The Representation Theory of Riemannian Curvature Tensors

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CUNY Graduate Center – Representation Theory Seminar

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Outline

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R. S. Kulkani: On the Bianchi Identities (1972)
A. Besse: Einstein Manifolds (1987)
R. S. Strichartz: Linear Algebra of Curvature Tensors and their Covariant Derivatives (1988)

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- covariant constant metric tensor:
  $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
- zero torsion: $\nabla_X Y - \nabla_Y X = [X, Y]$

Curvature tensor field $R_p(x,y) \in \mathrm{End}(T_pM) = T_pM \otimes (T_pM)^*$:

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(C3) Jacobi identity for $\mathcal{T}(M)$ + zero torsion $\implies$ Bianchi identity: $R_p(x,y)z + R_p(y,z)x + R_p(z,x)y = 0$

The Space of Curvature Tensors

Fix $p \in M$. Let $E = (T_pM)_{\mathbb{C}} \cong E^*$ (via $Q = (g_p)_{\mathbb{C}}$)
Define Riemann-Christoffel curvature tensor $R \in \bigotimes^4 E \cong \bigotimes^4 E^*$:

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- (C1) + (C2) + (C3) $\Longrightarrow$ $\sigma(12)R = -R$, $\sigma(34)R = -R$, $\sigma(13)\sigma(24)R = R$ $\Longrightarrow$ $R \in S^2(\bigwedge^2 E)$

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Hence $\mathrm{Curv}(E) = \mathrm{Ker}(b) \cap S^2(\bigwedge^2 E)$

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Fix Q-orthonormal basis $e_1, \ldots, e_n$ for $E \cong \mathbb{C}^n$ $(n \geq 2)$
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Weyl dim. formula + $(\star) \implies \dim F_n^{[2,2]} = \dim \mathrm{Curv}(\mathbb{C}^n)$ □

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Corollary

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- $\mathrm{Range}(\mathbf{p}_A) \subset S^2(\bigwedge^2 \mathbb{C}^n)$

Orthogonal Decomposition of Curvature Tensors

Let $R \in \mathrm{Curv}(\mathbb{C}^n)$.

Ricci curvature:

$$\mathrm{Ric}_Q(R)(v, w) = \sum_{i=1}^n R(e_i, v, e_i, w)$$

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$R \mapsto s_Q(R)$ gives $\mathrm{O}(Q)$ intertwining operator $s_Q : \mathrm{Curv}(\mathbb{C}^n) \to \mathbb{C}$ (trivial $\mathrm{O}(Q)$ module)

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Representation-theoretic description of decomposition $(\star\star\star)$

Define Weyl conformal curvature tensors:

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The space of curvature tensors decomposes under $\mathrm{O}(Q)$ *as*

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Proof.

Formula $(\star\star) \implies$ the sum $(\star\star\star)$ is direct $\implies$ dimension formula

The Space of Weyl Curvature Tensors

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Show $\mathrm{Weyl}_Q(\mathbb{C}^n)$ is irreducible under $\mathrm{O}(Q)$ ($n \geq 4$)

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$$f_k = \frac{1}{\sqrt{2}}\big(e_k + \sqrt{-1}e_{n+1-k}\big), \quad f_{-k} = \frac{1}{\sqrt{2}}\big(e_k - \sqrt{-1}e_{n+1-k}\big)$$

for $k = 1, \ldots, l$ ($l = \lfloor \frac{n}{2} \rfloor$) (For n odd: $f_0 = e_{l+1}$)
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Have $V_n \subset \mathrm{Weyl}_Q(\mathbb{C}^n)$, and Weyl dimension formula $\Longrightarrow$

$$\dim V_n = \frac{1}{12} n(n+1)(n+2)(n-3) = \dim \mathrm{Weyl}_Q(\mathbb{C}^n)$$

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$V_4 \cap \overline{V}_4 = 0$ (inequivalent for $\mathrm{SO}(Q)$) $\square$

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Special feature: $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ (semisimple)
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Set $\overline{C_0} = \tau \cdot w = (f_1 \wedge f_{-2}) \otimes (f_1 \wedge f_{-2}) \in \mathrm{Weyl}_Q(\mathbb{C}^4)$

- $\overline{C_0}$ eigenvector for B of weight $4\varpi_2$
- $\overline{V}_4 = \tau \cdot V_4 \subset \mathrm{Weyl}_Q(\mathbb{C}^4)$ is irreducible under $\mathrm{SO}(Q)$

Theorem

$\mathrm{Weyl}_Q(\mathbb{C}^4) = V_4 \oplus \overline{V}_4$ *and is irreducible under* $\mathrm{O}(\mathbb{C}^4, Q)$.

Proof.

$\tau : V_4 \leftrightarrow \overline{V}_4$ and $\dim V_4 = \dim \overline{V}_4 = 5$ while $\dim \mathrm{Weyl}_Q(\mathbb{C}^4) = 10$
$V_4 \cap \overline{V}_4 = 0$ (inequivalent for $\mathrm{SO}(Q)$) $\square$

Note: Likewise, $\bigwedge^2 \mathbb{C}^4$ is irreducible under $\mathrm{O}(Q)$, but decomposes under $\mathrm{SO}(Q)$ with highest weights $2\varpi_1$ and $2\varpi_2$.

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[Follows from $e^{-2f} X(e^{2f} g(Y, Z)) = g(\widetilde{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}_X Z)$ and cyclic permutation of X, Y, Z.]

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- Scalar curvature is multiplied by e^{2f} plus term

  $e^{2f}\{2(n-1)\Delta f - (n-1)(n-2)|Df|^2\}$

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  - Find other G orbits on V and orbit invariants (Strichartz)
    $n = 4$: Use classical invariant theory of binary quartics

Hermann Weyl:

"The wish to understand what really is the mathematical substance behind the formal apparatus of relativity theory led me to the study of representations and invariants of groups."

Relativity theory as a stimulus in mathematical research (1949)