April 15, 2002

Extended discussion of the answers to the Review Problems for the second exam in section 1 of Math 403

1. 0 is an isolated singularity. If 0 were a removable singularity, then $\lim_{z\to 0} f(z)$ would exist. But the hypotheses show that this limit (which must be a *unique* complex number), does **not** exist (because on one sequence which $\to 0$, the function's values $\to 0$ while on another sequence which $\to 0$, the function's values $\to 1$). So the singularity is not removable. It also cannot be a pole because the modulus of the function on the sequence $\{\frac{1}{n}\}$ does not tend to ∞. So the singularity must be essential. Consider the function in the punctured disc (this means the center is taken out) centered at 0 with radius $\frac{1}{n}$. Also consider in the range the disc centered at 2 with radius $\frac{1}{n}$. The Casorati-Weierstrass Theorem implies that there must be z_n in the punctured disc with radius $\frac{1}{n}$ so that $f(z_n)$ is inside the disc specified in the range. More algebraically, the theorem implies that for each positive integer n there is z_n with $0 < |z_n| < \frac{1}{n}$ so that $|f(z_n) - 2| < \frac{1}{n}$. But this means $\lim_{n\to\infty} f(z_n)$ exists and is 2. Also, of course, $\lim_{n\to\infty} z_n = 0$.

2. There are a variety of ways of computing these integrals including the Residue Theorem and the Cauchy Integral Formula for derivatives. I don't know one approach which is to be preferred among all of them, *except* that I would definitely avoid direct computation by parameterizing the integral!

a) If we use linearity of integration, then we need to compute $\int_\gamma \frac{5}{z}\,dz$ and $\int_\gamma \frac{-3}{(z-1)^2}\,dz$. The first integral was done repeatedly as early as chapter 1, and is $5\cdot 2\pi i = 10\pi i$. The second integral is 0 because, for example, the integrand has an antiderivative: $\frac{3}{z-1}$. Please recall that the integral of an analytic function with an antiderivative in an open set over a closed curve in that set must be 0.

b) Use the reasoning stated above ("The integral of an analytic function with an antiderivative in an open set containing a closed curve must be 0.") to see that the answer is 0. I suppose that $g'(z)$ can also be computed and then the observation made that none of the powers resulting are -1's, so the integral is 0.

c) Some computation needs to be done here. So $\big(g(z)\big)^2 = \frac{25}{z^2} + \frac{-30}{z(z-1)^2} + \frac{9}{(z-1)^4}$. This is a function with isolated singularities (all apparently poles) at 0 and 1. We can apply the Residue Theorem. The residue of $\big(g(z)\big)^2$ at 1 is the sum of the residue at 1 of $\frac{25}{z^2}$, which is 0 (since this part is analytic at 1), and the residue at 1 of $\frac{9}{(z-1)^4}$, which is 0 (since this is a negative fourth power), and the residue at 1 of $\frac{-30}{z(z-1)^2}$. If we think of this quotient as $H(z)\cdot\frac{1}{(z-1)^2}$ where $H(z)$ is analytic at 1, then the coefficient of $(z-1)^{-1}$ must be gotten from the coefficient of $z-1$ in the Taylor series of $H(z)$. This coefficient is $H'(1)$. If $H(z) = \frac{-30}{z}$ then $H'(z) = \frac{30}{z^2}$ and $H'(1) = 30$. So the residue of $\big(g(z)\big)^2$ at 1 is 30. What happens at 0? Now the terms $\frac{25}{z^2}$ and $\frac{9}{(z-1)^4}$ have residue 0. The residue of $\frac{-30}{z(z-1)^2}$ at 0 is -30. Here if we decompose $\frac{-30}{z(z-1)^2} = H(z)\cdot\frac{1}{z}$ (a *different* $H(z)$ here, with $H(z)$ analytic near 0!) then $H(z) = \frac{-30}{(z-1)^2}$ and since the pole is first order, the value of the residue is $H(0)$ which is -30. The Residue Theorem then applies to give the integral of $\big(g(z)\big)^2$ around γ as $2\pi i(30 + -30) = 0$.

There are other ways to do this problem. The Cauchy formula for derivatives, as mentioned above, can be used. Also a more sneaky way, not necessarily recommended, is to realize that we can *enlarge* the contour γ and not change the integral: this is a consequence of Cauchy's Theorem and the fact that the singularities are all inside γ. Then notice that on a really big circle, the ML inequality shows that the integral is less than $2\pi r\cdot$(the circle's length)$\cdot$[approximately] $\left(\frac{\text{Some constant}}{r^2}\right)$. The estimate for M is correct because there are at least two powers in the denominator of every piece of $\big(g(z)\big)^2$. Therefore as $r \to \infty$ this $\to 0$. Since the integral is constant, it must be 0!

3. The Cauchy integral formula for derivatives when used on a really big circle of radius R centered at 0 followed by the ML inequality gives $|k^{(n)}(0)| \leq \frac{n!}{R^n} \max_{|z|=R} |k(z)|$ if n is any non-negative integer. This is called the Cauchy estimates (page 133 of the text). We're given $|k(z)| \leq A\ln(|z|) + B$ which implies $\max_{|z|=R} |k(z)| \leq A\ln R + B$. Now compare rates of growth. That is, what is the limit of $\frac{A\ln R + B}{R^n}$ as $R \to \infty$? For $n > 0$ this limit is 0. To see this, you can use l'Hopital's Rule directly or just remember (I hope!) from calculus that log growth is slower than the growth of any positive power. Therefore $k^{(n)}(0) = 0$ for positive n. Since $k(z)$ is entire, we know from general results that $k(z) = \sum_{n=0}^{\infty} \frac{k^{(n)}(0)}{n!} z^n$ for all z. But all the coefficients are 0 except for the first, so that $k(z) = k(0)$ for all z: $k(z)$ is a constant function.

4. You can use the Cauchy estimates here, also. They can be applied to show that $h^{(n)}(0) = 0$ for $n > 7$. Then you know that $h(z)$ is a polynomial of degree 7. You still need to show that $h(z)$ is only a monomial of degree 7 (so it can't be $z^7 - 33z + 5$, for example). You will need to look at the behavior near 0 for this.

Here is another way to do the whole problem as suggested on the bald answer page. Suppose $K(z) = \frac{h(z)}{z^7}$. Then $K(z)$ is analytic away from 0: it has an isolated singularity at 0. What is the type of this singularity? Since $|h(z)| \leq |z^7|$ we see that $|K(z)| \leq 1$ and the Riemann Removable Singularity Theorem implies that the singularity is removable. So "remove it": redefine $K(z)$ as $\frac{h(z)}{z^7}$ for $z \neq 0$ with $K(0)$ defined appropriately, as the limit of $\frac{h(z)}{z^7}$ as $z \to 0$. The theorem cited shows that the limit exists. Note that since $|K(z)| \leq 1$ for $z \neq 0$, $|K(0)| \leq 1$. So $K(z)$ is now an entire function with $|K(z)| \leq 1$. Liouville's Theorem then applies to show that $K(z)$ is a constant, C, with $|C| \leq 1$. Therefore $h(z) = Cz^7$ where C is a complex number with $|C| \leq 1$.

5. Every way I see to do this involves some tedious computation. Begin by observing that $\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} + \ldots$, so that $z - \sin z = \frac{z^3}{6} - \frac{z^5}{120} + \ldots = \frac{z^3}{6}(1 - \frac{z^2}{20} + \ldots)$. What's inside the parentheses is a convergent power series with constant term 1, so its sum is an analytic function whose value at 0 is 1. Then $m(z)$ must be $\left(\frac{6}{z^3}\right) \cdot \frac{1}{1 - \frac{z^2}{20} + \ldots}$.

I'd like to find the first two non-zero terms of the power series for $\frac{1}{1 - \frac{z^2}{20} + \ldots}$ at 0. If we call this function $T(z)$, then I could compute $T(0) + T'(0)z + \ldots$ (as many terms as are needed to get two non-zero terms). Certainly $T(0) = 1$ (just plug in 0 for z). But it turns out that $T'(0)$ is 0 and we will (at least!) need to compute $T''(0)$, which looks tedious. Another way to compute the series for $T(z)$ is to use long division: divide 1 by the infinite "polynomial" $1 - \frac{z^2}{20} + \ldots$. This will work fine. Yet another way is the following: remember that $\frac{a}{1-r}$ is $a + ar + \ldots$. We can compute the beginning of the power series for $T(z)$ by considering $\frac{1}{1-\frac{z^2}{20}+\ldots}$ with $a = 1$ and $r = \frac{z^2}{20} + \ldots$. Then we see that $T(z) = 1 + \frac{z^2}{20} + \ldots$ where "$\ldots$" contains only terms involving powers of z with degree > 2.

We put it all together: $m(z) = \left(\frac{6}{z^3}\right) \cdot T(z) = \left(\frac{6}{z^3}\right) \cdot \left(1 + \frac{z^2}{20} + \ldots\right) = \frac{6}{z^3} + \frac{3}{10z} + \ldots$, which answers the question. The residue is $\frac{3}{10}$.

By the way, the `Maple` command `residue(1/(z-sin(z)),z=0);` gives the answer above (after you enter `readlib(residue);`) and the command `series(1/(z-sin(z)),z,5);` gives the initial piece of the Laurent series displayed above.

6. We apply Rouché's Theorem several times. Here the version given in the text on page 177 is used. Since I am writing this material somewhat before it will be covered in class, I'll give a different presentation below. We need a closed curve, γ, and two functions f and g analytic on and inside γ satisfying $|f(z) + g(z)| < |g(z)|$ on γ. With all this true, $f(z)$ and $g(z)$ must have the same number of zeros inside γ. Please note that in a) and b), different parts of $P(z)$ are selected for "big" and "little".

a) If $|z| = 1$ (the candidate for γ, then $|12z^2| = 12$ and $|z^5 + 2| \leq |z^5| + 2 = 3$. Take $f(z) = -12z^5$ and $g(z) = P(z) = z^5 + 12z^2 + 2$ so that $f(z) + g(z) = 12z^2 + 2$. Then $|g(z)| = |12z^2 + (z^5+2)| \overset{\nabla}{\geq} 12 - 3 = 9 > 3 \geq |z^5 + 2|$. Since $-12z^2$ has a zero of multiplicity 2 at 0 (inside γ) then $P(z)$ has two zeros (counting multiplicity!) inside γ.

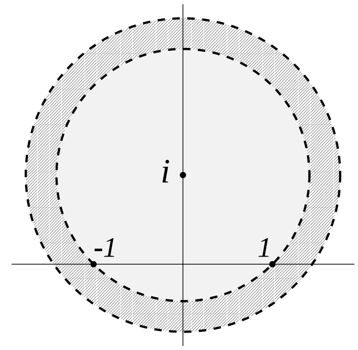
b) If $|z| = 10$, then $|z^5| = 100{,}000$ and $|12z^2 + 2| \leq 1{,}202$. Now the γ is $|z| = 10$ and $f(z) = -z^5$ and $g(z) = P(z)$ so $f(z) + g(z) = 12z^2 + 2$. On γ, $|g(z)| \overset{\nabla}{\geq} 100{,}000 - 1{,}202 > 1{,}202 \geq |f(z) + g(z)|$ so that the five roots at 0 of z^5 imply that $P(z)$ must have five roots within $|z| = 10$.

c) Since $P(z)$ has five roots inside the larger circle and only two inside the smaller one, it must have three roots inside the specified annulus.

Alternative approach The "walking the dog" version of Rouché's Theorem which I hope to present in class is easier for me to work with. The hypotheses are: f and g are analytic on and inside a simple closed curve, γ, and, for all z on γ, $|g(z)| < |f(z)|$. Here $|f(z)|$ is the distance of the dog-walker to the lamppost (the origin), and $|g(z)|$ is the length of the leash. The conclusion is that the number of zeros of $f(z)$ inside γ is the same as the number of zeros of $f(z) + g(z)$ inside γ. Part a) can then be done by taking $f(z) = 12z^2$ and $g(z) = z^5 + 2$. Easily $|f(z)| = 12$ and $|g(z)| \leq 3$ on γ,which is $|z| = 1$. Since $3 < 12$, the conclusion of a) follows. For part b), use $f(z) = z^5$ and $g(z) = 12z^2 + 2$. On $|z| = 10$, $|f(z)| = 100{,}000$ and $|g(z)| \leq 1{,}202$. Since $100,000 > 1,202$, b) is proven. Of course the estimates are the same, but somehow I find the psychology of this version easier to work with. Note that these are rather simple applications of Rouché's Theorem. It is amazing that such easy estimates often give enough information to handle real problems.

As far as I know, `Maple` can't use Rouché's Theorem (yet!).

7. The Taylor series will converge in a disc of largest radius inside of which the function $Q(z)$ is analytic. $Q(z)$ has isolated singularities at ± 1, 2, and 3. The closest of these to i is ± 1, so the radius is *at least* $\sqrt{2}$. It is exactly $\sqrt{2}$, however, because the singularities are all poles. They are poles because the top of the fraction defining $Q(z)$ is never 0 and the bottom is 0 at all the singularities. Therefore as $z \to 1$ from the inside of the disc $|z-i| < \sqrt{2}$ the function $|Q(z)| \to \infty$. The function can't be extended to be analytic in a disc of larger radius because it must have finite modulus (!) at each point.



Alternative approach Consider the Taylor series for $Q(z)$ centered at $z = i$. We know the sum of a power series must be an analytic function. Call this analytic function $QQ(z)$ (an irritating name). $QQ(z)$ must be defined in a disc centered at i because power series converge in discs. $QQ(z)$ and $Q(z)$ are equal in $|z-i| < \sqrt{2}$ because Taylor series of analytic functions must converge to the function's values in the largest disc which sits inside where the function (here, $Q(z)$) is defined. Suppose the radius of convergence of $QQ(z)$ were *greater than* $\sqrt{2}$. We know $QQ(z) = Q(z)$ for $|z-i| < 1$. $QQ(z)$ and $Q(z)$ are both defined and analytic in a larger connected open set: the set of all z's *except* ± 1 in $|z-i| < r$ (for some $r > 1$). Therefore by the Identity Theorem ("Two functions analytic in a connected open set which agree on a set with a limit point must actually agree everywhere in the set.") $QQ(z)$ and $Q(z)$ must be equal for all z's with $|z-i| < r$ when $z \neq \pm 1$. But $QQ(z)$ has a removable singularity at ± 1 and we know that $Q(z)$ does not (it has a pole). So the radius of convergence is exactly $\sqrt{2}$.
Some people may find this overelaborate. Please compare the situation in problem 14.

8. If $\frac{z+1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$ then $z+1 = A(z-1) + Bz$. $z = 0$ shows that $A = -1$ and $z = 1$ shows that $B = 2$. Then $R(z) = \frac{-1}{z} + \frac{2}{z-1}$. When $|z| < 1$, the geometric series with ratio z gives $\frac{2}{z-1} = -2\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} -2z^n$. Since we're dividing by z, we also need to know $0 < |z|$. So the Laurent series for R in the annulus $0 < |z| < 1$ is $-\frac{1}{z} + \sum_{n=0}^{\infty} -2z^n$. The coefficient of z^{10} is -2. The coefficient of z^{-10} is 0.

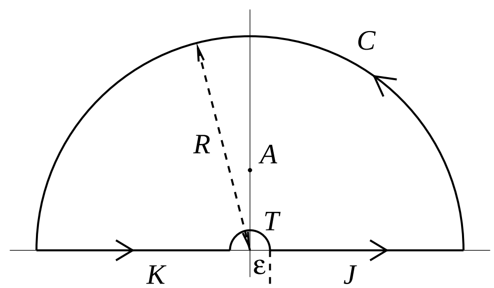
Another method to get the answer: $\frac{z+1}{z(z-1)} = -\frac{z+1}{z} \cdot \frac{1}{1-z} = -\frac{z+1}{z} \cdot \sum_{n=0}^{\infty} z^n$. There is some "overlapping" of powers because of the multiplication by $z+1$. This answer can be rewritten to be identical to the previous result.

9. We know that $e^{(z^2)} = 1 + z^2 + \frac{1}{2}z^4 +$ HIGHER ORDER TERMS, using the power series for exp centered at 0 which converges for all z. Multiply by $1+3z$ and discard all terms with degree > 4: $(1+3z)(1+z^2+\frac{1}{2}z^4) = 1+z^2+\frac{1}{2}z^4+3z+3z^3+\frac{3}{2}z^5$ "$=$" $1+3z+z^2+3z^3+\frac{1}{2}z^4$. $S^{(4)}(0)$ is the coefficient of z^4 multiplied by $4!$, so it is $\frac{1}{2} \cdot 4! = \frac{1}{2} \cdot 24 = 12$.

10. Use $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ so that $\int_0^{2\pi} \frac{2}{3+\sin\theta}\, d\theta = \int_0^{2\pi} \frac{4i}{6i+e^{i\theta}-e^{-i\theta}}\, d\theta = \int_0^{2\pi} \frac{4ie^{i\theta}}{6ie^{i\theta}+(e^{i\theta})^2-1}\, d\theta$. Now make the substitution $z = e^{i\theta}$ so $dz = ie^{i\theta}\, d\theta$ and the definite integral is recognized

as a parameterization of a line integral around the unit circle, $|z| = 1$: $\int_{|z|=1} \frac{4}{z^2+6iz-1}\,dz$. The function $\frac{4}{z^2+6iz-1}$ is analytic everywhere except where $z^2+6iz-1 = 0$. The quadratic formula tells us these z's: $z = \frac{-6i\pm\sqrt{(6i)^2-4(-1)}}{2} = \frac{-6i\pm\sqrt{-32}}{2} = \frac{-6i\pm 4\sqrt{2}i}{2} = (-3\pm 2\sqrt{2})i$. Since $2\sqrt{2} \approx 2.8$ we see that one singularity lies inside the circle (the one with $+$) and the other is outside. We compute the line integral with the Residue Theorem since its value will be $2\pi i$(the sum of the residues inside the unit circle).
We need the residue of $\frac{4}{z^2+6iz-1}$ at $z = (-3+2\sqrt{2})i$. Rewrite the integrand: $\frac{4}{z^2+6iz-1} = \frac{4}{\left(z-(-3+2\sqrt{2})i\right)\cdot\left(z-(-3-2\sqrt{2})i\right)} = \frac{1}{z-(-3+2\sqrt{2})i} \cdot \frac{4}{z-(-3-2\sqrt{2})i}$. If $H(z)$ has a simple pole at z_0 and we can write $H(z)$ as the product of $\frac{1}{z-z_0} \cdot K(z)$ where $K(z)$ is analytic near z_0, the residue of $H(z)$ at z_0 will be $K(z_0)$. Here $K(z) = \frac{4}{z-(-6-4\sqrt{2})i}$ and $z_0 = (-3+2\sqrt{2})i$. The desired residue is $K((-3+2\sqrt{2})i) = \frac{4}{(-3+2\sqrt{2})i-(-3-2\sqrt{2})i} = \frac{4}{4\sqrt{2}i} = \frac{1}{\sqrt{2}i}$. The Residue Theorem states that the value of the integral is $2\pi i \cdot \frac{1}{\sqrt{2}i} = \sqrt{2}\pi$, luckily agreeing with `Maple`'s value.

11. **I** We'll apply the Residue Theorem to the function $f(z) = \frac{1}{\sqrt{z}(4+z^2)}$ on the closed curve shown: this is an "indented contour". Here ε is a small positive number and R is a large positive number. J is the interval $[\varepsilon, R]$ on the positive real axis, and K is the interval $[-R, -\varepsilon]$ on the negative real axis. T is the "upper" semicircle of radius ε centered at 0, and C is the "upper" semicircle of radius R centered at 0. The orientations on the four pieces are as shown. We will consider what happens as $\varepsilon \to 0^+$ and $R \to +\infty$.



II We estimate $\int_T f(z)\,dz$. Remember that $\sqrt{z} = \sqrt{r}e^{i\theta/2}$ if $z = re^{i\theta}$ is the complex polar representation of z. Therefore when $|z| = \varepsilon$, $|\sqrt{z}| = \sqrt{\varepsilon}$. Also, $|4+z^2| \overset{\nabla}{\geq} 4 - |z|^2 \geq 4 - \varepsilon^2$. The ML inequality produces this: $\left|\int_T f(z)\,dz\right| \leq (\pi\varepsilon) \cdot \left(\frac{1}{\sqrt{\varepsilon}(4-\varepsilon^2)}\right) = \frac{4\sqrt{\varepsilon}}{4-\varepsilon^2}$. We just need $\pi\varepsilon$ because the curve is a <u>semi</u>circle. As $\varepsilon \to 0^+$, this integral $\to 0$ because the bottom $\to 4$ and the top (a positive power of ε!) $\to 0$.
III Let's estimate $\int_C f(z)\,dz$. Here $|z| = R$ so that $|\sqrt{z}| = \sqrt{R}$. The denominator gets handled a bit differently because what was "big" before becomes "little" here: $|4+z^2| \overset{\nabla}{\geq} |z|^2 - 4 \geq R^2 - 4$. Apply ML again to get: $\left|\int_C f(z)\,dz\right| \leq (\pi R) \cdot \left(\frac{1}{\sqrt{R}(R^2-4)}\right) = \frac{\pi\sqrt{R}}{R^2-4}$. As $R \to +\infty$, this integral also $\to 0$ because the power of R on the bottom (2) is larger than the power of R on the top (.5).
IV The Residue Theorem asserts that the integral of $f(z)$ around the closed curve shown will be $2\pi i$ multiplied by the sum of residues of the singularities inside the curve. For $R > 2$ and $\varepsilon < 2$ the function $f(z)$ has one isolated singularity inside the closed curve. Singularities occur when $4 + z^2 = 0$, which happens when $z = \pm 2i$. One of these ($2i$) is inside the closed curve, at the point labeled A in the diagram. This isolated singularity is a simple pole, because the bottom has two *distinct* roots. The Laurent expansion

of $f(z)$ at $2i$ must begin with $\frac{A}{z-2i}$ and we need A, which we can compute from the limit: $\lim_{z\to 2i}(z-2i)\frac{1}{\sqrt{z}(4+z^2)} = \lim_{z\to 2i}\frac{z-2i}{\sqrt{z}(4+z^2)} \overset{\text{L'H}}{=} \lim_{z\to 2i}\frac{1}{\left(\frac{1}{2\sqrt{z}}\right)(4+z^2)+\sqrt{z}(2z)} = \frac{1}{\sqrt{2i}(2\cdot 2i)}$. What is $\sqrt{2i}$? Since $2i = 2e^{i\pi/2}$, $\sqrt{2i} = \sqrt{2}e^{i\pi/4} = \sqrt{2}\left(\frac{1+i}{\sqrt{2}}\right) = 1+i$. The limit we want is $\frac{1}{(1+i)(4i)} = \frac{1}{4}\cdot\frac{1}{i}\cdot\frac{1}{1+i} = \frac{1}{4}\cdot\frac{1-i}{2}\cdot -i = \frac{1-i}{8}$, the residue of $f(z)$ at A. The integral will therefore be $(2\pi i)\cdot\left(\frac{1-i}{8}\right) = \frac{(1+i)\pi}{4}$.

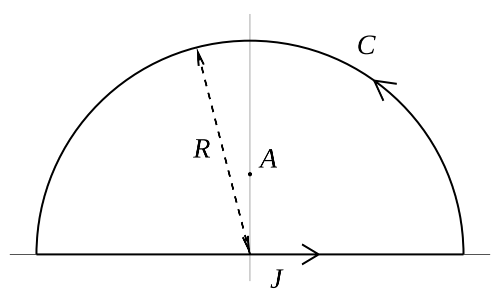
V Now for the integral over J: $\int_J f(z)\,dz = \int_\varepsilon^R \frac{dx}{\sqrt{x}(4+x^2)}$. As $\varepsilon\to 0^+$ and $R\to+\infty$, this integral $\to \int_0^\infty \frac{dx}{\sqrt{x}(4+x^2)}$. This is the quantity we want to compute.

VI The integral over K demands a bit of care. The orientation gives a minus sign, and the square root introduces $\frac{1}{i}$: $\int_J f(z)\,dz = -\frac{1}{i}\int_\varepsilon^R \frac{dx}{\sqrt{x}(4+x^2)} = i\int_\varepsilon^R \frac{dx}{\sqrt{x}(4+x^2)}$. Now as $\varepsilon\to 0^+$ and $R\to+\infty$, this integral $\to i\int_0^\infty \frac{dx}{\sqrt{x}(4+x^2)}$.

VII Let's put everything together. As $\varepsilon\to 0^+$ and $R\to+\infty$, the integral over the closed curve $\to (1+i)\int_0^\infty \frac{dx}{\sqrt{x}(4+x^2)}$. But according to the Residue Theorem, the value of the integral over the closed curve for *any* choices of $R>2$ and $0<\varepsilon<2$ is $\frac{(1+i)\pi}{4}$. Divide by $1+i$ to see that $\int_0^\infty \frac{dx}{\sqrt{x}(4+x^2)} = \frac{\pi}{4}$.

Comments How could an instructor expect students to write good solutions of such problems on an exam? There are many "opportunities" for error in almost every step. What will I look for in grading such a problem? I hope to find a clear, correct, and appropriate application of the Residue Theorem as in paragraphs **I** and **VII** above. Sometimes limiting values of certain integrals need to be computed. I want to see estimates supporting the limit assertions, as in **II** and **III** above. I also need to see some residue computation. I deliberately chose a different way of computing the residue in **IV**. I could have done the computation as in the solutions to problem 2 and 10. Paragraphs **V** and **VI** relate some integrals occurring in the Residue Theorem application to the original desired integral: of course something like that must be present, also. I want to also see some "putting together" or summary description (here **VII**). I want enough explanation so that simple errors will *not* subtract substantially from a student's credit on the problem.

12. **I** Clever people long ago suggested using $f(z) = \frac{e^{iaz}}{(1+z^2)^2}$, with the contour as shown: J is the interval from $-R$ to R on the real axis (where R is some large positive number) and C is the upper semicircle of radius R. The direction of integration is indicated in the picture. The strategy to solve this problem is similar to the preceding one. The details are simpler though, except for computing the residue.



II $f(z)$ is chosen so that for z real (so $z = x+i0$), $f(z) = f(x+i0) = \frac{e^{iax}}{(1+x^2)^2} = \frac{\cos(ax)+i\sin(ax)}{(1+x^2)^2}$. Here ax is a real number and the cosine and sine functions are the standard calculus functions. We can analyze the integral over J as follows: $\int_J f(z)\,dz = \int_{-R}^R \frac{\cos(ax)+i\sin(ax)}{(1+x^2)^2}\,dx = \int_{-R}^R \frac{\cos(ax)}{(1+x^2)^2}\,dx + i\int_{-R}^R \frac{\sin(ax)}{(1+x^2)^2}\,dx = \int_{-R}^R \frac{\cos(ax)}{(1+x^2)^2}\,dx$. The last equality is true because the integral of an antisymmetric (odd) function over an interval symmetric about the origin is 0. That is, $\int_{-R}^{R} g(x)\,dx = 0$ if $g(-x) = -g(x)$ always. The sign change is guaranteed here by the presence of sine with only even powers of x.

III The selection of the upper semicircle and the selection of $f(z)$ are directly related to the specification that a is positive. If a were negative, the lower semicircle would be used. The choice is made so that the integral over C can be estimated easily. For z on C, $\operatorname{Im} z \geq 0$ so that $|e^{iaz}| = |e^{ia(x+iy)}| = |e^{iax-ay}| = |e^{iax}e^{-ay}| = |e^{iax}||e^{-ay}| = 1 \cdot \exp(\text{a real number } \leq 0) \leq 1$. We needed to know that a was real and positive to guarantee the truth of this chain of relationships. Also $|1+z^2| \geq |z^2|-1 = R^2-1$. Therefore on C, $|f(z)| \leq \frac{1}{(R^2-1)^2}$ when R is large. Use ML again: $|\int_C f(z)\,dz| \leq (\pi R) \cdot \frac{1}{(R^2-1)^2}$ and this $\to 0$ as $R \to +\infty$ because the largest power of R "wins".

IV $f(z) = \frac{e^{iaz}}{(1+z^2)^2}$ has isolated singularities at $\pm i$ (where $(z^2+1=0)$. Since $\frac{e^{iaz}}{(1+z^2)^2} = \frac{e^{iaz}}{(z+i)^2(z-i)^2}$, the singularities are both double poles: poles of order 2. The singularity of interest is at i (the other singularity is outside the closed curve we are considering). The Laurent series at i begins with $\frac{A}{(z-i)^2} + \frac{B}{z-i} + \ldots$ and we need to know B, the residue of $f(z)$ at i. If $f(z) = \frac{H(z)}{(z-i)^2}$ where $H(z)$ is analytic near i, the initial segment of the Taylor series for $H(z)$ is $H(i) + H'(i)(z-i) + \ldots$ and therefore the residue, B, must also be $H'(i)$. Here $H(z) = \frac{e^{iaz}}{(z+i)^2}$ so that $H'(z) = \frac{iae^{iaz}(z+i)^2 - 2(z+i)e^{iaz}}{(z+i)^4}$. A short computation (!) shows that $H'(i) = \frac{iae^{-a}(2i)^2 - 2(2i)e^{-a}}{(2i)^4} = \frac{iae^{-a}(-4) - 4ie^{-a}}{16} = \frac{-ie^{-a}}{4}(a+1)$.

V The Residue Theorem can be applied when $R > 1$. The integral over the closed curve shown is $2\pi i \cdot \frac{-ie^{-a}}{4}(a+1) = \frac{(a+1)e^{-a}\pi}{2} = \frac{\pi e^{-a}}{2}(a+1)$.

VI When $R \to \infty$, the integral over $C \to 0$ and the integral over $J \to$ the requested definite improper integral. The sum of these two integrals is always $\frac{\pi e^{-a}}{2}(a+1)$, so that must be the value of the requested integral.

Comments The setup is introduced in **I** and the results summarized in paragraphs **V** and **VI**. The residue computation is done in **IV**. An estimate of an integral is done in **III**. The relation of the requested integral to a complex line integral is discussed in paragraph **II**.

13. Since $\left(q(0)\right)^2 = 0$, $q(0)$ must be 0. $q(z)$ can't always be 0 (otherwise its square couldn't be nonzero!). If $q(z)$ has a zero of order N at 0 (where N is some positive integer), then (as we saw in a homework assignment) $\left(q(z)\right)^2$ would have a zero of order $2N$ at 0. But z has a first order zero, so $2N$ must be 1: this is impossible.

This solution shows that there can be no analytic $\sqrt{z}$ which can be defined in any disc around 0. Maybe an easier solution to this problem is to differentiate the equation $\left(q(z)\right)^2 = z$ and plug in 0 for z to get $2q'(0)q(0) = 1$. But $q(0) = 0$ so this is impossible.

14. Since $\sin 0 = 0$, the Taylor series for sine centered at 0 is divisible by z. So factor out the z, and you can easily see that $M(z)$ has a removable singularity at 0. So: "remove it". That is, define $M(0)$ to be the value of $\frac{z - \frac{z^3}{3!} - \frac{z^5}{5!} + \ldots}{z} = 1 - \frac{z^2}{3!} - \frac{z^4}{5!} + \ldots$ at 0 (that value is 1). Then with that extended definition, $M(z)$ is an *entire* function: it is analytic in the whole complex plane. So the radius of convergence of the Taylor series of $M(z)$ at any

point must be infinity. But the Taylor series of $\frac{\sin z}{z}$ at $z = i$ is gotten by computations of values of derivatives at $z = i$ and is the same as the Taylor series of the "extended" $M(z)$. So the radius of convergence of the Taylor series of $\frac{\sin z}{z}$ at $z = i$ is infinity. There really is no singularity in this function in spite of the way it looks!