An Algebraic Group Approach to Compact Symmetric Spaces

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1. Classical Theory of Spherical Harmonics

(a) $L^2(S^{n-1}) = \oplus_j V_j$, with V_j irreducible under $\mathrm{SO}(n)$ and $V_j \not\cong V_m$ for $j \neq m$.

(b) $\mathcal{P}(\mathbf{R}^n) = \mathcal{J} \otimes \mathcal{H}$ (separation of variables), where $\mathcal{J} = \mathbf{C}[r^2]$ are the invariant polynomials, and $\mathcal{H}$ are the harmonic polynomials.

(c) Restrictions of $\mathcal{H}$ to S^{n-1} gives all irreducible G-invariant spaces in **(a)**.

Goal: Extend this theory to functions on compact Riemannian symmetric spaces as follows:

(Complex Algebraic Groups) Complexify the compact symmetric space to be an affine algebraic subset of the complexified isometry group of the space (Richardson).

(Matrix models) Give matrix realization of the complexified symmetric spaces of *classical type* (three linear algebra constructions).

(Spherical Varieties) Use theory of *multiplicity-free* spaces (Vinberg) to obtain **(a)**.

(Iwasawa Decomposition) Determine the highest weights of the irreducible spherical representations (Helgason).

(Linear Isotropy Representations) Change setting for **(b)** and **(c)** to the tangent space of a symmetric space (Kostant-Rallis).

2. Isotypic Decompositions

Given

- G - connected complex reductive algebraic group
- X - affine algebraic set on which G acts regularly

Let $\mathrm{Aff}(X)$ be the *regular functions* on X and define the representation

$$\rho_X(g)f(x) = f(g^{-1}x), \quad \text{for } f \in \mathrm{Aff}(X).$$

Notation:

- Borel subgroup $B = HN$ (H = maximal torus, N = unipotent radical)
- Weight lattice $P(G)$ of G
- Dominant weights $P_{++}(G)$, relative to the system of positive roots determined by N

Each $\lambda \in P(G)$ gives character $h \mapsto h^\lambda$ of H; extend this to a character of B by setting $(hn)^\lambda = h^\lambda$ for $h \in H$ and $n \in N$. An irreducible regular representation (π, V) of G is then determined (up to equivalence) by its *highest weight*:

- The subspace V^N of N-fixed vectors in V is one-dimensional
- H acts on V^N by a character $h \mapsto h^\lambda$ where $\lambda \in P_{++}(G)$.

Fix (π^λ, V^λ), an irreducible representation with highest weight λ. Let $\mathrm{Aff}(X)^N(\lambda)$ be the N-fixed regular functions on X of weight λ.

Theorem 1 *For $\lambda \in P_{++}(G)$, the isotypic subspace of type π^λ in $\mathrm{Aff}(X)$ is the span of $\rho_X(G)\mathrm{Aff}(X)^N(\lambda)$. This subspace is isomorphic to $V^\lambda \otimes \mathrm{Aff}(X)^N(\lambda)$ as a G-module, with action $\pi^\lambda(g) \otimes 1$. Thus*

$$\mathrm{Aff}(X) \cong \bigoplus_{\lambda \in P_{++}(G)} V^\lambda \otimes \mathrm{Aff}(X)^N(\lambda)$$

Define the *spectrum* of X to be

$$\mathcal{S}(X) = \{\lambda \in P_{++}(G) \,:\, \mathrm{Aff}(X)^N(\lambda) \neq 0\}.$$

Then $\mathcal{S}(X)$ is an additive semigroup, since

$$\mathrm{Aff}(X)^N(\lambda) \cdot \mathrm{Aff}(X)^N(\mu) \subset \mathrm{Aff}(X)^N(\lambda + \mu)$$

under pointwise multiplication. Thus $\mathcal{S}(X)$ determines the G-isotypic decomposition of $\mathrm{Aff}(X)$ as an abstract G-module.

3. Function Models for Irreducible Representations

Let $\bar{N}$ be the unipotent group *opposite* to N. We can choose the matrix form of G so that $\bar{N} = N^t$. For $\lambda \in P_{++}(G)$ let

$$\phi_\lambda : V^{\lambda^*} \otimes V^\lambda \to \mathrm{Aff}(G), \qquad \phi_\lambda(v^* \otimes v)(g) = \langle v^*, \pi^\lambda(g)v \rangle$$

Choose an N-fixed vector $v_\lambda \in V^\lambda$ and a $\bar{N}$-fixed vector $v^*_\lambda \in V^{\lambda^*}$, normalized so that $\langle v^*_\lambda, v_\lambda \rangle = 1$.

Theorem 2 (Algebraic Borel-Weil) *Let $\lambda \in P_{++}(G)$. Let $\mathcal{R}_\lambda \subset \mathrm{Aff}(G)$ be the subspace of functions such that*

$$f(\bar{n}hg) = h^\lambda f(g), \quad \text{for } \bar{n} \in \bar{N},\ h \in H,\ g \in G. \tag{1}$$

Then $\mathcal{R}_\lambda$ is spanned by the right translates of the function

$$f_\lambda(g) = \langle v^*_\lambda, \pi^\lambda(g)v_\lambda \rangle,$$

The restriction of the right regular representation R of G to $\mathcal{R}_\lambda$ is an irreducible representation with highest weight λ.

The function f_λ is uniquely determined by the property $f(\bar{n}hn) = h^\lambda$ for $\bar{n} \in \bar{N}$, $h \in H$, $n \in N$, and is called the *generating function* for the representation with highest weight λ.

Corollary 3 *Let $\lambda_1, \ldots, \lambda_r$ be generators for the additive semigroup $P_{++}(G)$. Set $f_i(g) = f_{\lambda_i}(g)$. Let $\lambda \in P_{++}(G)$ and write $\lambda = m_1\lambda_1 + \cdots + m_r\lambda_r$ with $m_i \in \mathbf{N}$. Then*

$$f_\lambda(g) = f_1(g)^{m_1} \cdots f_r(g)^{m_r} \quad \textit{for } g \in G. \tag{2}$$

Example

Suppose $G = \mathrm{GL}(n, \mathbf{C})$. Take B as the group of upper-triangular invertible matrices. Identify $P(G)$ with $\mathbf{Z}^n$, where $\lambda = [\lambda_1, \ldots, \lambda_n]$ gives the character

$$h^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}, \quad \text{for} \quad h = \mathrm{diag}[x_1, \ldots, x_n].$$

Then $P_{++}(G)$ consists of the monotone decreasing n-tuples

$$\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$$

and is generated by

$$\lambda_i = [\underbrace{1, \ldots, 1}_{i}, 0, \ldots, 0] \quad \text{for } i = 1, \ldots, n$$

and $\lambda_{n+1} = -\lambda_n$. Another model for the fundamental representations in this case is

$$V^{\lambda_i} = \bigwedge^i \mathbf{C}^n \quad \text{for i=1,\ldots, n.}$$

These models show that the generating functions are

$$f_{\lambda_i}(g) = \ i\text{th principal minor of } g.$$

for $i = 1, \ldots, n$ and $f_{\lambda_{n+1}}(g) = (\det g)^{-1}$.

4. Multiplicity-free Spaces

Def. X is *multiplicity-free* as a G-space if all the irreducible representations of G that occur in $\mathrm{Aff}(X)$ have multiplicity one.

For a subgroup $K \subset G$ and $x \in X$ we write $K_x = \{k \in K \,:\, k \cdot x = x\}$ for the isotropy group at x.

Theorem 4 (Vinberg-Kimelfeld) *Let X be a homogeneous affine G-space. Suppose there is a point $x_0 \in X$ such that $B \cdot x_0$ is open in X. Then*

- *X is multiplicity-free;*
- *if the representation π^λ with highest weight λ occurs in $\mathrm{Aff}(X)$, then $h^\lambda = 1$ for all $h \in H_{x_0}$.*

Proof. It suffices to show that

$$\dim \mathrm{Aff}(X)^N(\lambda) \leq 1 \quad \text{for all } \lambda \in P_{++}(G).$$

Suppose $B \cdot x_0$ is open in X. Then $f \in \mathrm{Aff}(X)^N(\lambda)$ is determined by $f(x_0)$, since it satisfies $f(b \cdot x_0) = b^{-\lambda} f(x_0)$ on the open set $B \cdot x_0$. □

Let $K \subset G$ be a reductive algebraic subgroup. Then $X = G/K$ has the structure of an affine algebraic set. The pair (G, K) will be called *spherical* if

$$\dim(V^\lambda)^K \leq 1.$$

for every $\lambda \in P_{++}(G)$. By Frobenius reciprocity

$$(G, K) \text{ spherical} \iff G/K \text{ is multiplicity-free.}$$

From the conjugacy of Borel subgroups in G and Theorem 4 we have the following criterion for spherical pairs.

Corollary 5 *Suppose there exists a connected solvable subgroup S of G so that $\mathbf{k} + \mathbf{s} = \mathbf{g}$. Then (G, K) is spherical.*

When (G, K) is a spherical pair an irreducible representation V^λ of G will be called K-*spherical* if $\dim(V^\lambda)^K = 1$.

Problem: When (G, K) is a spherical pair, determine the semigroup $\mathcal{S}(G/K)$ of highest weights of K-spherical representations of G.

5. Algebraic Models for Symmetric Spaces

Let θ be an involutive automorphism of G and let $K = G^\theta$. The space G/K can be embedded into G as an affine algebraic subset as follows.

Define the *θ-twisted conjugation*:

$$g \star y = gy\theta(g)^{-1}, \quad \text{for } g, y \in G.$$

Let

$$Q = \{y \in G \,:\, \theta(y) = y^{-1}\}.$$

Then Q is an algebraic subset of G and $G \star Q = Q$.

Theorem 6 (Richardson) *The θ-twisted action of G is transitive on each irreducible component of Q. Hence Q is a finite union of closed θ-twisted G-orbits.*

Proof. Show that the tangent space to a twisted G-orbit coincides with the tangent space to Q. □

Corollary 7 *Let $P = G \star 1 = \{g\theta(g)^{-1} \,:\, g \in G\}$ be the orbit of the identity element under the θ-twisted conjugation action. Then P is a closed irreducible subset of G isomorphic to G/K as a G-space (relative to the θ-twisted conjugation action of G).*

6. Classical Symmetric Spaces

Let $G \subset \mathrm{GL}(n, \mathbf{C})$ be a connected classical group with $\mathrm{Lie}(G)$ a simple Lie algebra. The involutions θ and associated symmetric spaces G/K for G can be described in terms of three kinds of geometric structures on $\mathbf{C}^n$:

- nondegenerate bilinear forms on $\mathbf{C}^n$: $G = \mathrm{SL}(n, \mathbf{C})$ and $K = \mathrm{SO}(n, \mathbf{C})$ (Type **AI**) or $K = \mathrm{Sp}(n, \mathbf{C})$ (Type **AII**)
- polarizations $\mathbf{C}^n = V_+ \oplus V_-$ with $V_\pm$ totally isotropic subspaces (Types **AIII, BDI,** and **CII**)
- orthogonal decompositions $\mathbf{C}^n = V_+ \oplus V_-$ (Types **DIII** and **CI**).

The proof that these seven types give all the possible involutive automorphisms of the classical groups (up to inner automorphisms) can be obtained from following characterization of automorphisms of the classical groups.

Proposition 8 *Let σ be a regular automorphism of the classical group G.*

(1) *If $G = \mathrm{SL}(n, \mathbf{C})$ then there exists $s \in G$ so that σ is either $\sigma(g) = sgs^{-1}$ or $\sigma(g) = s(g^t)^{-1}s^{-1}$.*

(2) *If G is $\mathrm{Sp}(n, \mathbf{C})$ then there exists $s \in G$ so that $\sigma(g) = sgs^{-1}$.*

(3) *If G is $\mathrm{SO}(n, \mathbf{C})$ with $n \neq 2, 4$, then there exists $s \in \mathrm{O}(n, \mathbf{C})$ so that $\sigma(g) = sgs^{-1}$.*

Proof. The Weyl dimension formula implies that the defining representation (and its dual, in the case $G = \mathrm{SL}(n, \mathbf{C})$) is the unique representation of smallest dimension. So this representation is sent to an equivalent representation (or its dual) by σ. □

Given the classical group G and involution θ, we set $K = G^\theta$ and

$$P = \{g\theta(g)^{-1} \,:\, g \in G\}, \qquad Q = \{y \in G \,:\, \theta(y) = y^{-1}\}.$$

Let $\tau(g) = (\bar{g}^t)^{-1}$. We may assume:

(1) $\tau(G) = G$ and G^τ is a compact real form of G.

(2) The diagonal subgroup H in G is a maximal torus and $\theta(H) = H$.

(3) $\tau\theta = \theta\tau$; hence $\sigma = \tau\theta$ is a conjugation on G.

Example–Type AI.

Let $G = \mathrm{SL}(n, \mathbf{C})$ and $\theta(g) = (g^t)^{-1}$. Then

$$K = G^\theta = \mathrm{SO}(\mathbf{C}^n, B), \quad G^\tau = \mathrm{SU}(n), \quad G^\theta \cap G^\tau = \mathrm{SO}(n).$$

Also

$$g \star y = gyg^t, \qquad Q = \{y \in G \,:\, y^t = y\} = P \quad \text{(one orbit)}.$$

Hence the map $gK \mapsto gg^t$ gives the algebraic embedding

$$\mathrm{SL}(n, \mathbf{C})/\mathrm{SO}(n, \mathbf{C}) \cong \{y \in M_n(\mathbb{C}) \,:\, y = y^t, \det y = 1\}.$$

The compact symmetric space is embedded in P as

$$\mathrm{SU}(n)/\mathrm{SO}(n) \cong \{y \in \mathrm{SU}(n) \,:\, y\bar{y} = I\}.$$

7. Iwasawa Decompositions

Set $A = (H \cap Q)^\circ$. This is a θ-*anisotropic* (algebraic) torus:

$$\theta(a) = a^{-1} \quad \text{for all } a \in A.$$

If χ is a regular character of A then

$$\chi(\sigma(a)) = \overline{\chi(a)} \quad \text{(complex conjugate)}.$$

One says that A is σ-*split*.

There is a solvable subgroup AN^+ of G (A a torus, N^+ a unipotent group), such that

$$K \cap (AN^+) \quad \text{is finite,} \quad KAN^+ \quad \text{is Zariski-dense in } G.$$

This is the (complexified) *Iwasawa decomposition* of G.

Set $T = H \cap K$. Then T° is a torus and there is a finite group C such that $T = T^\circ \times C$.

Define $M = \mathrm{Cent}_K(A)$ (the centralizer of A in K). Then $M = TM^\circ$

Call $\lambda \in P_{++}(G)$ θ-*admissible* if

$$t^\lambda = 1 \quad \text{for all } t \in T.$$

Theorem 9 *(G, K) is a spherical pair. If $\lambda \in P_{++}(G)$ is the highest weight of a K-spherical representation, then λ is θ-admissible*

Proof. Use the Iwasawa decomposition, Theorem 4 and Corollary 5. □

Example (Type AI).

$G = \mathrm{SL}(n, \mathbf{C})$ and $\theta(g) = (g^t)^{-1}$. Here $A = H$, $N^+ =$ all upper-triangular unipotent matrices, $T \cong (\mathbf{Z}/2\mathbf{Z})^{n-1}$ consists of all matrices

$$t = \mathrm{diag}[\delta_1, \ldots, \delta_n], \quad \delta_i = \pm 1, \, \det(t) = 1,$$

and $M = T$. The Iwasawa decomposition is the so-called QR factorization. $\lambda = [\lambda_1, \ldots, \lambda_n]$ is θ-admissible iff λ_i is even for all i.

8. Spherical Representations

Theorem 10 (Helgason) *Let (π^λ, V^λ) be an irreducible regular representation of G with highest weight λ. The following are equivalent:*
(1) *V^λ contains a nonzero K-fixed vector.*
(2) *V^λ contains a nonzero MN^+-fixed vector.*
(3) *λ is θ-admissible.*

Proof. We already showed (1) $\Longrightarrow$ (3). We show (3) $\Longrightarrow$ (1):

Write $G_0 = G^\sigma$ (non-compact), $K_0 = K^\sigma = K^\tau$ (compact) and $A_0 = A^\sigma$. Define

$$v_0 = \int_{K_0} \pi^\lambda(k) v_\lambda \, dk.$$

Then $v_0 \in (V^\lambda)^K$ by the unitarian trick. Must show: $v_0 \neq 0$. Let f_λ be the generating function for π^λ. Then

$$\langle v_\lambda^*, v_0 \rangle = \int_{K_0} f_\lambda(k) \, dk. \tag{3}$$

If λ is θ-admissible, then

- $a^\lambda > 0$ for all $a \in A_0$.
- $f_\lambda(g) \geq 0$ for $g \in G_0$.

Since $f_\lambda(1) = 1$ and $K_0 \subset G_0$, this shows that the integral (3) is nonzero.
□

Corollary 11 *As a G-module, $\mathrm{Aff}(G/K) \cong \oplus_\lambda V^\lambda$, where λ runs over all θ-admissible dominant weights of H.*

Example (Type AI): Here A_0 = real diagonal matrices, $G_0 = \mathrm{SL}(n, \mathbf{R})$ and

$$f_\lambda(g) = \Delta_1(g)^{m_1} \cdots \Delta_{n-1}(g)^{m_{n-1}},$$

where Δ_i is the ith principal minor and $m_i = \lambda_i - \lambda_{i+1}$. Since λ is θ-admissible $\Longleftrightarrow$ all λ_i are even, the conditions in the proof are obvious.

The highest weight $\lambda = [2, 0, \ldots, 0]$ is admissible, and $V^\lambda = S^2(\mathbb{C}^n)$. The K-fixed vector is the invariant bilinear form.

Satake Diagrams of Spherical Highest Weights

Here are the classical examples, where $l = \mathrm{rank}(G)$ and $p = \mathrm{rank}(G/K) = \dim(A)$:

Bilinear Forms

$G = \mathrm{SL}(n, \mathbb{C})$, $\quad K = \mathrm{SO}(n, \mathbb{C})$ ($l = n - 1$, $p = n - 1$):

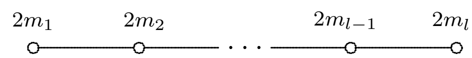

Figure 12.1: Satake Diagram of Type AI

$G = \mathrm{SL}(2n, \mathbb{C})$, $\quad K = \mathrm{Sp}(n, \mathbb{C})$ ($l = 2n - 1$, $p = n$):

Figure 12.2: Satake Diagram of Type AII

Polarizations

$G = \mathrm{SL}(n, \mathbb{C})$, $\quad K = \mathrm{S}(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(n - p, \mathbb{C}))$ ($l = n - 1$, $2p \leq n$):

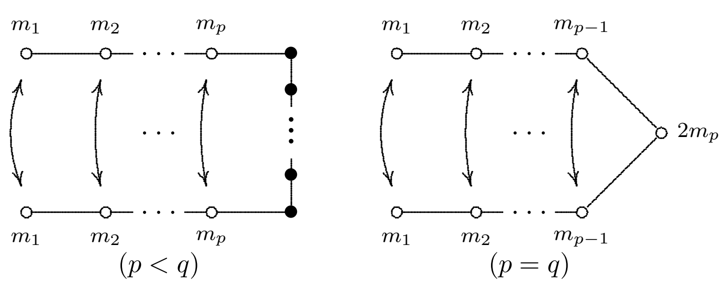

Figure 12.3: Satake Diagrams of Type AIII

$G = \mathrm{Sp}(n, \mathbb{C})$, $\quad K = \mathrm{GL}(n, \mathbb{C})$ ($l = n$, $p = n$):

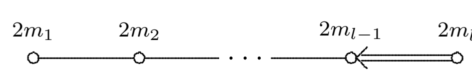

Figure 12.4: Satake Diagram of Type CI

$G = \mathrm{SO}(2n, \mathbb{C}), \quad K = \mathrm{GL}(n, \mathbb{C})$ $(l = n,\ p = n)$:

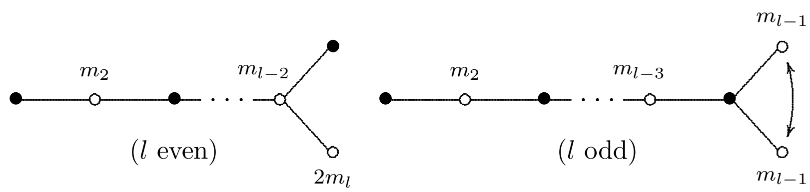

Figure 12.5: Satake Diagrams of Type DIII

Orthogonal Decompositions

$G = \mathrm{SO}(2l+1, \mathbb{C}), \quad K = \mathrm{S}(\mathrm{O}(p, \mathbb{C}) \times \mathrm{O}(q, \mathbb{C}))$ $(p + q = 2l + 1,\ p < q)$:

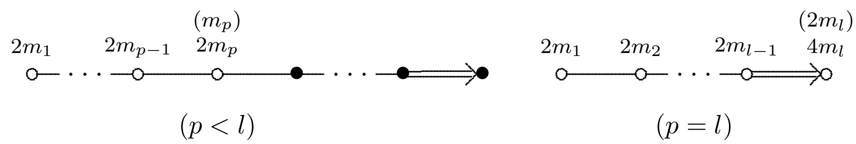

Figure 12.6: Satake Diagrams of Type BI

$G = \mathrm{SO}(2l, \mathbb{C}), \quad K = \mathrm{S}(\mathrm{O}(p, \mathbb{C}) \times \mathrm{O}(q, \mathbb{C}))$ $(p + q = 2l,\ p \leq q)$:

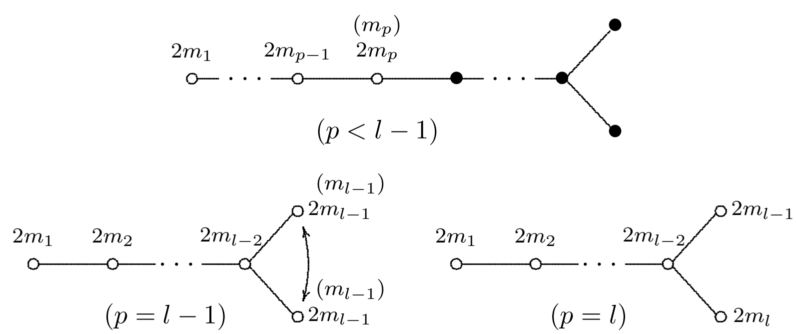

Figure 12.7: Satake Diagrams of Type DI

$G = \mathrm{Sp}(l, \mathbb{C}), \quad K = \mathrm{S}(\mathrm{O}(p, \mathbb{C}) \times \mathrm{O}(q, \mathbb{C}))$ $(p + q = l,\ p \leq q)$:

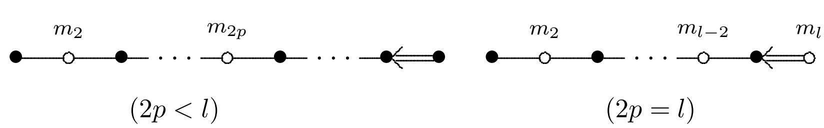

Figure 12.8: Satake Diagrams of Type CII

9. Kostant-Rallis Theorem

We now turn to the tangent space analysis. Set

$$V = \{X \in \mathbf{g} \,:\, \theta X = -X\}$$

Then

$$\mathbf{g} = \mathbf{k} \oplus V$$

as a K-module under $\mathrm{Ad}|_K$. Set $\sigma(k) = \mathrm{Ad}(k)|_V$ for $k \in K$. Then (σ, V) is a regular representation of K.

Let μ be the representation of K on $\mathcal{P}(V)$ given by

$$\mu(k)f(v) = f(\sigma(k)^{-1}v) \quad \text{for } f \in \mathcal{P}(V),\, k \in K \text{ and } v \in V.$$

By Chevalley's Theorem, there are homogenous polynomials u_i with

$$\mathcal{P}(V)^K = \mathbf{C}[u_1, \ldots, u_r] \qquad (r = \dim A).$$

Set

$$\mathcal{P}_+(V)^K = \{f \in \mathcal{P}(V)^K \,:\, f(0) = 0\}.$$

Choose a K-invariant subspace $\mathcal{H}^j$ in $\mathcal{P}^j(V)$ such that

$$\mathcal{P}^j(V) = \mathcal{H}^j \oplus \{\mathcal{P}^j(V) \cap (\mathcal{P}(V)\mathcal{P}_+(V)^K)\ \}.$$

and set

$$\mathcal{H} = \bigoplus_{j \geq 0} \mathcal{H}^j.$$

One choice for $\mathcal{H}$ is the K-*harmonic* polynomials:

$$\partial(u_i)f = 0 \quad \text{for } i = 1, \ldots, r.$$

Theorem 12 (Kostant-Rallis)

(a) *The map* $\mathcal{H} \otimes \mathcal{P}(V)^K \longrightarrow \mathcal{P}(V)$ *given by* $h \otimes f \mapsto hf$ *is a linear bijection.*

(b) $\mathcal{H}$ *is equivalent with* $\mathrm{Aff}(K/M)$ *as a* K*-module.*

In particular, if (ρ, F) *is an irreducible regular representation of* K *then* $\mathrm{Hom}_K(F, \mathcal{P}(V))$ *is a free* $\mathcal{P}(V)^K$ *module on* $\dim F^M$ *generators.*

Special case: $G = G_1 \times G_1$ (G_1 connected, reductive). Let $\theta(g, h) = (h, g)$. Then $K \cong G_1$ (embedded diagonally). Let $\mathbf{g}_1$ be the Lie algebra of G_1. Then

$$(\sigma, V) \cong (\mathrm{Ad}, \mathbf{g}_1), \qquad M \cong T_1 \quad \text{(maximal torus in } G_1.$$

10. Classical Examples

There are 16 pairs covered by the Kostant-Rallis Theorem, with $\mathbf{g}$ simple and K a product of classical groups (7 pairs with $\mathbf{g}$ classical and 9 with $\mathbf{g}$ exceptional).

Type AI

Here $G = \mathrm{SL}(n, \mathbb{C})$, $\theta(g) = (g^t)^{-1}$ and $K = \mathrm{SO}(n, \mathbb{C})$.

$$V = \text{symmetric } n \times n \text{ matrices of trace } 0$$

$$\sigma(k)X = kXk^{-1} \quad \text{for } k \in K,\ X \in V.$$

The polynomials $u_i(X) = \mathrm{tr}(X^{i+1})$, for $i = 1, \ldots, n-1$, generate $\mathcal{P}(V)^K$.

$$M = K \cap H \cong \times^{n-1}(\mathbf{Z}/2\mathbf{Z}).$$

Type G

Let $K = (\mathrm{SL}(2) \times \mathrm{SL}(2))/\{(I, I), (-I, -I)\}$ and let (σ, V) the representation with

$$V = \mathbf{C}^2 \widehat{\otimes} S^3(\mathbf{C}^2) \quad \text{(outer tensor product)}.$$

Here M is isomorphic with $\times^2(\mathbf{Z}/2\mathbf{Z})$. One has

$$\mathcal{P}(V)^K = \mathbb{C}[u_1, u_2] \quad \text{with } \deg u_1 = 2 \text{ and } \deg u_2 = 6.$$

This example comes from the exceptional group G_2.

11. Comments on the Proof and Further Examples

Let K be a connected reductive linear algebraic group and let (σ, V) be a regular representation of K.

Assume that there is a subspace $\mathbf{a}$ in V and an element $h \in \mathbf{a}$ so that the following conditions hold.

(1) The restriction $f \mapsto f|_{\mathbf{a}}$ defines an isomorphism of $\mathcal{P}(V)^K$ onto a subalgebra $\mathcal{R}$ of $\mathcal{P}(\mathbf{a})$.

(2) The subalgebra $\mathcal{R}$ of $\mathcal{P}(\mathbf{a})$ is generated by algebraically independent homogeneous elements $u_1, \ldots, u_l$ with $l = \dim \mathbf{a}$. Furthermore, there exists a graded subspace $\mathcal{A}$ of $\mathcal{P}(\mathbf{a})$ such that the map $\mathcal{A} \otimes \mathcal{R} \to \mathcal{P}(\mathbf{a})$ given by $a \otimes r \mapsto ar$ is a linear bijection.

(3) There exists $h \in \mathbf{a}$ such that $|\sigma(K)h \cap \mathbf{a}| \geq \dim \mathcal{A}$.

(4) Let h be as in (3) and set

$$\mathcal{X}_h = \{v \in V \, : \, f(v) = f(h) \text{ for all } f \in \mathcal{P}(V)^K\}.$$

If $v \in \mathcal{X}_h$ then $\dim Kv = \dim V - \dim \mathbf{a}$.

For the linear isotropy representation of a symmetric space, conditions (1) and (2) are proved using the Chevalley restriction theorem. The other conditions require new arguments.

In the general setting above, let

$$M = \{k \in K \, : \, \sigma(k)h = h\}.$$

Our proof of Theorem 12 then proves the following result.

Theorem 13 *Assume that* (σ, V) *satisfies* (1)-(4)*. Let* F *be an irreducible regular* K*-module. Then as a* $\mathcal{P}(V)^K$*-module, the space* $\mathrm{Hom}_K(F, \mathcal{P}(V))$ *is free on* $\dim \mathrm{Hom}_M(\mathbf{C}, F)$ *generators (here* $\mathbf{C}$ *denotes the trivial one-dimensional* M *representation).*

Here are some examples that are not isotropy representations for symmetric spaces but nevertheless conditions (1)-(4) are satisfied.

1. Let $K = \mathrm{SL}(2, \mathbf{C})$ and let (σ, V) be the representation of K on $S^3(\mathbf{C}^2)$ (i.e. the irreducible four-dimensional representation). Then $\mathcal{P}(V)^K = \mathbf{C}[f]$ with f irreducible and homogeneous of degree 4. Let e_1, e_2 be the usual basis of $\mathbf{C}^2$ and let $h = e_1^3 + e_2^3$. Set $\mathbf{a} = \mathbf{C}h$. Here

$$M = \left\{ \begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix} : \xi^3 = 1 \right\}.$$

2. Let $K = \mathrm{Sp}(3, \mathbf{C})$ and let $V \subset \wedge^3 \mathbf{C}^6$ be the irreducible K-submodule with highest weight $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$. Then $\mathcal{P}(V)^K = \mathbf{C}[f]$ with f an irreducible homogeneous polynomial of degree 4. Let $h = e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6$ and $\mathbf{a} = \mathbf{C}h$. Here M is the group of all matrices

$$k = \begin{bmatrix} b & 0 \\ 0 & (b^t)^{-1} \end{bmatrix}, \quad b \in \mathrm{SL}(3, \mathbf{C}).$$

3. Let $K = \mathrm{SL}(6, \mathbf{C})$ and let $V = \wedge^3 \mathbb{C}^6$. As in Examples 1 and 2, one has $\mathcal{P}(V)^K = \mathbf{C}[f]$ with f homogeneous of degree 4. Take h and u as in Example 2. Then M is the group of all

$$\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}, \quad b_1, b_2 \in \mathrm{SL}(3, \mathbf{C}).$$