Integral Transforms

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Introduction to Math at Rutgers

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Fourier Transform

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G has a Haar measure (translation invariant)
$L^2(G)$ – square integrable complex valued functions on G
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  norm: $\|\phi\|_2 = \sqrt{(\phi, \phi)}$
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Solution: Use characters of G and Fourier transform

Fourier Transform on $\mathbb{Z}/n\mathbb{Z}$

Example 1 $G = \mathbb{Z}/n\mathbb{Z}$ (additive group of integers mod n)
$L^2(G) = \{\phi : \mathbb{Z} \to \mathbb{C} \ : \ \phi(k+n) = \phi(k) \text{ for all } k \in \mathbb{Z}\}$
inner product $(\phi, \psi) = \frac{1}{n} \sum_{k=0}^{n-1} \phi(k)\overline{\psi(k)}$

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Characters $e_p(k) = w^{kp}$ for $k, p \in \mathbb{Z}$ $\quad (w = \mathrm{e}^{2\pi \mathrm{i}/n}, \;\; w^n = 1)$

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- Eigenfunctions for translations $T_k e_p = w^{-kp} e_p$
- Orthogonality relations

  $$(e_p, e_q) = \begin{cases} 1 & \text{if } p - q \equiv 0 \mod (n) \\ 0 & \text{else} \end{cases}$$

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- Diagonalization $\psi = T_k\phi \quad \Rightarrow \quad \widehat{\psi}(p) = w^{-kp}\widehat{\phi}(p)$
- Fourier inversion $\phi = \sum_{p=0}^{n-1} \widehat{\phi}(p) e_p$
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Diagonalization of Translation Invariant Operators

Theorem

Let $G = \mathbb{Z}/n\mathbb{Z}$. *Let* C *be a translation invariant operator on* $L^2(G)$. *There is a function* F *on* $\widehat{G} \cong \mathbb{Z}/n\mathbb{Z}$ *such that*

$(\star) \qquad \widehat{C\phi}(p) = F(p)\widehat{\phi}(p)$ *for all* $\phi \in L^2(G)$ *and* $p \in \mathbb{Z}$.

Conversely, every function F *on* $\mathbb{Z}/n\mathbb{Z}$ *defines a translation invariant operator* C *on* $L^2(G)$ *by* $(\star)$ *(*$C =$ *convolution by* f*, where* $\widehat{f} = F$*).*

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Conversely, every function F on $\mathbb{Z}/n\mathbb{Z}$ defines a translation invariant operator C on $L^2(G)$ by $(\star)$ ($C =$ convolution by f, where $\widehat{f} = F$).

Proof.

Let $S = T_1$ (shift operator). Then S has n distinct eigenvalues $\lambda_p = w^{-p}$ for $p = 0, \ldots, n-1$ with eigenvectors e_p. Since C commutes with S, the function Ce_p is an eigenvector for S with eigenvalue w^{-p}. Hence $Ce_p = F(p)e_p$ for some scalar $F(p) \in \mathbb{C}$. The Fourier inversion formula now implies $(\star)$. ☐

Character Group and Duality

General Version of Fourier Transform

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$\widehat{G}$ – all characters of G:

$\psi : G \to \mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$ (continuous)
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Example

$G = \mathbb{Z}/n\mathbb{Z} \qquad \widehat{G} = \{e_p \ : \ p \in \mathbb{Z}/n\mathbb{Z}\} \cong G$
Choose basic character e_1. Then $e_p(k) = e_1(pk)$

Fourier Transform on $\mathbb{R}/\mathbb{Z}$

Example 2 $G = \mathbb{R}/\mathbb{Z}$ (additive group of real numbers modulo 1)

$L^2(\mathbb{R}/\mathbb{Z})$ $\phi : \mathbb{R} \to \mathbb{C},\ \phi(x+1) = \phi(x)$ (periodic, measurable)

$\int_0^1 |\phi(x)|^2 \, \mathrm{d}x < \infty$ (Lebesgue integral)

Inner product $(\phi, \psi) = \int_0^1 \phi(x)\overline{\psi(x)} \, \mathrm{d}x$

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Bounded Translation Invariant Operators

Linear operator C on $L^2(\mathbb{R}/\mathbb{Z})$ is bounded if $\|C\phi\|_2 \leq M\|\phi\|_2$

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Then there is a bounded function F on $\mathbb{Z}$ such that
$(\star) \qquad \widehat{C\phi}(p) = F(p)\widehat{\phi}(p)$ *for all* $\phi \in L^2(\mathbb{R}/\mathbb{Z})$.
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$CS = SC \Rightarrow Ce_p = F(p)e_p$ with $F(p) \in \mathbb{C}$. Then C bounded $\Rightarrow$ $\|F\|_\infty := \sup_p |F(p)| < \infty$. Hence

$$C\phi = \sum_{p\in\mathbb{Z}} \widehat{\phi}(p) Ce_p = \sum_{p\in\mathbb{Z}} \widehat{\phi}(p) F(p) e_p \qquad \square$$

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- $(D\phi, \psi) = (\phi, D\psi)$ for $\phi, \psi \in C^\infty(\mathbb{R}/\mathbb{Z})$ (integrate by parts)
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- $\phi \in C^\infty(\mathbb{R}/\mathbb{Z}) \iff \widehat{\phi}$ is rapidly decreasing:

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Conversely, every such function F on $\mathbb{Z}$ defines a continuous translation invariant operator C on $C^\infty(\mathbb{R}/\mathbb{Z})$ by $(\star)$.

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Example 3 $G = \mathbb{R}$ (additive group of real numbers)

$L^2(\mathbb{R})$ $\phi : \mathbb{R} \to \mathbb{C}$, (measurable) $\int_{-\infty}^{\infty} |\phi(x)|^2 \, \mathrm{d}x < \infty$

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- Fix basic character e_1. Then $e_\xi(x) = e_1(x\xi)$
- $e_\xi(x)e_\tau(x) = e_{\xi+\tau}(x)$, so $\widehat{\mathbb{R}} \cong \mathbb{R}$ under $e_\xi \leftrightarrow \xi$
- $\mathbb{R}$ not compact $\Rightarrow e_\xi \notin L^2(\mathbb{R})$ (plane wave, frequency ξ)

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Example 3 $G = \mathbb{R}$ (additive group of real numbers)
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- Fourier transform extends to isometry $L^2(\mathbb{R}) \to L^2(\widehat{\mathbb{R}})$
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- Bounded translation invariant operator C on $L^2(\mathbb{R}) \longleftrightarrow$ multiplication by bounded measurable function F on $\widehat{\mathbb{R}}$

Tempered Fourier Analysis on $\mathbb{R}$

$\mathcal{S}(\mathbb{R}) =$ rapidly decreasing C^∞ functions on $\mathbb{R}$:

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Theorem

Let C be a continuous translation invariant operator on $\mathcal{S}(\mathbb{R})$. Then there is a C^∞ function F on $\mathbb{R}$ with all derivatives of polynomial growth at ∞ such that

$(\star)$ $\qquad \widehat{C\phi}(\xi) = F(\xi)\widehat{\phi}(\xi)$ *for all* $\phi \in \mathcal{S}(\mathbb{R})$.

Conversely, every such function F on $\mathbb{R}$ defines a continuous translation invariant operator C on $\mathcal{S}(\mathbb{R})$ by $(\star)$.

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$\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ – locally compact group under multiplication

- Invariant integral $\displaystyle\int_{-\infty}^{\infty} f(x)\,\frac{\mathrm{d}x}{|x|}$
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  for $f \in L^1(\mathbb{R}, \frac{\mathrm{d}x}{|x|})$
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Log Trick: Use group homomorphism $x \mapsto (\log|x|, \mathrm{sgn}(x))$ to turn Fourier-Mellin transform into Fourier transform on $\mathbb{R} \times (\mathbb{Z}/2\mathbb{Z})$.

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Use integral transform that turns Δ into $(\partial/\partial p)^2$ on even functions of $p \in \mathbb{R}$ with parameter $\omega \in \mathbb{S}^{n-1}$ (no singularity). Then diagonalize by one-dimensional Fourier transform.

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Radon Transform

$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \;:\; x \cdot x = 1\}$ unit sphere

$x \cdot y = x_1 y_1 + \cdots + x_n y_n$ inner product on $\mathbb{R}^n$

Hyperplane with oriented normal $\omega \in \mathbb{S}^{n-1}$ and height $p \in \mathbb{R}$:

$H(\omega, p) = \{x \in \mathbb{R}^n \;:\; x \cdot \omega = p\}$

Write $\xi = H(\omega, p) \cong \mathbb{R}^{n-1}$

$\mathrm{d}m = (n-1)$-dimensional Lebesgue measure on ξ

$\mathbb{P}^n =$ set of all hyperplanes ξ in $\mathbb{R}^n$ (smooth n-dim manifold)

two-sheeted covering $\mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{P}^n$ (no singularities)

$(\omega, p) \mapsto H(\omega, p) = H(-\omega, -p)$

Radon transform of $f \in \mathcal{S}(\mathbb{R}^n)$:

$F(\omega, p) = \int_{x \cdot \omega = p} f(x)\, \mathrm{d}m(x)$

- Integral converges since $f|_{H(\omega,p)}$ is rapidly decreasing
- $F(\xi) = F(\omega, p)$ defined on $\mathbb{P}^n$ since $F(\omega, \xi) = F(-\omega, -\xi)$
- Fourier transform $\widehat{f}(r\omega) = \int_{-\infty}^{\infty} F(\omega, p) \mathrm{e}^{-2\pi \mathrm{i} r p}\, \mathrm{d}p$
- Radon transform of $\Delta f(x)$ is $(\partial/\partial p)^2\, F(\omega, p)$

Inverse Radon Transform

For $x \in \mathbb{R}^n$

$$\begin{aligned} K(x) &= \text{all hyperplanes } \xi \text{ containing } x \\ &= \{(\omega, p) \;:\; x \cdot \omega = p\} \cong \mathbb{S}^{n-1} / \pm 1 \end{aligned}$$

Let $\mathrm{d}\mu =$ invariant measure on $K(x)$ (total mass 1)

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For $F \in \mathcal{S}(\mathbb{P}^n)$ define dual Radon transform

$$\widetilde{F}(x) = \int_{\xi \in K(x)} F(\xi)\,\mathrm{d}\mu(\xi) = \int_{\omega \in \mathbb{S}^{n-1}} F(\omega, \omega \cdot x)\,\mathrm{d}\omega$$

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Radon Inversion Formula

If $f \in \mathcal{S}(\mathbb{R}^n)$ and F = Radon transform of f, then

$$f(x) = c\,(-\Delta)^{(n-1)/2}\,\widetilde{F}(x) \qquad (c = \text{normalizing constant})$$

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odd dimensions: Inversion formula is local - differential operator applied to $\widetilde{F}(x)$

even dimensions: Inversion formula is non-local - square root of differential operator (Hilbert transform) applied to $\widetilde{F}(x)$

Further Reading

The Wikipedia articles on Fourier Analysis, p-adic Numbers, and Radon Transform are good starting points. Here are some books:

- Fourier analysis on locally compact abelian groups:
  W. Rudin, **Fourier Analysis on Groups**, Wiley (1962)
  G. Folland, **A Course in Abstract Harmonic Analysis**, CRC Press (1995)
- Finite Fourier transform:
  A. Terras, **Fourier Analysis on Finite Groups and applications**, Cambridge (1999)
- Fourier analysis on $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}$:
  G. Folland, **Real Analysis: Modern Techniques and Their Applications**, Wiley (1999)
- Radon Transform:
  S. Helgason, **Groups and Geometric Analysis**, Academic Press (1984)