Preface

Symmetry, in the title of this book, should be understood as the geometry of Lie (and algebraic) group actions. The basic algebraic and analytic tools in the study of symmetry are representation and invariant theory. These three threads are precisely the topics of this book. The earlier chapters can be studied at several levels. An advanced undergraduate or beginning graduate student can learn the theory for the classical groups using only linear algebra, elementary abstract algebra, and advanced calculus, with further exploration of the key examples and concepts in the numerous exercises following each section. The more sophisticated reader can progress through the first ten chapters with occasional forward references to Chapter 11 for general results about algebraic groups. This allows great flexibility in the use of this book as a course text. The authors have used various chapters in a variety of courses; we suggest ways in which courses can be based on the book later in this preface. Finally, we have taken care to make the main theorems and applications meaningful for the reader who wishes to use the book as a reference to this vast subject.

The authors are gratified that their earlier text, *Representations and Invariants of the Classical Groups* [56], was well received. The present book has the same aim: an entry into the powerful techniques of Lie and algebraic group theory. The parts of the previous book that have withstood the authors' many revisions as they lectured from its material have been retained; these parts appear here after substantial rewriting and reorganization. The first four chapters are, in large part, newly written and offer a more direct and elementary approach to the subject. Several of the later parts of the book are also new. While we continue to look upon the classical groups as both fundamental in their own right and as important examples for the general theory, the results are now stated and proved in their natural generality. These changes justify the more accurate new title for the present book.

We have taken special care to make the book readable at many levels of detail. A reader desiring only the statement of a pertinent result can find it through the table of contents and index, and then read and study it through the examples of its use that are generally given. A more serious reader wishing to delve into a proof of the result can read in detail a more computational proof that uses special properties of the classical groups, or, perhaps in a second reading, the proof in the general case (with occasional forward references to results from later chapters). Usually, there is a third possibility of a proof using analytic methods. Some material in the earlier book, although important in its own right, has been eliminated or replaced. There are new proofs of some of the key results of the theory such as the theorem of the highest weight, the theorem on complete reducibility, the duality theorem, and the Weyl character formula. We hope that our new presentation will make these fundamental tools more accessible.

The last two chapters of the book develop, via a basic introduction to complex algebraic groups, what has come to be called *geometric invariant theory*. This includes the notion of quotient space and the representation-theoretic analysis of the regular functions on a space with an algebraic group action. A full description of the material covered in the book is given later in the preface.

When our earlier text appeared there were few other introductions to the area. The most prominent included the fundamental text of Hermann Weyl, *The Classical Groups: Their Invariants and Representations* [164] and Chevalley's *The Theory of Lie groups I* [33], together with the more recent text *Lie Algebras* by Humphreys [76]. These remarkable volumes should be on the bookshelf of any serious student of the subject. In the interim, several other texts have appeared that cover, for the most part, the material in Chevalley's classic with extensions of his analytic group theory to Lie group theory and that also incorporate much of the material in Humphrey's text. Two books with a more substantial overlap but philosophically very different from ours are those by Knapp [86] and Procesi [123]. There is much for a student to learn from both of these books, which give an exposition of Weyl's methods in invariant theory that is different in emphasis from our book. We have developed the combinatorial aspects of the subject as consequences of the representations and invariants of the classical groups. In Hermann Weyl (and the book of Procesi) the opposite route is followed: the representations and invariants of the classical groups rest on a combinatorial determination of the representations of the symmetric group. Knapp's book is more oriented toward Lie group theory.

Organization

The logical organization of the book is illustrated in the chapter and section dependency chart at the end of the preface. A chapter or section listed in the chart depends on the chapters to which it is connected by a horizontal or rising line. This chart has a central spine; to the right are the more geometric aspects of the subject and on the left the more algebraic aspects. There are several intermediate terminal nodes in this chart (such as Sections 5.6 and 5.7, Chapter 6, and Chapters 9–10) that can serve as goals for courses or self study.

Chapter 1 gives an elementary approach to the classical groups, viewed either as Lie groups or algebraic groups, without using any deep results from differentiable manifold theory or algebraic geometry. Chapter 2 develops the basic structure of the classical groups and their Lie algebras, taking advantage of the defining representations. The complete reducibility of representations of $\mathfrak{sl}(2,\mathbb{C})$ is established by a variant of Cartan's original proof. The key Lie algebra results (Cartan subalgebras and root space decomposition) are then extended to arbitrary semisimple Lie algebras.

Chapter 3 is devoted to Cartan's highest-weight theory and the Weyl group. We give a new algebraic proof of complete reducibility for semisimple Lie algebras following an argument of V. Kac; the only tools needed are the complete reducibility for $\mathfrak{sl}(2,\mathbb{C})$ and the Casimir operator. The general treatment of associative algebras and their representations occurs in Chapter 4, where the key result is the general duality theorem for locally regular representations of a reductive algebraic group. The unifying role of the duality theorem is even more prominent throughout the book than it was in our previous book.

The machinery of Chapters 1–4 is then applied in Chapter 5 to obtain the principal results in classical representations and invariant theory: the first fundamental theorems for the classical groups and the application of invariant theory to representation theory via the duality theorem.

Chapters 6, on spinors, follows the corresponding chapter from our previous book, with some corrections and additional exercises. For the main result in Chapter 7—the Weyl character formula—we give a new algebraic group proof using the radial component of the Casimir operator (replacing the proof via Lie algebra cohomology in the previous book). This proof is a differential operator analogue of Weyl's original proof using compact real forms and the integration formula, which we also present in detail. The treatment of branching laws in Chapter 8 follows the same approach (due to Kostant) as in the previous book.

Chapters 9–10 apply all the machinery developed in previous chapters to analyze the tensor representations of the classical groups. In Chapter 9 we have added a discussion of the Littlewood–Richardson rule (including the role of the $\mathbf{GL}(n,\mathbb{C})$ branching law to reduce the proof to a well-known combinatorial construction). We have removed the partial harmonic decomposition of tensor space under orthogonal and symplectic groups that was treated in Chapter 10 of the previous book, and replaced it with a representation-theoretic treatment of the symmetry properties of curvature tensors for pseudo-Riemannian manifolds.

The general study of algebraic groups over $\mathbb{C}$ and homogeneous spaces begins in Chapter 11 (with the necessary background material from algebraic geometry in Appendix A). In Lie theory the examples are, in many cases, more difficult than the general theorems. As in our previous book, every new concept is detailed with its meaning for each of the classical groups. For example, in Chapter 11 every classical symmetric pair is described and a model is given for the corresponding affine variety, and in Chapter 12 the (complexified) Iwasawa decomposition is worked out explicitly. Also in Chapter 12 a proof of the celebrated Kostant–Rallis theorem for symmetric spaces is given and every implication for the invariant theory of classical groups is explained.

This book can serve for several different courses. An introductory one-term course in Lie groups, algebraic groups, and representation theory with emphasis on the classical groups can be based on Chapters 1–3 (with reference to Appendix D as needed). Chapters 1–3 and 11 (with reference to Appendix A as needed) can be the core of a one-term introductory course on algebraic groups in characteristic zero. For students who have already had an introductory course in Lie algebras and Lie groups, Chapters 3 and 4 together with Chapters 6–10 contain ample material for a second course emphasizing representations, character formulas, and their applications. An alternative (more advanced) second-term course emphasizing the geometric side of the subject can be based on topics from Chapters 3, 4, 11, and 12. A year-long course on representations and classical invariant theory along the lines of Weyl's book would follow Chapters 1–5, 7, 9, and 10. The exercises have been revised and many new ones added (there are now more than 350, most with several parts and detailed hints for solution). Although none of the exercises are used in the proofs of the results in the book, we consider them an essential part of courses based on this book. Working through a significant number of the exercises helps a student learn the general concepts, fine structure, and applications of representation and invariant theory.

Acknowledgments

In the end-of-chapter notes we have attempted to give credits for the results in the book and some idea of the historical development of the subject. We apologize to those whose works we have neglected to cite and for incorrect attributions. We are indebted to many people for finding errors and misprints in the many versions of the material in this book and for suggesting ways to improve the exposition. In particular we would like to thank Ilka Agricola, Laura Barberis, Bachir Bekka, Enriqueta Rodríguez Carrington, Friedrich Knop, Hanspeter Kraft, Peter Landweber, and Tomasz Przebinda. Chapters of the book have been used in many courses, and the interaction with the students was very helpful in arriving at the final version. We thank them all for their patience, comments, and sharp eyes. During the first year that we were writing our previous book (1989–1990), Roger Howe gave a course at Rutgers University on basic invariant theory. We thank him for many interesting conversations on this subject.

The first-named author is grateful for sabbatical leaves from Rutgers University and visiting appointments at the University of California, San Diego; the Université de Metz; Hong Kong University; and the National University of Singapore that were devoted to this book. He thanks the colleagues who arranged programs in which he lectured on this material, including Andrzei Hulanicki and Ewa Damek (University of Wrocław), Bachir Bekka and Jean Ludwig (Université de Metz), Ngai-Ming Mok (Hong Kong University), and Eng-Chye Tan and Chen-Bo Zhu (National University of Singapore). He also is indebted to the mathematics department of the University of California, Berkeley, for its hospitality during several summers of writing the book. The second-named author would like to thank the University of California, San Diego, for sabbatical leaves that were dedicated to this project, and to thank the National Science Foundation for summer support. We thank David Kramer for a superb job of copyediting, and we are especially grateful to our editor Ann Kostant, who has steadfastly supported our efforts on the book from inception to publication.

New Brunswick, New Jersey *Roe Goodman*
San Diego, California *Nolan R. Wallach*
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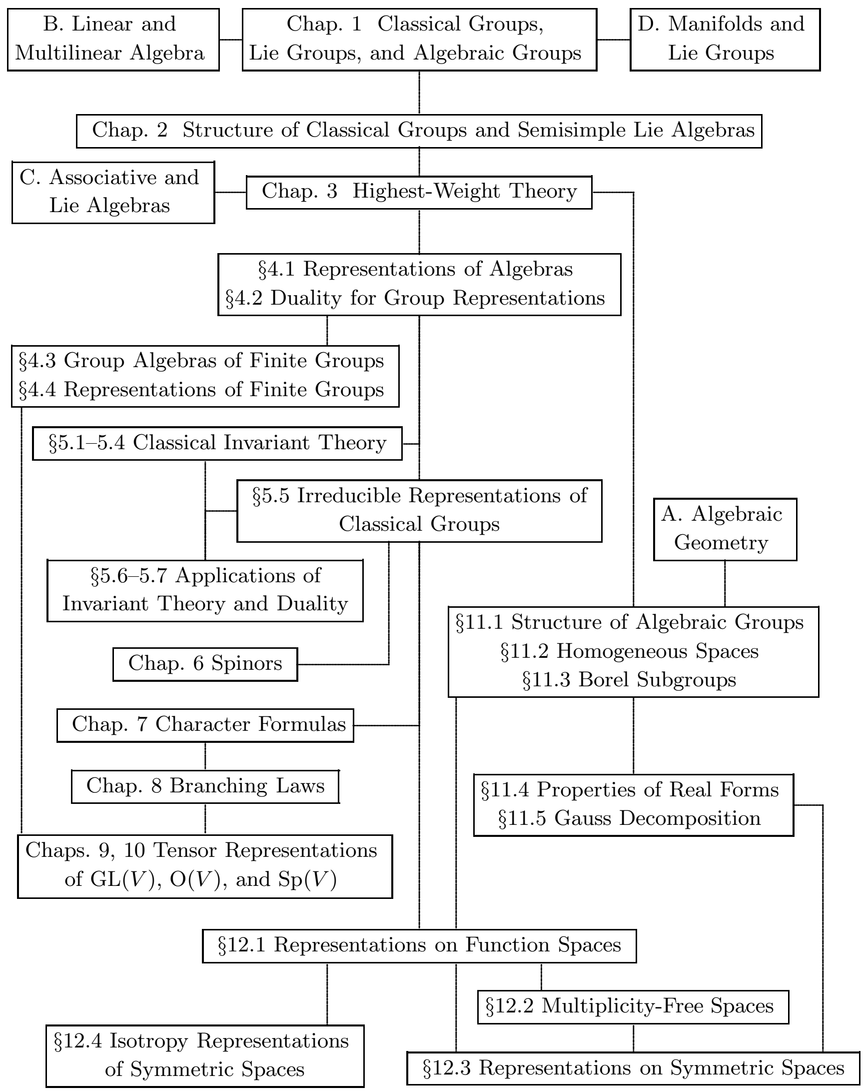
Organization and Notation



Dependency Chart among Chapters and Sections

"O," said Maggie, pouting, "I dare say I could make it out, if I'd learned what goes before, as you have." "But that's what you just couldn't, Miss Wisdom," said Tom. "For it's all the harder when you know what goes before: for then you've got to say what Definition 3. is and what Axiom V. is." George Eliot, *The Mill on the Floss*

Some Standard Notation

$\#S$ number of elements in set S (also denoted by $\mathrm{Card}(S)$ and $|S|$)

δ_{ij} Kronecker delta (1 if $i = j$, 0 otherwise)

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ nonnegative integers, integers, rational numbers

$\mathbb{R}, \mathbb{C}, \mathbb{H}$ real numbers, complex numbers, quaternions

$\mathbb{C}^\times$ multiplicative group of nonzero complex numbers

$[x]$ greatest integer $\leq x$ if x is real

$\mathbb{F}^n$ $n \times 1$ column vectors with entries in field $\mathbb{F}$

$M_{k,n}$ $k \times n$ complex matrices (M_n when $k = n$)

$M_n(\mathbb{F})$ $n \times n$ matrices with entries in field $\mathbb{F}$

$\mathbf{GL}(n, \mathbb{F})$ invertible $n \times n$ matrices with entries from field $\mathbb{F}$

I_n $n \times n$ identity matrix (or I when n understood)

$\dim V$ dimension of a vector space V

V^* dual space to vector space V

$\langle v^*, v \rangle$ natural duality pairing between V^* and V

$\mathrm{Span}(S)$ linear span of subset S in a vector space.

$\mathrm{End}(V)$ linear transformations on vector space V

$\mathbf{GL}(V)$ invertible linear transformations on vector space V

$\mathrm{tr}(A)$ trace of square matrix A

$\det(A)$ determinant of square matrix A

A^t transpose of matrix A

A^* conjugate transpose of matrix A

$\mathrm{diag}[a_1, \ldots, a_n]$ diagonal matrix

$\bigoplus V_i$ direct sum of vector spaces V_i

$\bigotimes^k V$ k-fold tensor product of vector space V (also denoted by $V^{\otimes k}$)

$S^k(V)$ k-fold symmetric tensor product of vector space V

$\bigwedge^k(V)$ k-fold skew-symmetric tensor product of vector space V

$\mathcal{O}[X]$ regular functions on algebraic set X

Other notation is generally defined at its first occurrence and appears in the index of notation at the end of the book.