The Representation Theory of Riemannian Curvature Tensors

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October 12, 2007

Outline

H. Weyl:

Reine Infinitesimalgeometrie (1918)

Das gruppentheretische Fundament der Tensorrechnung (1924)

Riemannian Connection and Curvature Tensor

(M, g) – smooth (pseudo) Riemannian manifold:
nondegenerate bilinear form g_p on tangent space $T_p(M)$
(Riemannian if g_p positive definite)
$\mathcal{C}(M)$ – smooth functions $\quad \mathcal{T}(M)$ – smooth vector fields
Riemannian connection:
$X \in \mathcal{T}(M)$ acts as covariant derivative ∇_X on tensor fields:

- $\nabla_{\varphi X} Y = \varphi \nabla_X Y \qquad \nabla_X(\varphi Y) = X(\varphi) Y + \nabla_X Y$

for $\varphi \in \mathcal{C}(M)$ and $Y \in \mathcal{T}(M)$
∇ uniquely determined from g by requiring

- covariant constant metric tensor:
  $X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0$
- zero torsion: $\quad \nabla_X Y - \nabla_Y X = [X, Y]$

Curvature tensor field $R_p(x,y) \in \mathrm{End}(T_pM) = T_pM \otimes (T_pM)^*$:

$$R_p(x,y)z = \big(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\big)_p$$

- Only depends on $X_p = x$, $Y_p = y$, and $Z_p = z$ (tensorial) (unlike Lie derivative $\theta(X)$ – not tensorial)
- Measures failure of $X \mapsto \nabla_X$ to be Lie algebra homomorphism

Algebraic Symmetries of the curvature tensor:

(C1) $R_p(x,y) = -R_p(y,x)$, so $R_p : \bigwedge^2 T_p(M) \to \mathrm{End}(T_p(M))$

(C2) $R_p(x,y)^* = -R_p(x,y)$ ($*$ via g_p), so $R_p(x,y) \in \mathrm{Lie}(\mathrm{O}(g_p))$

(C3) Jacobi identity for $\mathcal{T}(M)$ + zero torsion $\Longrightarrow$ Bianchi identity:
$R_p(x,y)z + R_p(y,z)x + R_p(z,x)y = 0$

The Space of Curvature Tensors

Fix $p \in M$. Let $E = (T_pM)_{\mathbb{C}} \cong E^*$ (via $Q = (g_p)_{\mathbb{C}}$)
Define Riemann-Christoffel curvature tensor $R \in \bigotimes^4 E \cong \bigotimes^4 E^*$:

$$R(v, w, x, y) = g_p(R_p(v, w)x, y) \quad \text{for } v, w, x, y \in E$$

Program: Study subspace $\mathrm{Curv}(E) \subset \bigotimes^4 E$ of tensors with the same algebraic symmetries as R.
Let σ be the permutation representation of $\mathfrak{S}_4$ on $\bigotimes^4 E$

- (C1) + (C2) + (C3) $\implies$ $\sigma(12)R = -R, \quad \sigma(34)R = -R,$ $\sigma(13)\sigma(24)R = R \quad \implies \quad R \in S^2(\bigwedge^2 E)$
- $S^2(\bigwedge^2 E)$ is invariant under the Bianchi operator $b = \frac{1}{3}(I + \sigma(123) + \sigma(123)^2) = \frac{1}{3}(I + \sigma(13)\sigma(12) + \sigma(23)\sigma(12))$

Hence $\mathrm{Curv}(E) = \mathrm{Ker}(b) \cap S^2(\bigwedge^2 E)$

Curv(E) as GL(E) module

- Range(b) $\cap S^2(\bigwedge^2 E) = \bigwedge^4 E$ (irreducible for GL(E))
- $b^2 = b \Longrightarrow S^2(\bigwedge^2 E) = \mathrm{Curv}(E) \oplus \left(\bigwedge^4 E\right)$ (second summand zero if $\dim E < 4$)
- $\dim \mathrm{Curv}(\mathbb{C}^n) = \frac{1}{12}\, n^2(n+1)(n-1)$ $(\star)$

Fix Q-orthonormal basis $e_1, \ldots, e_n$ for $E \cong \mathbb{C}^n$ $(n \geq 2)$
$\lambda \in \mathbb{N}^k, \quad \lambda_1 \geq \cdots \geq \lambda_k > 0 \quad (k \leq n)$
F_n^λ – irreducible GL$(n, \mathbb{C})$ representation, highest weight λ

Theorem

Curv$(\mathbb{C}^n) \cong F_n^{[2,2]}$ *is an irreducible* GL$(n, \mathbb{C})$ *module.*

Proof.

High wt vector $R = (e_1 \wedge e_2) \otimes (e_1 \wedge e_2) \Longrightarrow F_n^{[2,2]} \subset \mathrm{Curv}(\mathbb{C}^n)$.
Weyl dim. formula + $(\star) \Longrightarrow \dim F_n^{[2,2]} = \dim \mathrm{Curv}(\mathbb{C}^n)$ □

Curvature and Young Symmetrizers

Young tableau A (k boxes) has row group and column group

Young symmetrizer: $\mathbf{p}_A \in \mathrm{End}_{\mathrm{GL}(n,\mathbb{C})}(\bigotimes^k \mathbb{C}^n)$
(alternate over column group)·(symmetrize over row group)

Weyl module: Range $\mathbf{p}_A$ is an irreducible $\mathrm{GL}(n,\mathbb{C})$ module,
highest weight $\lambda = \mathrm{shape}(A)$

Corollary

$$\mathrm{Curv}(\mathbb{C}^n) = \mathrm{Range}\,\mathbf{p}_A \quad \textit{where} \quad A = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

Proof.

$$\mathbf{p}_A = \tfrac{1}{12}\left\{\big(1-\sigma(12)\big)\big(1-\sigma(34)\big)\right\}\cdot\left\{\big(1+\sigma(13)\big)\big(1+\sigma(24)\big)\right\}$$

- $\mathrm{Range}(\mathbf{p}_A) \cong F_n^{[2,2]}$ $\quad$ ($\mathrm{shape}(A) = [2,2]$)
- $\mathrm{Range}(\mathbf{p}_A) \subset S^2(\bigwedge^2 \mathbb{C}^n)$

$\square$

Orthogonal Decomposition of Curvature Tensors

Let $R \in \mathrm{Curv}(\mathbb{C}^n)$.

Ricci curvature:

$$\mathrm{Ric}_Q(R)(v, w) = \sum_{i=1}^{n} R(e_i, v, e_i, w)$$

- Ricci contraction operator $\mathrm{Ric}_Q : S^2(\bigwedge^2 \mathbb{C}^n) \to S^2(\mathbb{C}^n)$
- Ric_Q intertwines $\mathrm{O}(Q)$ action on $\mathrm{Curv}(\mathbb{C}^n)$ and $S^2(\mathbb{C}^n)$
- All nonzero contraction operators on $S^2(\bigwedge^2 \mathbb{C}^n)$ are $\pm\,\mathrm{Ric}_Q$

Scalar curvature:

$$s_Q(R) = \mathrm{tr}_Q(\mathrm{Ric}_Q(R)) = \sum_{i,j=1}^{n} R(e_i, e_j, e_i, e_j)$$

$R \mapsto s_Q(R)$ gives $\mathrm{O}(Q)$ intertwining operator $s_Q : \mathrm{Curv}(\mathbb{C}^n) \to \mathbb{C}$ (trivial $\mathrm{O}(Q)$ module)

Construct $\mathrm{O}(Q)$ intertwining operator $S^2(\mathbb{C}^n) \to S^2(\bigwedge^2 \mathbb{C}^n)$:

- $S^2(\mathbb{C}^n) \cong Q$-symmetric linear maps of $\mathbb{C}^n$ $(Q \leftrightarrow I)$
- $S^2(\bigwedge^2 \mathbb{C}^n) \cong Q \otimes Q$-symmetric linear maps of $\bigwedge^2 \mathbb{C}^n$

Given $A, B \in S^2(\mathbb{C}^n)$, define linear map $A \owedge B$ on $\bigwedge^2 \mathbb{C}^n$:

$$(A \owedge B)(v \wedge w) = Av \wedge Bw + Bv \wedge Aw \quad \text{for } v, w \in \mathbb{C}^n$$

- $A \owedge B \in S^2(\bigwedge^2 \mathbb{C}^n)$
- $A \owedge B$ satisfies Bianchi identity, so $A \owedge B \in \mathrm{Curv}(\mathbb{C}^n)$
- $\mathrm{Ric}_Q(A \owedge Q) = \mathrm{tr}_Q(A)Q + (n-2)A \quad (\star\star)$

$n = 2$:

- $\dim \mathrm{Curv}(\mathbb{C}^2) = 1$
- $R = \frac{1}{4} s_Q(R)\, Q \owedge Q$
- $\mathrm{Ric}_Q(R) = \frac{1}{2} s_Q(R) Q$

Assume $n \geq 3$. Given $R \in \mathrm{Curv}(\mathbb{C}^n)$, set

$$A = \frac{1}{n-2}\Big\{ \mathrm{Ric}_Q(R) - \frac{1}{n}s_Q(R)Q \Big\} \in S^2(\mathbb{C}^n)$$

$$C = R - A \owedge Q - \gamma s_Q(R)Q \owedge Q \in \mathrm{Curv}(\mathbb{C}^n) \quad (\gamma = \frac{1}{n(2n-1)})$$

The normalizing constants are chosen so that

- $\mathrm{tr}_Q(A) = 0$
- $\mathrm{Ric}_Q(C) = 0$ by $(\star\star)$
- $R = \gamma s_Q(R)Q \owedge Q + A \owedge Q + C \qquad (\star\star\star)$
  scalar part + traceless Ricci part + Weyl part

Representation-theoretic description of decomposition $(\star\star\star)$

Define Weyl conformal curvature tensors:

$$\mathrm{Weyl}_Q(\mathbb{C}^n) = \{C \in \mathrm{Curv}(\mathbb{C}^n) \,:\, \mathrm{Ric}_Q(C) = 0\}$$

Q-harmonic (traceless) symmetric two-tensors:

$$\mathcal{H}^2_{\mathrm{sym}}(\mathbb{C}^n, Q) = \{A \in S^2(\mathbb{C}^n) \,:\, \mathrm{tr}_Q(A) = 0\}$$

Properties of $\mathcal{H}^2_{\mathrm{sym}}(\mathbb{C}^n, Q)$:

- Irreducible under $\mathrm{SO}(Q)$ (Cartan component of $\mathbb{C}^n \otimes \mathbb{C}^n$)
- Highest weight $2\varpi_1$ ($n \neq 4$) or $2(\varpi_1 + \varpi_2)$ ($n = 4$)
- Dimension $= \frac{1}{2}n(n+1) - 1 = \frac{1}{2}(n+2)(n-1)$
- $\mathcal{H}^2_{\mathrm{sym}}(\mathbb{C}^n, Q) \hookrightarrow \mathrm{Curv}(C^n)$ by $A \mapsto A \owedge Q$

Theorem

The space of curvature tensors decomposes under $\mathrm{O}(Q)$ *as*

$$\mathrm{Curv}(\mathbb{C}^n) = \mathbb{C}(Q \owedge Q) \oplus \left(\mathcal{H}^2_{\mathrm{sym}}(\mathbb{C}^n, Q) \owedge Q\right) \oplus \mathrm{Weyl}_Q(\mathbb{C}^n)$$

Hence dim $\mathrm{Weyl}_Q(\mathbb{C}^n) = \frac{1}{12}n(n+1)(n+2)(n-3)$.
In particular, Ricci curvature determines curvature when $n = 3$.

Proof.

Formula $(\star\star) \Longrightarrow$ the sum $(\star\star\star)$ is direct $\Longrightarrow$ dimension formula

$\square$

The Space of Weyl Curvature Tensors

Goal:
Show $\mathrm{Weyl}_Q(\mathbb{C}^n)$ is irreducible under $\mathrm{O}(Q)$ ($n \geq 4$)
Method:
Find a highest weight vector and use Weyl dimension formula.

Take Q-isotropic basis for $\mathbb{C}^n$

$$f_k = \tfrac{1}{\sqrt{2}}\big(e_k + \sqrt{-1}e_{n+1-k}\big), \quad f_{-k} = \tfrac{1}{\sqrt{2}}\big(e_k - \sqrt{-1}e_{n+1-k}\big)$$

for $k = 1, \ldots, l$ ($l = \left\lfloor \frac{n}{2} \right\rfloor$) When n is odd let $f_0 = e_{l+1}$.
Set $F_k = \mathrm{Span}\{f_1, \ldots, f_k\}$ for $k = 1, \ldots, l$ (isotropic subspace)
Take Borel subgroup $B \subset \mathrm{SO}(Q)$ as stabilizer of isotropic flag

$$F_1 \subset F_2 \subset \cdots \subset F_l$$

(include $F_{l+1} = \mathrm{Span}\{f_0, f_1, \ldots, f_l\}$ when n odd)
$\varpi_1, \ldots, \varpi_l$ fundamental highest weights for $\mathrm{SO}(Q)$

Set $C_0 = (f_1 \wedge f_2) \otimes (f_1 \wedge f_2) \in S^2(\bigwedge^2 \mathbb{C}^n)$

- $C_0 \in \mathrm{Weyl}_Q(\mathbb{C}^n)$
- C_0 is a B eigenvector of weight $2\varpi_2$ (if $n > 4$) or $4\varpi_1$ (if $n = 4$)
- $V_n = \mathrm{Span}\,\mathrm{SO}(Q) \cdot C_0$ is irreducible under $\mathrm{SO}(Q)$ (theorem of the highest weight)

Theorem

Assume $n > 4$. $\mathrm{Weyl}_Q(\mathbb{C}^n)$ *is irreducible under* $\mathrm{SO}(Q)$ *and has highest weight* $2\varpi_2$*.*

Proof.

Have $V_n \subset \mathrm{Weyl}_Q(\mathbb{C}^n)$, and Weyl dimension formula $\Longrightarrow$

$$\dim V_n = \tfrac{1}{12} n(n+1)(n+2)(n-3) = \dim \mathrm{Weyl}_Q(\mathbb{C}^n)$$

☐

Note: ϖ_2 is the highest weight of $\bigwedge^2 \mathbb{C}^n$ if $n > 4$.

Assume $n = 4$:

Special feature: $\mathrm{SO}_4(\mathbb{C})$ is semisimple but not simple.
Let $\tau \in \mathrm{O}(\mathbb{C}^4, Q)$ fix $f_{\pm 1}$ and interchange $f_2 \leftrightarrow f_{-2}$ $\quad$ $(\det \tau = -1)$
Set $\overline{C_0} = \tau \cdot w = (f_1 \wedge f_{-2}) \otimes (f_1 \wedge f_{-2}) \in \mathrm{Weyl}_Q(\mathbb{C}^4)$

- $\overline{C_0}$ is an eigenvector for B of weight $4\varpi_2$.
- $\overline{V}_4 = \tau \cdot V_4 \subset \mathrm{Weyl}_Q(\mathbb{C}^4)$ is irreducible under $\mathrm{SO}(Q)$.

Theorem

$\mathrm{Weyl}_Q(\mathbb{C}^4) = V_4 \oplus \overline{V}_4$ *and is irreducible under* $\mathrm{O}(\mathbb{C}^4, Q)$.

Proof.

$\tau : V_4 \leftrightarrow \overline{V}_4$ and $\dim V_4 = \dim \overline{V}_4 = 5$ while $\dim \mathrm{Weyl}_Q(\mathbb{C}^4) = 10$
$V_4 \cap \overline{V}_4 = 0$ (inequivalent for $\mathrm{SO}(Q)$)

☐

Note: Likewise, $\bigwedge^2 \mathbb{C}^4$ is irreducible under $\mathrm{O}(Q)$, but decomposes under $\mathrm{SO}(Q)$ with highest weights $2\varpi_1$ and $2\varpi_2$.

Conformal Change of Metric Tensor

Replace g by $\widetilde{g} = e^{2f} g$ where $f \in \mathcal{C}(M)$.
Orthogonal group $\mathrm{O}(g) = \mathrm{O}(\widetilde{g})$ is unchanged.

Problem: Determine the change in the Weyl, traceless Ricci, and scalar parts of the Riemann curvature tensor.

Riemannian connection:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \Phi(X, Y) \quad \text{with } \Phi(X, Y) = \Phi(Y, X) \in \mathcal{T}(M)$$

Explicit formula:

$$\Phi(X, Y) = df(X)Y + df(Y)X - g(X, Y)Df \quad (Df = \mathrm{grad}_g\, f)$$

[Follows from $e^{-2f} X(e^{2f} g(Y, Z)) = g(\widetilde{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}_X Z)$ and cyclic permutation of X, Y, Z.]

Curvature Tensor:

$$\widetilde{R} = e^{2f} R - e^{2f} Q \owedge A \quad \text{with } A \in S^2(\mathbb{C}^n) \cong S^2(\mathbb{C}^n)^* \text{ via } Q$$

Explicit formula:

$$A = D^2 f - df \otimes df + \tfrac{1}{2}|Df|^2 Q \quad \text{(long calculation)}$$

where $D^2 f(X, Y) = XY(f) - (\nabla_X Y)(f)$ (Q-Hessian of f)

$|Df|^2 = Q(Df, Df)$

Conclusion: Under conformal change of metric $g \to e^{2f} g$

- Weyl part of the curvature is multiplied by e^{2f}.
- Traceless Ricci curvature is modified by adding term

  $(n-2)D^2 f - (n-2)(df \otimes df) + \frac{n-2}{n}\{\Delta f + |Df|^2\}Q$

  where $\Delta f = -\operatorname{tr}_Q(D^2 f)$ is the Q-Laplacian.
- Scalar curvature is multiplied by e^{2f} plus term

  $e^{2f}\{2(n-1)\Delta f - (n-1)(n-2)|Df|^2\}$