Multilinear Algebra and Tensor Symmetries

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Introduction to Math at Rutgers

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Vector Spaces and Duality

$\mathbb{F} =$ field: $\mathbb{R}$, $\mathbb{C}$, $\ldots$

$V =$ finite-dimensional vector space over $\mathbb{F}$ ($\dim V = n$)

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Duality pairing $V^* \times V \to \mathbb{F}$ (bilinear): $\langle \mathbf{v}^*, \mathbf{u} \rangle \overset{def}{=} \mathbf{v}^*(\mathbf{u})$
$V^* \longleftrightarrow 1 \times n$ row vectors using dual basis: $\langle \mathbf{v}_j^*, \mathbf{v}_i \rangle = \delta_{ij}$
When basis/dual basis fixed, then $\langle \mathbf{v}^*, \mathbf{u} \rangle = \mathbf{v}^* \mathbf{u}$.
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If $g \in \mathrm{GL}(V)$, then transpose ${}^t g \in \mathrm{GL}(V^*)$: $\langle {}^t g \mathbf{v}^*, \mathbf{u} \rangle \overset{def}{=} \langle \mathbf{v}^*, g\mathbf{u} \rangle$.
Calculate as $\mathbf{v}^* g$ (matrix product) when $\mathbf{v}^*$ = row vector.
Same as: Use (transposed matrix) × (column vector)

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Components of $\mathbf{u}$ relative to basis $\{\mathbf{v}_i\}$ are $x_i \overset{def}{=} \langle \mathbf{v}_i^*, \mathbf{u} \rangle$.

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Key Property: $\langle {}^t g^{-1}\mathbf{v}^*,\ g\mathbf{u}\rangle = \langle \mathbf{v}^*,\ g^{-1}g\mathbf{u}\rangle = \langle \mathbf{v}^*, \mathbf{u}\rangle$

Hence $f(\mathbf{v}^*, \mathbf{u}) \overset{def}{=} \langle \mathbf{v}^*, \mathbf{u}\rangle$ is a $\mathrm{GL}(V)$-invariant function on $V^* \times V$.

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Consequences:

- $\{{}^t g^{-1} \mathbf{v}_i^*\}$ is the dual basis to $\{g\mathbf{v}_i\}$
- Components of $g\mathbf{u}$ relative to basis $\{g\mathbf{v}_i\}$ are the same $\{x_i\}$.
- Components of ${}^t g^{-1} \mathbf{v}^*$ relative to basis $\{{}^t g^{-1} \mathbf{v}_i^*\}$ are the same $\{y^i\}$.
- $\langle \mathbf{v}^*, \mathbf{u} \rangle = \sum_i y_i x^i$ (contraction of covector and vector)

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- $\dim U \otimes V = \dim U \dim V$.
- Use dual bases to get bilinear map $\tau : U \times V \to U \otimes V$
  $\tau(\mathbf{u}, \mathbf{v}) = \sum_{i,j} x_i y_j\, \mathbf{u}_i \otimes \mathbf{v}_j \overset{def}{=} \mathbf{u} \otimes \mathbf{v}$ Kronecker product
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Universal Linearization Property: Let W be any vector space, and $\beta : U \times V \to W$ a bilinear map (linear in each variable)
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- $B : U \otimes V \to W$ is linear
- $B(u \otimes v) = \beta(u, v)$

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Theorem

If X, Y are vector spaces and $S \in \mathrm{Hom}(U, X)$, $T \in \mathrm{Hom}(V, Y)$, then there exists a unique $S \otimes T \in \mathrm{Hom}(U \otimes V, X \otimes Y)$ such that

$$(S \otimes T)(\mathbf{u} \otimes \mathbf{v}) = (S\mathbf{u}) \otimes (T\mathbf{v}) \quad \text{for } \mathbf{u} \in U, \mathbf{v} \in V.$$

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$$(\star) \qquad \mathrm{Hom}(U, X) \otimes \mathrm{Hom}(V, Y) \cong \mathrm{Hom}(U \otimes V, X \otimes Y)$$

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- $U \otimes V^* \cong \mathrm{Hom}(V, U)$:
  $\mathbf{u} \otimes \mathbf{v}^*$ gives transformation $T_{\mathbf{u},\mathbf{v}^*} : \mathbf{x} \mapsto \langle \mathbf{v}^*, \mathbf{x} \rangle \mathbf{u}$
  $U = \mathbb{F}^m$, $V = \mathbb{F}^n$: $\mathrm{Hom}(V, U) = m \times n$ matrices
  $T_{\mathbf{u},\mathbf{v}^*} = \mathbf{u}\mathbf{v}^*$ (column × row) rank one matrix

Iterated Tensor Products: Linearizing Multilinear Maps

Associativity of Tensor Product: U, V, W vector spaces

Define bilinear map $\tau : (U \otimes V) \times W \longrightarrow U \otimes (V \otimes W)$ by

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General Case: For vector spaces $V_1, \ldots, V_p, Z$ the tensor product $V_1 \otimes \cdots \otimes V_p$ has basis $\{\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_p}\}$ and linearizes p-multilinear maps $f : V_1 \times \cdots \times V_p \to Z$.

Mixed Tensors

Notation: $V^{\otimes p} = V \otimes \cdots \otimes V$ (p factors)

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Basis for $V^{\otimes(p,q)}$ from basis/dual basis for V and V^*:

$$\{\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_p} \otimes \mathbf{v}^*_{k_1} \otimes \cdots \otimes \mathbf{v}^*_{k_q}\} \quad (i_j, k_j = 1, \ldots, n = \dim V)$$

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Classic Tensor Notation: Components of $\mathbf{x} \in V^{\otimes(p,q)}$ are written

$$\mathbf{x}^{k_1 \cdots k_q}_{i_1 \cdots i_p} \overset{def}{=} \langle \mathbf{v}^*_{i_1} \otimes \cdots \otimes \mathbf{v}^*_{i_p} \otimes \mathbf{v}_{k_1} \otimes \cdots \otimes \mathbf{v}_{k_q}, \mathbf{x}\rangle$$

Identify $(V^*)^* = V$ as usual; $\{\mathbf{v}_i\}$ is dual basis to $\{\mathbf{v}^*_i\}$.

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Call $i_1, \ldots, i_p$ the covariant indices of $\mathbf{x}$ and
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Need ${}^t g^{-1}$ on $\mathbf{y}^*_i$ to have $\rho(gh) = \rho(g)\rho(h)$ for $g, h \in \mathrm{GL}(V)$.

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For each $1 \le r \le p$ and $1 \le s \le q$, define a linear map

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"Contraction is an operation of almost magical efficiency"

(*Tensor Analysis*, Encyclopedia Britannica, 14th ed.)

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Symmetrizer operator **Sym** : $V^{\otimes k} \to V^{\otimes k}$ (assume $\mathrm{char}(\mathbb{F}) = 0$)

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Define $S^k(V) = \mathbf{Sym}(V^{\otimes k})$ (*symmetric k-tensors*)
If $\mathbf{x} \in S^k(V)$ then the components $x_{i_1 \ldots i_k}$ are symmetric in the indices (unchanged under any transposition of indices), and conversely.

Universal Linearization Property of $S^k(V)$

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Let $f : V \times \cdots \times V \longrightarrow W$ (k factors) be any k-multilinear map that is symmetric in its arguments.

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Then $\mathbf{b} \in S^2(V^*) \iff b^{ij} = b^{ji} \iff B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x})$

Alternating Tensors

For $s \in \mathfrak{S}_k$ define $\operatorname{sgn}(s) = (-1)^m$ if s is a product of m transpositions. Then $\operatorname{sgn}(s)\operatorname{sgn}(t) = \operatorname{sgn}(st)$, $\operatorname{sgn}(1) = 1$.

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Notation: $\mathbf{Alt}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k$ (*exterior product* of vectors)

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- $\mathfrak{S}_k$ acts on the second tensor factor in $E_\lambda \otimes F_\lambda$

Schur–Weyl Duality

As a module for $\mathrm{GL}(V) \times \mathfrak{S}_k$, $\quad V^{\otimes k} \cong \bigoplus_\lambda E_\lambda \otimes F_\lambda$

- λ runs over all *partitions* of k with at most $\dim V$ parts
- E_λ is an irreducible representation of $\mathrm{GL}(V)$, and only occurs in the decomposition paired with F_λ
- F_λ is an irreducible representation of $\mathfrak{S}_k$, and only occurs in the decomposition paired with E_λ
  - trivial representation of $\mathfrak{S}_k \longleftrightarrow S^k(V)$
  - sgn representation of $\mathfrak{S}_k \longleftrightarrow \bigwedge^k(V)$
- $\mathrm{GL}(V)$ acts on the first tensor factor in $E_\lambda \otimes F_\lambda$
- $\mathfrak{S}_k$ acts on the second tensor factor in $E_\lambda \otimes F_\lambda$
- There are explicit operators (*Young symmetrizers*) that project on the subspaces E_λ and $E_\lambda \otimes F_\lambda$

Lots of interesting algebra, analysis, and combinatorics!