:

$$A = \begin{bmatrix} a & b \\ c & -s_l a^t s_l \end{bmatrix},$$

p. 25, l.-6 REPLACE: such that $b^t = -s_0 b s_0$ and $c^t = -s_0 c s_0$

BY: such that $b^t = -s_l b s_l$ and $c^t = -s_l c s_l$

p.25, l.-3 REPLACE:

$$A = \begin{bmatrix} a & b \\ c & -s_0 a^t s_0 \end{bmatrix},$$

BY:

$$A = \begin{bmatrix} a & b \\ c & -s_l a^t s_l \end{bmatrix},$$

p. 25, l.-2 REPLACE: such that $b^t = s_0 b s_0$ and $c^t = s_0 c s_0$

BY: such that $b^t = s_l b s_l$ and $c^t = s_l c s_l$

p.26, l.6 REPLACE:

$$S = \begin{bmatrix} 0 & 0 & s_0 \\ 0 & 1 & 0 \\ s_0 & 0 & 0 \end{bmatrix}.$$

BY:

$$S = \begin{bmatrix} 0 & 0 & s_l \\ 0 & 1 & 0 \\ s_l & 0 & 0 \end{bmatrix}.$$

p.26, l.12 REPLACE:

$$A = \begin{bmatrix} a & w & b \\ u & 0 & -w^t s_0 \\ c & -s_0 u^t & -s_0 a^t s_0 \end{bmatrix},$$

BY:

$$A = \begin{bmatrix} a & w & b \\ u & 0 & -w^t s_l \\ c & -s_l u^t & -s_l a^t s_l \end{bmatrix},$$

p.26, l.13 REPLACE: such that $b^t = -s_0 b s_0$ and $c^t = -s_0 c s_0$

BY: such that $b^t = -s_l b s_l$ and $c^t = -s_l c s_l$

p.31, l.-8 **to l.**-1 REPLACE PRINTED TEXT BY:

$X_A \mathcal{I}_G \subset \mathcal{I}_G$. Write $\sigma = \pi|_G$ and take $f = f_C \circ \pi$ for $C \in \mathrm{End}(V)$. Then $X_A(f_C \circ \sigma)(I) = X_A(f_C \circ \pi)(I)$, and hence $d\sigma(A) = d\pi(A)$ by (1.2.9).

(3): Write $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. By (1), $\mathrm{Lie}(G \cap H) \subset \mathfrak{g} \cap \mathfrak{h}$. Let $X = G \times H$ and define $\varphi : X \to \mathrm{GL}(n, \mathbb{C})$ by $\varphi(g, h) = gh^{-1}$. Set $Y = \overline{\varphi(X)}$ and $F_y = \varphi^{-1}\{y\}$. Then $F_{gh^{-1}} = \{(gz, hz) : z \in G \cap H\}$, and hence $\dim F_{gh^{-1}} = \dim(G \cap H)$ for all $(g, h) \in X$. Since $\mathrm{Ker}\, d\varphi_{(1,1)} = \{(A, -A) : A \in \mathfrak{g} \cap \mathfrak{h}\}$ and $d\varphi_{(g,h)} = dL_g dR_{h^{-1}} d\varphi_{(1,1)}$, we have $\dim \mathrm{Ker}\, d\varphi_{(g,h)} = \dim(\mathfrak{g} \cap \mathfrak{h})$ for all $(g, h) \in X$. Proposition A.3.6 now implies that $\dim(G \cap H) = \dim(\mathfrak{g} \cap \mathfrak{h})$, hence $\mathrm{Lie}(G \cap H) = \mathfrak{g} \cap \mathfrak{h}$. □

p.30, l.-4 REPLACE: Corollary A.3.6 BY: Corollary A.3.5

p.32, l.-2 REPLACE: $= [\mathrm{Ad}(g)A, \mathrm{Ad}(g)A]$, BY: $= [\mathrm{Ad}(g)A, \mathrm{Ad}(g)B]$,

p.39, l.-15 REPLACE: $\mathfrak{g}_u$ BY: $\mathfrak{g}_n$

p.39, l.-13 REPLACE:

subset of $\mathrm{End}(V)$ and G_u is an algebraic subset of $\mathrm{GL}(V)$.

BY:

subset of $M_n(\mathbb{C})$ and G_u is an algebraic subset of $\mathrm{GL}(n, \mathbb{C})$.

p.39, l.-4 **and l.**-3 REPLACE:

Decompose $\mathbb{C}^n$ into spaces $W_\lambda = \{w \in \mathbb{C}^n : (H - \lambda I)^p w = 0$ for some $p\}$. Show that $XW_\lambda \subset W_{\lambda+2}$.)

BY:

Show that $[H, X^k] = 2kX^k$. Then consider the eigenvalues of $\mathrm{ad}H$ on $M_n(\mathbb{C})$.)

p.44, l.9 REPLACE:

Hence ρ^{-1} is regular by Theorem 1.1.14.

BY:

Clearly $\rho^*(\mathrm{Aff}(H)) = \mathrm{Aff}(G)$, so ρ^{-1} is regular.

p.49, l.4 (Exercise #1) REPLACE:

1. Check the assertion in (1.4.2) above.

BY:

1. Define a real form $\mathrm{Sp}(p, q)$ of $\mathrm{Sp}(p + q, \mathbb{C})$ analogous to the real form $\mathrm{U}(p, q)$ of $\mathrm{GL}(p + q, \mathbb{C})$.

p.49, l.7 and l.8 (Exercise #3) REPLACE:

Let $\psi \in \mathrm{End}(\mathbb{C}^{2n})$ act by

$$\psi[z_1, \ldots, z_n, z_{n+1}, \ldots, z_{2n}] = [\bar{z}_{n+1}, \ldots, z_{2n}, -\bar{z}_1, \ldots, -\bar{z}_n]$$

BY:

Let ψ be the real linear transformation of $\mathbb{C}^{2n}$ defined by

$$\psi[z_1, \ldots, z_n, z_{n+1}, \ldots, z_{2n}] = [\bar{z}_{n+1}, \ldots, \bar{z}_{2n}, -\bar{z}_1, \ldots, -\bar{z}_n]$$

p.51, formula (2.1.1) REPLACE: $\prod_{k=1}^{n}$ BY: $\prod_{k=1}^{l}$

p.66, l.−7 REPLACE:

$\sigma_k(g)f(x) = (-cx+d)^k f\left(\dfrac{ax-b}{-cx+d}\right).$

BY:

$\sigma_k(g)f(x) = (cx+a)^k f\left(\dfrac{dx+b}{cx+a}\right).$

p.68, l.10 REPLACE: $P(G) = \mathrm{Span}\{d\theta : \theta \in \mathcal{X}(H)\}$ BY: $P(G) = \{d\theta : \theta \in \mathcal{X}(H)\}$

p.77, Figure 2.2 REPLACE: $\varepsilon_l - \varepsilon_{l+1}$ BY: $\varepsilon_{l-1} - \varepsilon_l$

p.77, l.−13 and −12 REPLACE:

as in Type A,

BY:

and $\varepsilon_i + \varepsilon_l = \alpha_i + \cdots + \alpha_l$,

p.78, Figure 2.3 REPLACE: $\varepsilon_l - \varepsilon_{l+1}$ BY: $\varepsilon_{l-1} - \varepsilon_l$

p.82, l.−17 REPLACE:

$\alpha_i + \cdots + \alpha_j$ for $1 \le i < j < l$

BY:

$\alpha_i + \cdots + \alpha_j$ for $1 \le i < j \le l$

p.94, l.6 REPLACE: Let $s_0 \in \mathrm{GL}(2l, \mathbb{C})$

BY: Let $s_l \in \mathrm{GL}(l, \mathbb{C})$

p.94, l.10 REPLACE:

$$\pi(\sigma) = \begin{bmatrix} s_\sigma & 0 \\ 0 & s_0 s_\sigma s_0 \end{bmatrix},$$

BY:

$$\pi(\sigma) = \begin{bmatrix} s_\sigma & 0 \\ 0 & s_l s_\sigma s_l \end{bmatrix},$$

p.95, l.8 REPLACE:

$$\phi(\sigma) = \begin{bmatrix} s_\sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s_0 s_\sigma s_0 \end{bmatrix},$$

BY:

$$\phi(\sigma) = \begin{bmatrix} s_\sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s_l s_\sigma s_l \end{bmatrix},$$

p.95, l.−14 REPLACE: $\mathrm{O}(2l+1,\mathbb{C})$ BY: $\mathrm{O}(B,\mathbb{C})$

p.169, l.−14 REPLACE:

From Theorem 3.3.6 we have a

BY:

From Proposition 3.1.6 we have the

p.170, l.12 REPLACE: if $\phi \in \mathcal{J}$ then there exist

BY: if $\phi \in \mathcal{J}_+$ then there exist

p.172, l.7 REPLACE:

$$\sigma_I - x_1 = \sigma_1 - x_1 = x_2 + \cdots + x_n$$

BY:

$$\sigma_I - x^I = \sigma_1 - x_1 = x_2 + \cdots + x_n$$

p.172, l.−14 REPLACE: $f(x) - a_I \sigma^I$ BY: $f(x) - a\sigma^I$

p.174, l.−7 REPLACE: induction that $\mathcal{H} \cdot (\mathcal{P}\mathcal{J}_+)$ contains all polynomials

BY: induction that $\mathcal{H} \cdot (1 + \mathcal{P}\mathcal{J}_+)$ contains all polynomials

p.175, l.7 REPLACE:

4.1.4(1), which contradicts

BY:

4.1.4, which contradicts

p.176, l.2 REPLACE: $g = 0.$ □ BY: $g = 0.$

p.180, l.14 REPLACE: $\rho(g^{-1})v_n$ BY: $\rho(g^{-1})v_m$

p.181, l.2 REPLACE: $f(x\rho(g^{-1}), \rho(g)y), \quad x \in X, \quad y \in Y.$ BY: $f(x\rho(g^{-1}), \rho(g)y).$

p.181, l.7 REPLACE: for $g \in G$ and $x \in X$, $y \in Y$. BY: for $g \in \mathrm{GL}(V)$.

p.181, l.−15 REPLACE: $i = 1, \ldots, m,\ j = 1, \ldots, k$ BY: $i = 1, \ldots, k,\ j = 1, \ldots, m$

p.182, l.−5 REPLACE: $i \neq j$ BY: $i < j$

p.183, l.−7 **display** REPLACE:

$$uZw = \begin{bmatrix} I_r & O_{r,m-r} \\ O_{m-r,r} & O_{m-r} \end{bmatrix}$$

BY:

$$uZw = \begin{bmatrix} I_r & O_{r,m-r} \\ O_{k-r,r} & O_{k-r,m-r} \end{bmatrix}$$

p.183, l.−5 display REPLACE:

$$X = \begin{bmatrix} I_r & O_{r,m-r} \\ O_{m-r,r} & O_{k-r} \end{bmatrix}, \quad Y = \begin{bmatrix} I_r & O_{r,n-r} \\ O_{n-r,r} & O_{n-r} \end{bmatrix},$$

BY:

$$X = \begin{bmatrix} I_r & O_{r,n-r} \\ O_{k-r,r} & O_{k-r,n-r} \end{bmatrix}, \quad Y = \begin{bmatrix} I_r & O_{r,m-r} \\ O_{n-r,r} & O_{n-r,m-r} \end{bmatrix},$$

p.184, l.−8 display REPLACE:

$$X = \begin{bmatrix} J_r & O_{r,k-r} \\ O_{k-r,r} & O_{n-r,k-r} \end{bmatrix} g.$$

BY:

$$X = \begin{bmatrix} J_r & O_{r,k-r} \\ O_{n-r,r} & O_{n-r,k-r} \end{bmatrix} g.$$

p.184, l.−3 REPLACE: (SFT, Free Case) BY: (SFT, Free Case) Let $V = \mathbb{C}^n$.

p.184, l.−2 REPLACE: $\dim V \geq \min(k, m)$ BY: $n \geq \min(k, m)$

p.185, l.6, l.7, l.10 REPLACE: $(\mathbb{C}^n)^k$ BY: V^k

p.189, l.10 display REPLACE: $\prod_{j=1}^{k} y_j^{q_j}$

BY: $\prod_{j=1}^{m} y_j^{q_j}$

p.189, l.13 REPLACE: $z = (v_1, \ldots, v_k, v_1^*, \ldots, v_k^*)$

BY: $z = (v_1, \ldots, v_k, v_1^*, \ldots, v_m^*)$

p.198, l.−7 REPLACE: representation on $\mathbb{C}^n$

BY: representation on V

p.198, l.−5 REPLACE: space $\mathcal{P}^{[p,q]}(V^k \otimes (V^*)^m)^{\mathrm{GL}(V)}$

BY: space $\mathcal{P}^{[p,q]}(V^k \oplus (V^*)^m)^{\mathrm{GL}(V)}$

p.198, l.−3 REPLACE: acts on $\mathcal{P}^{[p,q]}(V^k \otimes (V^*)^m)$

BY: acts on $\mathcal{P}^{[p,q]}(V^k \oplus (V^*)^m)$

p.198, l.−1 display REPLACE: $\mathcal{P}^{[p,q]}(V^k \otimes (V^*)^m)^{\mathrm{GL}(V)} = 0$

BY: $\mathcal{P}^{[p,q]}(V^k \oplus (V^*)^m)^{\mathrm{GL}(V)} = 0$

p.199, l.2 display REPLACE: $\mathcal{P}^{[p,q]}(V^k \otimes (V^*)^m)^{\mathrm{GL}(V)}$

BY: $\mathcal{P}^{[p,q]}(V^k \oplus (V^*)^m)^{\mathrm{GL}(V)}$

p.199, l.4 REPLACE: complete contractions C_s

BY: complete contractions λ_s

p.199, l.6 display REPLACE: C_s

BY: λ_s

p.199, l.9 display REPLACE: C_s

BY: λ_s

p.211, l.7 REPLACE: $EM = M$ BY: $EM \subset M$

p.211, l.−5 REPLACE: $\mathrm{Span}\{\rho(G)u\} = Z_\lambda$ BY: $\mathrm{Span}\{\rho(G)f\} = Z_\lambda$

p.211, l.−1 REPLACE: $u \in \mathcal{R}^G$ BY: $r \in \mathcal{R}^G$

p.218, l.−10 REPLACE:

$(V^k)^*$ dual to the coordinates x_{ij} on V^k.

BY:

V^* dual to the coordinates x_{ij} on V.

p.219, l.13 REPLACE: $\rho(g)D_{ij}\rho(g^{-1})$ BY: $\rho(g)\Delta_{ij}\rho(g^{-1})$

p.224, l.9 REPLACE: $\xi^* \in V^*$ BY: $\xi \in V^*$

p.226 between l.5 and l.6 INSERT:

4.5.8 Exercises

1. Let $G = \mathrm{GL}(n, \mathbb{C})$ and $V = M_{n,p}(\mathbb{C}) \oplus M_{n,q}(\mathbb{C})$. Let $g \in G$ act on V by $g \cdot (x \oplus y) = gx \oplus (g^t)^{-1}y$ for $x \in M_{n,p}(\mathbb{C})$ and $y \in M_{n,q}(\mathbb{C})$. Note that the columns x_i of x transform as vectors in $\mathbb{C}^n$ and the columns y_j of y transform as covectors in $(\mathbb{C}^n)^*$.

(a) Let $\mathfrak{p}_-$ be the subspace of $\mathbb{D}(V)$ spanned by the operators of multiplication by $(x_i)^t \cdot y_j$ for $1 \le i \le p$, $1 \le j \le q$. Let $\mathfrak{p}_+$ be the subspace of $\mathbb{D}(V)$ spanned by the operators $\Delta_{ij} = \sum_{r=1}^n \frac{\partial}{\partial x_{ri}} \frac{\partial}{\partial y_{rj}}$ for $1 \le i \le p$, $1 \le j \le q$. Prove that $\mathfrak{p}_\pm \subset \mathbb{D}(V)^G$.

(b) Let $\mathfrak{k}$ be the subspace of $\mathbb{D}(V)$ spanned by the operators $E_{ij}^{(x)} + \frac{k}{2}\delta_{ij}$ (with $1 \le i, j \le p$) and $E_{ij}^{(y)} + \frac{k}{2}\delta_{ij}$ (with $1 \le i, j \le q$), where $E_{ij}^{(x)}$ is defined by equation (4.5.27) and $E_{ij}^{(y)}$ is similarly defined with x_{ij} replaced by y_{ij}. Prove that $\mathfrak{k} \subset \mathbb{D}(V)^G$.

(c) Prove the commutation relations $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}_\pm] = \mathfrak{p}_\pm$, $[\mathfrak{p}_-, \mathfrak{p}_+] \subset \mathfrak{k}$.

(d) Set $\mathfrak{g}' = \mathfrak{p}_- + \mathfrak{k} + \mathfrak{p}_+$. Prove that $\mathfrak{g}'$ is isomorphic to $\mathfrak{gl}(p+q, \mathbb{C})$, and that $\mathfrak{k} \cong \mathfrak{gl}(p, \mathbb{C}) \oplus \mathfrak{gl}(q, \mathbb{C})$.

(e) Prove that $\mathbb{D}(V)^G$ is generated by $\mathfrak{g}'$. (HINT: Use Theorems 4.2.1 and 4.5.16. Note that there are four possibilities for contractions to obtain G-invariant polynomials on $V \oplus V^*$: (1) vector and covector in V; (2) vector and covector in V^*; (3) vector from

V and covector from V^*; (4) covector from V and vector from V^*. Show that the contractions of types (1) and (2) furnish symbols for bases of $\mathfrak{p}_\pm$, and that contractions of type (3) and (4) furnish symbols for a basis of $\mathfrak{k}$.)

p.226, l.7 REPLACE:

The finiteness result in Theorem 4.1.1, due to Hilbert, was a major

BY:

Theorem 4.1.1 (the proof given is due to Hurwitz) was a major

p.227, l.-1 REPLACE: general Capelli problem."

BY: general "Capelli problem."

p.237, l.10 REPLACE: $p = 0, 1, \ldots, [k/2]$. BY: $p = 0, 1, \ldots, [k/2]$ (where $\varpi_0 = 0$).

p.237, l.12 REPLACE: $\bigoplus_{k=0}^{[2l-p]}$ BY: $\bigoplus_{k=0}^{l-p}$

p.243, l.2 REPLACE: If we choose $-\Phi_+$ BY: If we choose $-\Phi^+$

p.249, l.9 REPLACE: $z^{m_1+\cdots m_n}$ BY: $z^{m_1+\cdots+m_n}$

p.250, l.8 REPLACE: $\mathrm{O}(n, \mathbb{C})$ BY: $\mathrm{O}(B, \mathbb{C})$

p.254, l.−13 REPLACE:

We can choose $g_1 \in G$ so that $G = G^\circ \bigcup g_1 G^\circ$ and $\rho(g_1)\varphi^k = \pm\varphi^k$

BY:

We can choose $g_0 \in G$ so that $G = G^\circ \bigcup g_0 G^\circ$ and $\rho(g_0)\varphi^k = \varphi^k$

p.255, l.−8 REPLACE: $\sum \mu_i$ BY: $\sum i\mu_i$

p.256, l.9 REPLACE: $\mathrm{depth}(\mu) \le r$ BY: $\mathrm{depth}(\nu) \le r$

p.257, l.−3 REPLACE: *of size r such that* BY: *of size $2r$ such that*

p.258, l.18 REPLACE: it has degree $|\mu|$ BY: it has degree $|\mu|/2$

p.259, l.5 REPLACE: *such that $|\mu| = r$ and* BY: *such that $|\mu| = 2r$ and*

p.270, l.−3 REPLACE: $2\gamma(v_i)^2 = \beta(v_i, v_i)$ BY: $\{\gamma(v_i), \gamma(v_j)\} = \beta(v_i, v_j)$

p.272, l.−1 REPLACE: $\epsilon(x^*)\epsilon(y^*) = -\epsilon(x^*)\epsilon(y^*)$

BY: $\epsilon(x^*)\epsilon(y^*) = -\epsilon(y^*)\epsilon(x^*)$

p.273, l.12 REPLACE:

We combine them into a linear map

BY:

When $\dim V$ is even, we combine these operators to obtain a linear map

p.274, l.1 REPLACE:

Let $\{e_1, \ldots, e_k\}$ be a basis for W, where $k = n/2$, and let $\{e_{-1}, \ldots, e_{-k}\}$ be the basis

BY:

Let $\{e_1, \ldots, e_l\}$ be a basis for W, where $l = n/2$, and let $\{e_{-1}, \ldots, e_{-l}\}$ be the basis

p.274, l.3 REPLACE: with $1 \le j_1 < \cdots < j_p \le k$ BY: with $1 \le j_1 < \cdots < j_p \le l$

p.275, l.2,l.3,l.4 REPLACE:

Since the range of T is spanned by 2^l vectors and $\dim(\bigwedge W^*) = 2^l$, we conclude that T is bijective.

BY:

We will prove that $T\gamma(w + w^*) = \gamma'(w + w^*)T$ for $w \in W$ and $w^* \in W^*$. This will imply that $\mathrm{Ker}T = 0$, since $\gamma(W + W^*)$ acts irreducibly, and hence that $\dim Z = 1$.

p.276, l.−1 REPLACE: $(1)^r e_{j_1} \wedge \cdots$ BY: $(-1)^r e_{j_1} \wedge \cdots$

p.277, l.7 REPLACE: $\dim V = 2l + 1$ is odd, BY: $\dim V = 2l + 1$ is odd with $l \ge 1$,

p.277, l.−9 REPLACE:

We use the tensor-product model

BY:

Let $l \ge 1$ (the case $\dim V = 1$ is left to the reader) and use the model

p.278, l.−7 REPLACE: dimension $2^{\dim V}$, BY: dimension $2^{\dim V_0}$,

p.279, l.12 REPLACE: $(x_1e_1 + \cdots + x_ne_n)^2$ BY: $2(x_1e_1 + \cdots + x_ne_n)^2$

p.279, l.16 REPLACE: $\sum_{i=1}^n$ BY: $\frac{1}{2}\sum_{i=1}^n$

p.279, l.−13 REPLACE: $(2\sum R_{ijji})I$ BY: $(1/2)\sum R_{ijji}$

p.279, l.−11 REPLACE: algbera BY: algebra

p.281, l.6 REPLACE: $[\phi(X), \lambda(v)]$ BY: $[\phi(X), \gamma(v)]$

p.281, l.−1 REPLACE: spin representation BY: space of spinors

p.284, l.−15 REPLACE: dominant weight BY: highest weight

p.285, l.−12 REPLACE: $c : V \to \mathrm{Cliff}(V, \beta)$ BY: $\gamma : V \to \mathrm{Cliff}(V, \beta)$

p.286, l.4 REPLACE: $\rho(x_1) = 0$ BY: $\widetilde{\gamma}(x_1) = 0$

p.286, l.5 REPLACE: $\rho(x_1)$ BY: $\widetilde{\gamma}(x_1)$

p.286, l.8 REPLACE:

$\rho_\pm(x_1) = \pm\mu I$ for some $\mu \in \mathbb{C}$.

BY:

$\widetilde{\gamma}_\pm(x_1) = \mu_\pm I$ for some $\mu_\pm \in \mathbb{C}$.

p.286, l.9 REPLACE:

$\rho_\pm(e_0)$ is invertible, so $\mu = 0$.

BY:

$\widetilde{\gamma}_\pm(e_0)$ is invertible, so $\mu_\pm = 0$.

p.286, l.18 REPLACE:

Hence $\mathrm{O}(V, \beta)$ *is generated by reflections.*

BY:

(3) $\mathrm{O}(V, \beta)$ *is generated by reflections.*

p.286, l.-5 REPLACE:

a product of reflections.

BY:

a product of reflections, proving (3).

p.288, l.-15 and l.-14 REPLACE:

These subalgebras are spanned by elements of the form $R_{x,y}$ where $x, y \in V$ satisfy

BY:

By Lemma 6.2.1 these subalgebras are spanned by elements $R_{x,y}$ where $x, y \in V$ satisfy

p.288, l.-5 REPLACE:

$$= \frac{1}{2}\beta(y, y)\beta(x, z)\gamma(x),$$

BY:

$$= \frac{1}{2}\beta(y, y)\beta(x, z)\gamma(x) = 0,$$

p.288, l.-3 REPLACE:

$$u(t)\gamma(z)u(-t) = \gamma(z) + t[\gamma(x)\gamma(y), \gamma(z)] + \frac{t^2}{2}\beta(y, y)\beta(x, z)\gamma(x)$$

BY:

$$u(t)\gamma(z)u(-t) = \gamma(z) + t[\gamma(x)\gamma(y), \gamma(z)]$$

p.288, l.-2 REPLACE:

$$= \gamma(z) + t\gamma(R_{x,y}z) + \frac{t^2}{2}\beta(y, y)\beta(x, z)\gamma(x)$$

BY:

$$= \gamma(z) + t\gamma(R_{x,y}z)$$

p.294, l.-6 REPLACE:

(g) $\mathrm{Spin}(5, 1)^\circ \cong \mathrm{SU}(1, 3)$.

BY:

(g) $\mathrm{Spin}(5, 1)^\circ \cong \mathrm{SU}^*(4) \cong \mathrm{SL}(2, \mathbb{H})$ (see 1.4.6, Exercise # 3).

p.333, l.3 REPLACE: $Q \in \Phi^+$ BY: $Q \subset \Phi^+$

p.336, l.-7 REPLACE:

for every $Q \subset \Phi^+$ and has multiplicity one.

BY:

for every $Q \subset \Phi^+$.

p.340, l.-16 REPLACE:

$$\gamma s_0 \gamma^t = I_{2l}$$

BY:

$$\gamma s_{2l} \gamma^t = I_{2l}$$

p.340, l-15 REPLACE: where s_0 is the matrix

BY: where s_{2l} is the matrix

p.340, l-14 REPLACE: corresponding to s_0 as in

BY: corresponding to s_{2l} as in

p.340, l-12 REPLACE:

$$\gamma g \gamma^{-1}(\gamma g \gamma^{-1})^t = \gamma g s_0 g^t \gamma^t = \gamma s_0 \gamma^t = I_{2l}.$$

BY:

$$\gamma g \gamma^{-1}(\gamma g \gamma^{-1})^t = \gamma g s_{2l} g^t \gamma^t = \gamma s_{2l} \gamma^t = I_{2l}.$$

p.340, l-9 REPLACE: defined by the equation $g^t g = I$.

BY: defined by the equation $g^t g = I_{2l}$.

p.354, l.-1 REPLACE: irreducible $\mathfrak{g}$-module BY: irreducible $\mathfrak{h}$-module

p.434, l.11 REPLACE:

$$\mathcal{H}\mathcal{T}_r^{\otimes k} = \{u \in \mathcal{T}_r^{\otimes k} \,:\, u \cdot u = 0 \text{ for all } u \in \mathcal{B}_{k,r+1}(V, \omega)\}$$

BY:

$$\mathcal{H}\mathcal{T}_r^{\otimes k} = \{u \in \mathcal{T}_r^{\otimes k} \,:\, z \cdot u = 0 \text{ for all } z \in \mathcal{B}_{k,r+1}(V, \omega)\}$$

p.436, equation (10.3.4) REPLACE:

$$1 \le m(r, \lambda) \le \dim(G^\lambda)|\mathcal{M}(k, r)|$$

BY:

$$\dim(G^\lambda) \le m(r, \lambda) \le \dim(G^\lambda)|\mathcal{M}(k, r)|$$

p.436, l.-8 REPLACE: Let $r \ge 0$ BY: Let $r > 0$

p.467, l.-3 REPLACE: $\mathrm{Aff}(G/N)$ BY: $\pi^*\mathrm{Aff}(G/N)$

p.467, l.-1 REPLACE: translates if f BY: translates of f

p.485, l.-4 REPLACE: X_A BY: X_G

p.486, l.-4 REPLACE: $V_i \subset V_{i-1}$ BY: $V_i^0 \subset V_{i-1}^0$

p.487, l.-5 **and l.**-6 REPLACE:

and

$$\frac{d}{dt}(y^{-1}\theta(y)(I + t\theta(B))y(I + tB))|_{t=0} = \mathrm{Ad}(y^{-1})\theta(B) + B.$$

BY:

whereas the curve $t \mapsto y(I + tB)$ is tangent to Q at y provided

$$0 = \frac{d}{dt}(y^{-1}\theta(y)(I + t\theta(B))y(I + tB))|_{t=0} = \mathrm{Ad}(y^{-1})\theta(B) + B.$$

p.492, l.-12 REPLACE: $\mathrm{Sp}(\omega)$ BY: $\mathrm{Sp}(\mathbb{C}^{2n}, \omega)$

p.500, l.15 REPLACE:

and distinct regular homomorphisms

BY:

and regular homomorphisms

p.500, l.-10 REPLACE:

Then we have distinct regular characters

BY:

Then we have regular characters

p.500, l.-10 REPLACE: $\cdots \supset V_r$ with BY: $\cdots \supset V_r \supset V_{r+1} = \{0\}$ with

p.501, l.5 REPLACE:

Given $v \in V_r$, $x \in \mathcal{D}(G)$, and $g \in G$ we have

BY:

If $v \in V$ and $\pi(x)v = \theta_r(x)v$ for all $x \in \mathcal{D}(G)$, then

p.501, l.7 **and l.**8 REPLACE:

Thus $\pi(g)v \in V_r$. since π is an irreducible representation, this implies that $V = V_r$. We conclude that $r = 1$ and $\pi(x) = \theta_1(x)I$ for all $x \in \mathcal{D}(G)$.

BY:

Thus $\pi(x)v = \theta_r(x)v$ for all $v \in V$ and $x \in \mathcal{D}(G)$, since the space of vectors with this property contains $V_r \neq 0$ and is G-invariant. Write $\theta_r = \theta$.

p.501, l.9**, l.**13**, and l.**14 REPLACE: θ_1 BY: θ

p.502, l.8 REPLACE: element BY: elements

p.502, l.−5 REPLACE:

$$(\exp yX_0)g(\exp -yX_0) = t\exp[(t^{-\alpha}-1)y + zX_0].$$

BY:

$$(\exp yX_0)g(\exp -yX_0) = t\exp[((t^{-\alpha}-1)y + z)X_0].$$

p.504, l.13 to l.17 REPLACE:

Proof of Theorem 11.3.7 We may take G to be a closed subgroup of $\mathrm{GL}(n,\mathbb{C})$. Let X be the projective variety of full flags in $\mathbb{C}^n$. Let B be a Borel subgroup of G of maximum dimension. Then Theorem 11.3.8 implies that the set

$$Y = \{x \in X \,:\, bx = x \text{ for all } b \in B\}$$

is nonempty. Fix $y \in Y$ and set $\mathcal{O} = G \cdot y$. Set $Z = \overline{\mathcal{O}}$ (Zariski closure in X).

BY:

Proof of Theorem 11.3.7 Let B be a Borel subgroup of G of maximum dimension. By Theorem 11.1.1 there is a representation (π, V) of G and a point $y \in \mathbb{P}(V)$ so that B is the stabilizer of y. Set $X = \mathbb{P}(V)$ and $\mathcal{O} = G \cdot y \subset X$. Then $G/B \cong \mathcal{O}$ as a quasi-projective set. Set $Z = \overline{\mathcal{O}}$ (Zariski closure in X).

p.505, l.10 REPLACE: $y \cdot (gB) - ygB$ BY: $y \cdot (gB) = ygB$

p.505, l.15 REPLACE: $\phi_k(x) = x^k$ BY: $\Phi_k(x) = x^k$

p.505, l.16 REPLACE: $G(k) \subset G(k+1)$ BY: $G(2^k) \subset G(2^{k+1})$

p.515, l.10 REPLACE: Theorem A.3.4 BY: Theorem A.3.3

p.527, l.−7 REPLACE: Theorem A.3.4 BY: Theorem A.3.3

p.532, l.15 to l. 19 (Exercise #1) REPLACE:

1. Let L be a reductive group, and set $G = L \times L$. Let $K = \{(g,g) \,:\, g \in L\}$ be the diagonal embedding of L in G. Show that (G,K) is a spherical pair. (HINT: The irreducible representations of G are of the form $\pi = \sigma\widehat{\otimes}\mu$, where σ and μ are irreducible representations of L. Use Schur's Lemma to show that the K-spherical representations of G are the representations $\pi = \sigma\widehat{\otimes}\sigma^*$.)

BY:

1. Use Theorem 12.2.1 to show that the following spaces are multiplicity-free:

(a) $G = \mathrm{GL}(n) \times \mathrm{GL}(k)$, $X = M_{n,k}(\mathbb{C})$, $(g,h)\cdot x = gxh^{-1}$. (HINT: Lemma B.2.8.)

(b) $G = \mathrm{GL}(n)$, $X = SM_n(\mathbb{C})$, $g \cdot x = gxg^t$. (HINT: Lemma B.2.9.)

(c) $G = \mathrm{GL}(n)$, $X = AM_n(\mathbb{C})$, $g \cdot x = gxg^t$. (HINT: Lemma B.2.10.)

p.534, l.16 REPLACE: $\tau(g) = (g^{-t)^{-1}}$ BY: $\tau(g) = (\bar{g}^t)^{-1}$

p.538, l.8 REPLACE: Theorem A.3.4 BY: Theorem A.3.3

p.540, l.23 REPLACE: note that $l = n - 1$ is odd. BY: note that $l = 2n - 1$ is odd.

p.550, l.-1 REPLACE:

TYPE AII: $\{\varpi_2, \varpi_4, \ldots, \varpi_l\}$ $(p = l/2)$,

BY:

TYPE AII: $\{\varpi_2, \varpi_4, \ldots, \varpi_{l-1}\}$ $(p = (l-1)/2)$,

p.558, l.-13 DELETE: Then

p.566, l.4 and l.5 REPLACE: $T_{f,j}$ BY: $T_{j,f}$

p.566, l.8 REPLACE: $\mathcal{V}(f) \neq 0$ BY: $\mathcal{V}(\mathcal{I}(f)) \neq \{0\}$

p.582, l.8 to l.17 REPLACE STATEMENT AND PROOF OF LEMMA A.1.3 BY:

LEMMA A.1.3 *An element $b \in B$ is integral over A if and only if there exists a finitely-generated A-submodule $C \subset B$ such that $b \cdot C \subset C$.*

Proof. Let b satisfy (A.1.2). Then $A[b] = A \cdot 1 + A \cdot b + \cdots + A \cdot b^{n-1}$ is a finitely-generated A-submodule, so we may take $C = A[b]$. Conversely, suppose C exists as stated and is generated by $\{x_1, \ldots, x_n\}$ as an A-module. Since $bx_i \in C$, there are elements $a_{ij} \in A$ so that

$$bx_i - \sum_{j=1}^{n} a_{ij}\, x_j = 0 \qquad \text{for } i = 1, \ldots, n.$$

Since $x_i \neq 0$ and B has no zero divisors, the determinant of the coefficient array of the x_i must vanish. This determinant is a monic polynomial in b, with coefficients in A. Hence b is integral over A. □

p.582, l.20 to l.23 REPLACE:

The submodule $A[b]$ of B is therefore also finitely generated, for any $b \in B$, and hence b is integral over A.

BY:

Now apply Lemma A.1.3 with $C = B$.

p.587, l.2 REPLACE: but f_1 not vanishing BY: but f_i not vanishing

p.587, l.3 REPLACE: and $X \neq X_1$. BY: and $X \neq X_i$.

p.588, l.9 REPLACE: $\widetilde{v}(x)/f(x) = 0.$ BY: $\widetilde{v}(x)/f(x)^k = 0.$

p.592, l.1 REPLACE: Let $\phi \in \mathrm{Aff}(X)$. BY: Let $\phi \in \mathrm{Aff}(Y)$.

p.592, l.6 REPLACE: all $\phi \in \mathrm{Aff}(X)$. BY: all $\phi \in \mathrm{Aff}(Y)$.

p.592, l.−14 to l.−10 REPLACE:

every $\phi \in \mathrm{Hom}(A, \mathbb{C})^a$ *extends to* $\psi \in \mathrm{Hom}(B, \mathbb{C})^b$.

Proof. We start with the case $B = A[u]$ for some element $u \in B$. Let $b = f(u)$ be given, where

$$f(X) = a_n X^n + \cdots + a_0, \quad a_i \in A.$$

BY:

every $\phi \in \mathrm{Hom}(A, \mathbb{C})^a$ *extends to* $\psi \in \mathrm{Hom}(B, \mathbb{C})^b$. *If* B *is integral over* A *and* $b = 1$, *then* $a = 1$.

Proof. We start with the case $B = A[u]$ for some element $u \in B$. Let $b = f(u)$ be given, where $f(X) = a_n X^n + \cdots + a_0$ with $a_i \in A$.

p.593, l.3 REPLACE:

element $a = a_m c_0$ has the desired property in this case.

BY:

element $a = a_m c_0$ has the desired property. Note that if u is integral over A and $b = 1$ then $a = 1$.

p.593, l.5 and l.6 REPLACE: $q(X)$ BY: $h(X)$

p.599, l.15 and l.16 REPLACE:

a map $x \mapsto L_x$ from X to $T(X)_x$

BY:

a correspondence $x \mapsto L_x \in T(X)_x$

p.601, l.−13 **to** −8 DELETE: Statement and proof of Corollary A.3.3

p.601, l.−7 REPLACE: **Theorem A.3.4** BY: **Theorem A.3.3**

p.602, l.1 REPLACE: **Lemma A.3.5** BY: **Lemma A.3.4**

p.603, l.9 REPLACE: *Theorem A.3.4* BY: *Theorem A.3.3*

p.602, l.9 REPLACE: Lemma A.3.5 BY: Lemma A.3.4

p.602, l.−12 REPLACE: **Corollary A.3.6** BY: **Corollary A.3.5**

p.602, l.−9 REPLACE: Lemma A.3.5 BY: Lemma A.3.4

p.603, l.−8 REPLACE: Theorem A.3.4 BY: Theorem A.3.3

p.604, l.12 to l.25 DELETE EXERCISES A.3.5 AND REPLACE BY:

Proposition A.3.6 *Let* $\varphi : X \to Y$ *be a dominant regular map of irreducible affine algebraic sets. For* $y \in Y$ *let* $F_y = \varphi^{-1}\{y\}$. *Then there is a nonempty open set* $U \subset X$ *such that* $\dim X = \dim Y + \dim F_{\varphi(x)}$ *and* $\dim F_{\varphi(x)} = \dim \mathrm{Ker}(d\varphi_x)$ *for all* $x \in U$.

Proof. Let $d = \dim X - \dim Y$, $S = \varphi^*\mathrm{Aff}(Y)$, and $R = \mathrm{Aff}(X)$. Set $k = \mathrm{Quot}(S)$ and let $B \subset \mathrm{Quot}(R)$ be the subalgebra generated by k and R (the rational functions on X with denominators in $S \setminus \{0\}$). Since B has transcendence degree d over k, Lemma A.1.17 furnishes an algebraically independent set $\{f_1, \ldots, f_d\} \subset R$ such that B is integral over $k[f_1, \ldots, f_d]$. Taking the common denominator of a set of generators of the algebra B, we obtain $f = \varphi^* g \in S$ such that R_f is integral over $S_f[f_1, \ldots, f_d]$, where $R_f = \mathrm{Aff}(X^f)$ and $S_f = \varphi^*\mathrm{Aff}(Y^g)$. By Theorem A.2.5 we can take g so that $\varphi(Y^g) = X^f$.

Define $\psi : X^f \to Y^g \times \mathbb{C}^d$ by $\psi(x) = (\varphi(x), f_1(x), \ldots, f_d(x))$. Then $\psi^*\mathrm{Aff}(Y^g \times \mathbb{C}^d) = S_f[f_1, \ldots, f_d]$, and hence $\mathrm{Aff}(X^f)$ is integral over $\psi^*\mathrm{Aff}(Y^g \times \mathbb{C}^d)$. By Theorem A.2.5 every homomorphism from $S_f[f_1, \ldots, f_d]$ to $\mathbb{C}$ extends to a homomorphism from R_f to $\mathbb{C}$. Hence ψ is surjective. Let $\pi : Y^g \times \mathbb{C}^d \to Y^g$ by $\pi(y, z) = y$. Then $\varphi = \pi \circ \psi$ and $F_y = \psi^{-1}(\{y\} \times \mathbb{C}^d)$. If W is any irreducible component of F_y then $\mathrm{Aff}(W)$ is integral over $\psi^*\mathrm{Aff}(\{y\} \times \mathbb{C}^d)$, and hence $\dim W = d$.

We have $d\varphi_x = d\pi_{\psi(x)} \circ d\psi_x$. By integrality, every derivation of $\mathrm{Quot}(\psi^*(Y^g \times \mathbb{C}^d))$ extends uniquely to a derivation of $\mathrm{Rat}(X^f)$, as in the proof of Theorem A.3.1. Hence $d\psi_x$ is bijective for x in a nonempty dense open set U by Lemma A.3.4. For such x, $\mathrm{Ker}(d\varphi_x) = \mathrm{Ker}(d\pi_{\psi(x)})$ has dimension d. □

p.606, l.−2 and l.−1 REPLACE:

$(x, y) \mapsto x^t y$, where x^t is the transpose of x.

BY:

$(x, y) \mapsto x y^t$, where y^t is the transpose of y.

p.607, l.8, l.12, and l.−3 REPLACE: $x^t y$ BY: $x y^t$

p.607, l.−5 REPLACE: $x^t x$ BY: $x x^t$

p.609, l.−4 to l.−1 REPLACE:

Corollary A.4.6 *Let X be a quasiprojective algebraic set and $\phi : X \to X$ a regular map. Then the fixed-point set $\{x \in X : \phi(x) = x\}$ is closed in X.*

Proof. The fixed-point set of ϕ is the intersection of the closed sets Γ_ϕ and Δ, where Δ is the diagonal in $X \times X$. □

BY:

Corollary A.4.6 *Let X, Y be quasiprojective algebraic sets and $\phi : X \times Y \to X$ a regular map. Then $\{(x, y) \in X \times Y : \phi(x, y) = x\}$ is closed in $X \times Y$.*

Proof. Use the same argument as for Proposition A.4.5. □

p.664, l.8, l.9, and l.10 REPLACE:

$d\varphi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ by

$$d\varphi(X)_1 = d\varphi_1(X_1).$$

The content of the following result is that $d\varphi$ is a Lie algebra homomorphism.

BY:

$d\varphi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ by $d\varphi(X)_1 = d\varphi_1(X_1)$.

p.664, l.12 REPLACE:

Proof. Use the same argument as in Theorem 1.2.10 □

BY:

Proof. If $f \in C^\infty(H)$ then $X(f \circ \varphi) = (d\varphi(X)f) \circ \varphi$ by the left-invariance of X. Hence $[X, Y](f \circ \varphi) = ([d\varphi(X), d\varphi(Y)]f) \circ \varphi$. This implies that $d\varphi([X, Y])_1 = ([d\varphi(X), d\varphi(Y)])_1$. □

p.664, l.17 REPLACE:

Thus Lemma D.2.5 implies that

BY:

Thus Theorem D.2.3 implies that