Multilinear Algebra and Tensor Symmetries

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Vector Spaces and Duality

$\mathbb{F}$ = field: $\mathbb{R}$, $\mathbb{C}$, ... (characteristic zero later for symmetry types)
V = vector space over $\mathbb{F}$ (assume $\dim V < \infty$)
$V \leftrightarrow$ column vectors when basis $\{\mathbf{v}_i\}$ fixed
$\mathrm{GL}(V)$ = group of invertible linear transformations $g : V \to V$
V^* = dual space of linear functions $\mathbf{v}^* : V \to \mathbb{F}$
$V^* \leftrightarrow$ row vectors when dual basis $\{\mathbf{v}_j^*\}$ fixed.

Duality pairing $V^* \times V \to \mathbb{F}$ (bilinear): $\langle \mathbf{v}^*, \mathbf{v} \rangle = \mathbf{v}^*(\mathbf{v})$
When basis/dual basis fixed, then $\langle \mathbf{v}^*, \mathbf{v} \rangle = \mathbf{v}^*\mathbf{v}$
(row vector × column vector = scalar)

If $g \in \mathrm{GL}(V)$, then $g^t \in \mathrm{GL}(V^*)$: $\langle g^t\mathbf{v}^*, \mathbf{v} \rangle \overset{def}{=} \langle \mathbf{v}^*, g\mathbf{v} \rangle$.
(calculate as $\mathbf{v}^* g$ (matrix product) when $\mathbf{v}^*$ row vector)

Change of Basis (two points of view):

- Express $\mathbf{v}$ in terms of basis $\{\mathbf{v}_i\}$ or basis $\{g\mathbf{v}_i\}$ $(g \in \mathrm{GL}(V))$.
- Change vector $\mathbf{v}$ to $g\mathbf{v}$ (orbit of $\mathbf{v}$ under action of $\mathrm{GL}(V)$).

Relation: $\mathbf{v} = \sum_i \langle \mathbf{v}_i^*, \mathbf{v} \rangle \mathbf{v}_i$ (basis-dual basis expansion formula)

Components of $\mathbf{v}$ relative to basis $\{\mathbf{v}_i\}$ are $x_i = \langle \mathbf{v}_i^*, \mathbf{v} \rangle$.
But $\langle (g^t)^{-1}\mathbf{v}^*, g\mathbf{v} \rangle = \langle \mathbf{v}^*, \mathbf{v} \rangle = f(v^*, v)$
(f is a $\mathrm{GL}(V)$-invariant function on $V^* \times V$)
Hence $\{(g^t)^{-1}\mathbf{v}_i^*\}$ is the dual basis to $\{g\mathbf{v}_i\}$
Expand

$$g\mathbf{v} = \sum_i \langle \mathbf{v}_i^*, \mathbf{v} \rangle g\mathbf{v}_i = \sum_i \langle (g^t)^{-1}\mathbf{v}_i^*, g\mathbf{v} \rangle g\mathbf{v}_i.$$

Thus the components of $g\mathbf{v}$ w.r.t. basis $\{g\mathbf{v}_i\}$ are also x_i.

Tensor Products and Universal Mapping Property

Given: U, V, W (vector spaces over $\mathbb{F}$) and a bilinear map $\beta : U \times V \to W$

Find:

- a vector space $U \otimes V$
- a bilinear mapping $\tau : \mathbf{u}, \mathbf{v} \mapsto \mathbf{u} \otimes \mathbf{v}$ from $U \times V \to U \otimes V$
- a linear map $B : U \otimes V \to W$

such that $\beta(u, v) = B\tau(u, v)$.
Want $U \otimes V$ to be universal (not depend on W).
Existence: Take basis/dual basis $\{\mathbf{u}_i\}$, $\{\mathbf{u}_i^*\}$ for U, U^* and basis/dual basis $\{\mathbf{v}_i\}$, $\{\mathbf{v}_i^*\}$ for V, V^*
Let $U \otimes V$ be vector space with basis $\{\mathbf{u}_i \otimes \mathbf{v}_j\}$
Then $\dim U \otimes V = \dim U \dim V$.
Define $\tau(\mathbf{u}, \mathbf{v}) = \sum_{i,j} \langle \mathbf{u}_i^*, \mathbf{u} \rangle \langle \mathbf{v}_j^*, \mathbf{v} \rangle \, \mathbf{u}_i \otimes \mathbf{v}_j$
$B(\mathbf{u}_i \otimes \mathbf{v}_j) = \beta(\mathbf{u}_i, \mathbf{v}_j)$ (extend by linearity)

Uniqueness: General property of solutions to universal mapping problems

Write $\tau(\mathbf{u},\mathbf{v}) = \mathbf{u} \otimes \mathbf{v}$ (spanning set for $U \otimes V$)

If $\mathbf{u}$ has components $\{x_i\}$ and $\mathbf{v}$ has components $\{y_j\}$, then $\mathbf{u} \otimes \mathbf{v}$ has components $\{x_i y_j\}$ (Kronecker product of $\mathbf{u}$ and $\mathbf{v}$).

$\mathbf{x} \in U \otimes V$ has expansion $\mathbf{x} = \sum_{i,j} x_{ij} \mathbf{u}_i \otimes \mathbf{v}_j$ with components

$$x_{ij} = \langle \mathbf{u}_i^* \otimes \mathbf{v}_j^*, \mathbf{x} \rangle.$$

Functoriality: Let $\mathrm{Hom}(U, V)$ = all linear maps $T : U \to V$.

If X, Y vector spaces, and $S \in \mathrm{Hom}(U, X)$, $T \in \mathrm{Hom}(V, Y)$, then we have unique map $S \otimes T \in \mathrm{Hom}(U \otimes V, X \otimes Y)$ with

$$(S \otimes T)(\mathbf{u} \otimes \mathbf{v}) = (S\mathbf{u}) \otimes (T\mathbf{v})$$

$$\mathrm{Hom}(U, X) \otimes \mathrm{Hom}(V, Y) \cong \mathrm{Hom}(U \otimes V, X \otimes Y)$$

(in matrix form: arrays with four indices)

Special Cases

1) $X = Y = \mathbb{F}$:

$\mathbb{F} \otimes \mathbb{F} \cong \mathbb{F}$ (canonical basis $1 \in \mathbb{F}$) and $U^* \otimes V^* \cong (U \otimes V)^*$

2) $U^* \otimes V \cong \mathrm{Hom}(U, V)$:

$\mathbf{u}^* \otimes \mathbf{v} \longleftrightarrow T_{\mathbf{u}^*,\mathbf{v}}(u) = \langle \mathbf{u}^*, \mathbf{u} \rangle \mathbf{v}$ (as matrix: $T_{\mathbf{u}^*,\mathbf{v}} = \mathbf{u}^* \mathbf{v}$)

Iterated Tensor Products: Linearizing Multilinear Maps

Associativity of Tensor Product: U, V, W vector spaces
Define bilinear map $\tau : (U \otimes V) \times W \to U \otimes (V \otimes W)$ by

$$\tau(\mathbf{u} \otimes \mathbf{v}, \mathbf{w}) = \mathbf{u} \otimes (\mathbf{v} \otimes \mathbf{w}).$$

Universal property gives isomorphism

$$T : (U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W)$$

On basis: $T(\mathbf{u}_i \otimes \mathbf{v}_j) \otimes \mathbf{w}_k = \mathbf{u}_i \otimes (\mathbf{v}_j \otimes \mathbf{w}_k)$
Write $(U \otimes V) \otimes W = U \otimes (V \otimes W) = U \otimes V \otimes W$ (note order)
$\mathbf{x} = \sum_{i,j,k} x_{ijk} \mathbf{u}_i \otimes \mathbf{v}_j \otimes \mathbf{w}_k$ components $x_{ijk} = \langle \mathbf{u}_i^* \otimes \mathbf{v}_j^* \otimes \mathbf{w}_k^*, \mathbf{x} \rangle$
Universal Property: If $f : U \times V \times W \to Z$ is a trilinear map, then there exists unique linear map $F : U \otimes V \otimes W \to Z$ with

$$f(\mathbf{u}, \mathbf{v}, \mathbf{w}) = F(\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w})$$

General Case: For vector spaces $V_1, \ldots, V_p$, the tensor product $V_1 \otimes \cdots \otimes V_p$ has basis $\{\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_p}\}$ and linearizes p-multilinear maps $f : V_1 \times \cdots \times V_p \to Z$

Mixed Tensors

Notation: $V^{\otimes p} = V \otimes \cdots \otimes V$ (p factors)

$V^{\otimes(p,q)} = V^{\otimes p} \otimes (V^*)^{\otimes q}$ *mixed tensors* of type (p, q)

Basis for $V^{\otimes(p,q)}$ from basis/dual basis for V and V^*:

$$\{\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_p} \otimes \mathbf{v}^*_{k_1} \otimes \cdots \otimes \mathbf{v}^*_{k_q}\} \qquad (i_j, k_j = 1, \ldots, n = \dim V)$$

Classic Tensor Notation: Components of $\mathbf{x} \in V^{\otimes(p,q)}$ are written

$$\mathbf{x}^{k_1 \cdots k_q}_{i_1 \cdots i_p} = \langle \mathbf{v}^*_{i_1} \otimes \cdots \otimes \mathbf{v}^*_{i_p} \otimes \mathbf{v}_{k_1} \otimes \cdots \otimes \mathbf{v}_{k_q}, \mathbf{x} \rangle$$

Note: Identify $(V^*)^* = V$ as usual; $\{\mathbf{v}_i\}$ is dual basis to $\{\mathbf{v}^*_i\}$. Call $i_1, \ldots, i_p$ the *covariant indices* of $\mathbf{x}$, and $k_1, \ldots, k_q$ the *contravariant indices* of $\mathbf{x}$.

Action of $\mathrm{GL}(V)$ on mixed tensors:

Define a group homomorphism $\rho : \mathrm{GL}(V) \to \mathrm{GL}(V^{\otimes(p,q)})$ by

$$\rho(g)(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_p \otimes \mathbf{u}^*_1 \otimes \cdots \otimes \mathbf{u}^*_q) = (g\mathbf{u}_1) \otimes \cdots \otimes (g\mathbf{u}_p) \otimes (g^t)^{-1}\mathbf{u}^*_1 \otimes \cdots \otimes (g^t)^{-1}\mathbf{u}^*_q$$

(for any $\mathbf{u}_i \in V$ and $\mathbf{u}^*_j \in V^*$).

Note: Need $(g^t)^{-1}$ to have $\rho(gh) = \rho(g)\rho(h)$ for $g, h \in \mathrm{GL}(V)$. Call ρ a representation of $\mathrm{GL}(V)$.

Contractions

For each $1 \leq r \leq p$ and $1 \leq s \leq q$, define a linear map

$C_r^s : V^{\otimes(p,q)} \longrightarrow V^{\otimes(p-1,q-1)}$ (r,s) contraction

by taking components of $C_r^s\mathbf{x}$ as

$$\sum_{1 \leq j \leq n} \mathbf{x}^{k_1 \cdots k_{s-1} j k_{s+1} \cdots k_{q-1}}_{i_1 \cdots i_{r-1} j i_{r+1} \cdots i_{p-1}} \qquad (\text{set } i_r = k_s = j \text{ and sum on } j)$$

Theorem

Contractions commute with the action of $\mathrm{GL}(V)$ *on mixed tensors.*

Examples: 1) $V^{\otimes(1,1)} = V \otimes V^* \cong \mathrm{End}(V)$, and
$C_1^1 : V^{\otimes(1,1)} \longrightarrow V^{\otimes(0,0)} = \mathbb{F}$ by $C_1^1\mathbf{x} = \sum_j x_j^j = \mathrm{tr}(\mathbf{x})$ (trace of $\mathbf{x}$)
If $\mathbf{x} = \mathbf{u} \otimes \mathbf{u}^*$ then $C_1^1(\mathbf{x}) = \langle \mathbf{u}^*, \mathbf{u} \rangle$.

2) Take $\mathbb{F} = \mathbb{R}$ and $V =$ tangent space at a point of a Riemannian manifold. Then the curvature tensor $R \in V^{\otimes(2,2)}$.

contraction of R gives Ricci curvature $\mathrm{Ric} \in V^{\otimes(1,1)}$

contraction (trace) of Ric gives scalar curvature in $V^{\otimes(0,0)} = \mathbb{R}$

"Contraction is an operation of almost magical efficiency"

(*Tensor Analysis*, Encyclopedia Britannica, 14th ed.)

Symmetry Properties of Tensors

$\mathfrak{S}_k$ = symmetric group (permutations of $\{1, \ldots, k\}$

Action of $s \in \mathfrak{S}_k$ on $V^{\otimes k}$ (permuting the positions of the factors):

$$\sigma(s)(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = \mathbf{u}_{s^{-1}(1)} \otimes \cdots \otimes \mathbf{u}_{s^{-1}(k)}$$

(vector in ith position → vector in position $s(i)$)

Properties:

- $\sigma : \mathfrak{S}_k \to \mathrm{GL}(V^{\otimes k})$ is a *group representation*:
  $\sigma(st) = \sigma(s)\sigma(t)$ and $\sigma_k(1) = I$.
- Representation σ of $\mathfrak{S}_k$ on $V^{\otimes k}$ commutes with representation ρ of $\mathrm{GL}(V)$.

Theorem (Schur duality)

(1) Any linear transformation on $V^{\otimes k}$ that commutes with $\rho(g)$ for all $g \in \mathrm{GL}(V)$ is a linear combination of $\{\sigma(s) : s \in \mathfrak{S}_k\}$.
(2) Any linear transformation on $V^{\otimes k}$ that commutes with $\sigma(s)$ for all $s \in \mathfrak{S}_k$ is a linear combination of $\{\rho(g) : g \in \mathrm{GL}(V)\}$.

Symmetric Tensors

Symmetrizer operator **Sym** : $V^{\otimes k} \to V^{\otimes k}$ (assume char$\mathbb{F} = 0$)

$$\mathbf{Sym}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \sigma(s)(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k)$$

Properties:

- $\mathbf{Sym}^2 = \mathbf{Sym}$ (projection operator)

Proof:

$$\mathbf{Sym}^2 = \frac{1}{(k!)^2} \sum_{s,t \in \mathfrak{S}_k} \sigma(s)\sigma(t) = \frac{1}{(k!)^2} \sum_{s,t \in \mathfrak{S}_k} \sigma(st)$$
$$= \frac{1}{k!} \sum_{r \in \mathfrak{S}_k} \sigma(r) = \mathbf{Sym}$$

- $\sigma(t)\,\mathbf{Sym} = \mathbf{Sym}$ for all $t \in \mathfrak{S}_k$

Proof:

$$\sigma(t)\,\mathbf{Sym} = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} \sigma(t)\sigma(s) = \frac{1}{k!} \sum_{r \in \mathfrak{S}_k} \sigma(r) = \mathbf{Sym}$$

Define $S^k(V) = \mathbf{Sym}(V^{\otimes k})$ (*symmetric k-tensors*)

If $\mathbf{x} \in S^k(V)$ then the components $x_{i_1 \dots i_k}$ are symmetric in the indices (unchanged under any transposition of indices), and conversely.

Universal Mapping Property of $S^k(V)$

Let $f : V \times \cdots \times V \to W$ (k factors) be any k-multilinear map that is *symmetric* in its arguments.

There is a unique *linear* map $F : S^k(V) \to W$ such that

$$F(\mathbf{Sym}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k)) = f(\mathbf{u}_1, \ldots, \mathbf{u}_k).$$

Proof: There exists linear map $F : V^{\otimes k} \to W$ with

$$F(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = f(\mathbf{u}_1, \ldots, \mathbf{u}_k)$$

Then $F \circ \mathbf{Sym} = F$ since f is symmetric, so $F \circ (I - \mathbf{Sym}) = 0$.

Example

$$(V^*)^{\otimes 2} \longleftrightarrow \text{bilinear forms on } V$$

$$\mathbf{b} = \sum_{i,j} b^{ij} \mathbf{v}_i^* \otimes \mathbf{v}_j^* \longleftrightarrow B(\mathbf{x}, \mathbf{y}) = \sum_{ij} b^{ij} x_i y_j$$

(written as $b^{ij} x_i y_j$ in *Einstein summation notation*)

Then $\mathbf{b} \in S^2(V^*) \iff b^{ij} = b^{ji} \iff B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x})$

Alternating Tensors

For $s \in \mathfrak{S}_k$ define *sign* $\mathrm{sgn}(s) = +1$ if s is a product of an *even* number of transposition, and $\mathrm{sgn}(s) = -1$ if s is the product of an *odd* number of transpositions (well defined)

Alternation operator $\mathbf{Alt} : V^{\otimes k} \to V^{\otimes k}$

$$\mathbf{Alt}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = \tfrac{1}{k!} \textstyle\sum_{s \in \mathfrak{S}_k} \mathrm{sgn}(s)\sigma(s)(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k)$$

Properties:

- $\mathbf{Alt}^2 = \mathbf{Alt}$ (projection operator)

Proof: Since $\mathrm{sgn}(s)\,\mathrm{sgn}(t) = \mathrm{sgn}(st)$, we have

$$\begin{aligned}\mathbf{Alt}^2 &= \tfrac{1}{(k!)^2} \textstyle\sum_{s,t \in \mathfrak{S}_k} \mathrm{sgn}(s)\,\mathrm{sgn}(t)\sigma(s)\sigma(t) \\ &= \tfrac{1}{(k!)^2} \textstyle\sum_{s,t \in \mathfrak{S}_k} \mathrm{sgn}(st)\sigma(st) = \tfrac{1}{k!} \textstyle\sum_{r \in \mathfrak{S}_k} \mathrm{sgn}(r)\sigma(r) \\ &= \mathbf{Alt}\end{aligned}$$

- $\sigma(t)\,\mathbf{Alt} = \mathrm{sgn}(t)\,\mathbf{Alt}$ for all $t \in \mathfrak{S}_k$

Proof:

$$\begin{aligned}\sigma(t)\,\mathbf{Alt} &= \tfrac{1}{k!} \textstyle\sum_{s \in \mathfrak{S}_k} \mathrm{sgn}(s)\sigma(t)\sigma(s) = \tfrac{1}{k!} \textstyle\sum_{r \in \mathfrak{S}_k} \mathrm{sgn}(t)^{-1}\sigma(r) \\ &= \mathrm{sgn}(t)\,\mathbf{Alt}\end{aligned}$$

Define $\bigwedge^k(V) = \mathbf{Alt}(V^{\otimes k})$ (*alternating k-tensors*)

If $\mathbf{x} \in \bigwedge^k(V)$ then the components $x_{i_1 \ldots i_k}$ are skew-symmetric in the indices (change sign under any transposition), and conversely.

Universal Mapping Property of $\bigwedge^k(V)$

Let $f : V \times \cdots \times V \to W$ (k factors) be any k-multilinear map that is *alternating* in its arguments (changes sign when two arguments are permuted).

There is a unique *linear* map $F : \bigwedge^k(V) \to W$ such that

$$F(\mathbf{Alt}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k)) = f(\mathbf{u}_1, \ldots, \mathbf{u}_k).$$

Proof: There exists linear map $F : V^{\otimes k} \to W$ with

$$F(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = f(\mathbf{u}_1, \ldots, \mathbf{u}_k)$$

Then $F \circ \mathbf{Alt} = F$ since f is alternating, so $F \circ (I - \mathbf{Alt}) = 0$.

Notation: $\mathbf{Alt}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k$ (*exterior product* of vectors)

Comparisons between Symmetric and Alternating Tensors

$S^k(V)$ $\quad$ $(\dim V = n)$

- basis $\{\mathbf{Sym}(\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_k}\}$, with all indices $1 \le i_1 \le i_2 \le \cdots \le i_k \le n$
- $\dim S^k(V) = \binom{n+k-1}{k} = \#\{k \text{ balls in } n \text{ boxes}\}$
- $S^k(V) \cong \mathcal{P}^k(V^*) =$ homogeneous polynomials of degree k on V^* (nonzero space for all k):
  $\mathbf{Sym}(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_k) \longleftrightarrow$ monomial $\langle \mathbf{u}^*, \mathbf{u}_1 \rangle \cdots \langle \mathbf{u}^*, \mathbf{u}_k \rangle$
- $\mathfrak{S}_k$ acts by identity on $S^k(V)$
- $S^k(V)$ is invariant under $\mathrm{GL}(V)$, and contains no proper subspace that is invariant under $\mathrm{GL}(V)$ (irreducible representation of $\mathrm{GL}(V)$)
- Quantum mechanics: $S^k(V)$ describes systems of k *bosons*

$\bigwedge^k(V)$ $\quad$ ($\dim V = n$)

- basis $\{\mathbf{v}_{i_1} \wedge \cdots \wedge \mathbf{v}_{i_k}\}$, with all indices $1 \le i_1 < i_2 < \cdots < i_k \le n$
- $\dim \bigwedge^k(V) = \binom{n}{k} = \#\{\text{subsets of size } k \text{ in } n\text{-element set}\}$
- $\bigwedge^k(V) \cong$ homogeneous alternating functions of degree k on V^* (zero if $k > n$)
- $\mathfrak{S}_k$ acts by sgn on $\bigwedge^k(V)$
- $\bigwedge^k(V)$ is invariant under GL(V), and contains no proper subspace that is invariant under GL(V) (irreducible representation of GL(V))
- $\dim \bigwedge^n V = 1$. Fix basis vector $\mathbf{u}$. Then $\rho(g)\mathbf{u} = \det(g)\mathbf{u}$ for $g \in \mathrm{GL}(V)$.
- Geometry: $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k \ne 0 \Longleftrightarrow \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ are linearly independent (span a k-plane).
- Quantum Mechanics: $\bigwedge^k(V)$ is used to describe systems of k *fermions* (Pauli exclusion principle)

Other Symmetry Types of Tensors

The group $\mathrm{GL}(V) \times \mathfrak{S}_k$ acts on $V^{\otimes k}$.
Program: Decompose $V^{\otimes k}$ into a direct sum of subspaces that are irreducible under the action of $\mathrm{GL}(V) \times \mathfrak{S}_k$
We already have two such spaces, namely $S^k(V)$ and $\bigwedge^k(V)$.

Example When $k = 2$, then $V^{\otimes 2} = S^2(V) \oplus \bigwedge^2(V)$
(Every bilinear form is the sum of a symmetric form and a skew-symmetric form)

For $k > 2$? $\dim V^{\otimes k} = n^k >> \dim S^2(V) + \dim \bigwedge^2(V)$

Schur–Weyl Duality

As a module for $\mathrm{GL}(V) \times \mathfrak{S}_k$, $\quad V^{\otimes k} \cong \bigoplus_\lambda E_\lambda \otimes F_\lambda$

- λ runs over all *partitions* of k with at most $\dim V$ parts
- E_λ is an irreducible representation of $\mathrm{GL}(V)$, and only occurs in the decomposition paired with F_λ
- F_λ is an irreducible representation of $\mathfrak{S}_k$, and only occurs in the decomposition paired with E_λ

  trivial representation of $\mathfrak{S}_k \longleftrightarrow S^k(V)$
  sgn representation of $\mathfrak{S}_k \longleftrightarrow \bigwedge^k(V)$
- $\mathrm{GL}(V)$ acts on the first tensor factor in $E_\lambda \otimes F_\lambda$
- $\mathfrak{S}_k$ acts on the second tensor factor in $E_\lambda \otimes F_\lambda$
- There are explicit operators (*Young symmetrizers*) that project on the subspaces E_λ and $E_\lambda \otimes F_\lambda$

Lots of interesting algebra, analysis, and combinatorics!