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Answers to the First Exam

2/19/2006

(8) 1. Suppose $p = (1, 0, 2)$, $q = (0, 2, 2)$, and $r = (1, 1, 1)$ are points in $\mathbb{R}^3$.

a) Find a vector orthogonal to the plane through the points p, q, and r.

Answer The vector $\overrightarrow{pq}$ is $\langle -1, 2, 0\rangle$ and the vector $\overrightarrow{pr}$ is $\langle 0, 1, -1\rangle$. The cross product is a vector perpendicular to these vectors: $\overrightarrow{pq} \times \overrightarrow{pq} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix} = -2\mathbf{i} - \mathbf{j} - \mathbf{k}$.

b) Find the area of triangle pqr.

Answer The area of triangle pqr is half the magnitude of $\overrightarrow{pq} \times \overrightarrow{pq}$, so it is $\frac{1}{2}\sqrt{(-2)^2 + (-1)^1 + 1^1} = \frac{1}{2}\sqrt{6}$.

(12) 2. Suppose $\mathbf{V} = \langle 1, 3, -2\rangle$ and $\mathbf{W} = \langle -1, 2, 1\rangle$. Find vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ so that $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$, $\mathbf{V}_1$ is parallel to $\mathbf{W}$, and $\mathbf{V}_2$ is perpendicular to $\mathbf{W}$.

Answer $\mathbf{V}_1 = \frac{\mathbf{V}\cdot\mathbf{W}}{|\mathbf{W}|^2}\mathbf{W}$ so compute $\mathbf{V}\cdot\mathbf{W} = 1\cdot(-1) + 3\cdot 2 + (-2)\cdot 1 = -1 + 6 - 2 = 3$ and $|\mathbf{W}|^2 = (-1)^2 + 2^2 + 1^2 = 6$. Then $\mathbf{V}_1 = \frac{3}{6}\langle -1, 2, 1\rangle = \langle -\frac{1}{2}, 1, \frac{1}{2}\rangle$. Since $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$, $\mathbf{V}_1 = \mathbf{V} - \mathbf{V}_2 = \langle 1, 3, -2\rangle - \langle -\frac{1}{2}, 1, \frac{1}{2}\rangle = \langle \frac{3}{2}, 2, -\frac{5}{2}\rangle$. If desired, a check that $\mathbf{W}\perp\mathbf{V}_1$ is: $\mathbf{W}\cdot\mathbf{V}_1 = (-1)\left(\frac{3}{2}\right) + 2\cdot 2 + 1\left(-\frac{5}{2}\right) = 0$.

(10) 3. The point $(1, 0, 2)$ is on the plane P whose equation is given by $2x - 2y + z = 4$.

a) Find parametric equations for the line perpendicular to P through $(1, 0, 2)$.

Answer The normal vector of P is $\langle 2, -2, 1\rangle$, the line's direction. Parametric equations are $\begin{cases} x = 1 + 2t \\ y = 0 - 2t \\ z = 2 + 1t \end{cases}$.

b) Find a point on the line in a) which has distance 5 to P.

Answer The distance from $(1 + 2t, -2t, 2 + t)$ to P is just the distance to $(1, 0, 2)$, and this distance is $\sqrt{(2t)^2 + (-2t)^2 + t^2} = 3|t|$. This will be 5 when $|t| = \frac{5}{3}$ so $t = \pm\frac{5}{3}$. The positive root substituted in the parametric equations gives the point $(1 + \frac{10}{3}, -\frac{10}{3}, 2 + \frac{5}{3}) = (\frac{13}{3}, -\frac{10}{3}, \frac{11}{3})$.

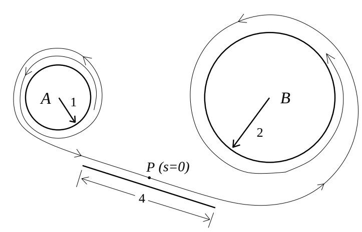
c) Write an equation for a plane parallel to P whose distance to P is 5.

Answer A normal vector is $\langle 2, -2, 1\rangle$ and one such plane passes through $(\frac{13}{3}, -\frac{10}{3}, \frac{11}{3})$. Therefore an equation for the plane is $2(x - \frac{13}{3}) - 2(y + \frac{10}{3}) + 1(z - \frac{11}{3}) = 0$.

(12) 4. Suppose the position vector of a curve is given by $\mathbf{r}(t) = \langle e^t, e^{-t}, \sqrt{2}t\rangle$. Find the unit tangent and unit normal vectors when the parameter $t = 1$. That is, find $\mathbf{T}(1)$ and $\mathbf{N}(1)$.

Answer Since $\mathbf{r}(t) = \langle e^t, e^{-t}, \sqrt{2}t\rangle$, $\mathbf{r}'(t) = \langle e^t, -e^{-t}, \sqrt{2}\rangle$ and $|\mathbf{r}'(t)| = \sqrt{(e^t)^2 + (-e^{-t})^2 + (\sqrt{2})^2} = \sqrt{e^{2t} + e^{-2t} + 2} = \sqrt{\left(e^t + e^{-t}\right)^2} = e^t + e^{-t}$. So $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \left\langle \frac{e^t}{e^t+e^{-t}}, \frac{-e^{-t}}{e^t+e^{-t}}, \frac{\sqrt{2}}{e^t+e^{-t}}\right\rangle$, and $\frac{d}{dt}\mathbf{T}(t) = \left\langle \frac{e^t(e^t+e^{-t})-(e^t-e^{-t})e^t}{(e^t+e^{-t})^2}, \frac{(-1)^2e^{-t}(e^t+e^{-t})-(e^t-e^{-t})(-e^{-t})}{(e^t+e^{-t})^2}, \frac{-(e^t-e^{-t})\sqrt{2}}{(e^t+e^{-t})^2}\right\rangle$. $\mathbf{T}(1) = \left\langle \frac{e}{e+e^{-1}}, \frac{-e^{-1}}{e+e^{-1}}, \frac{\sqrt{2}}{e+e^{-1}}\right\rangle$. $\mathbf{T}'(1) = \left\langle \frac{e(e+e^{-1})-(e-e^{-1})e}{(e+e^{-1})^2}, \frac{e^{-1}(e+e^{-1})+e^{-1}(e-e^{-1})}{(e+e^{-1})^2}, \frac{-(e-e^{-1})\sqrt{2}}{(e+e^{-1})^2}\right\rangle = \left\langle \frac{2}{(e+e^{-1})^2}, \frac{2}{(e+e^{-1})^2}, \frac{-(e-e^{-1})\sqrt{2}}{(e+e^{-1})^2}\right\rangle$. $\mathbf{N}(1)$ is a unit vector in the direction of $\mathbf{T}'(1)$, so consider $\left\langle 2e, 2e, -(e^2-1)\sqrt{2}\right\rangle$, a positive multiple of $\mathbf{T}'(1)$ whose length is $\sqrt{8e^2 + 2(e^4 - 2e^2 + 1)} = \sqrt{2(e^4 + 2e^2 + 1)} = \sqrt{2}(e^2+1)$. "Thus" $\mathbf{N}(1) = \left\langle \frac{2e}{\sqrt{2}(e^2+1)}, \frac{2e}{\sqrt{2}(e^2+1)}, \frac{-(e^2-1)\sqrt{2}}{\sqrt{2}(e^2+1)}\right\rangle$*.

(12) 5. A point is moving along the curve below in the direction indicated. Its motion is parameterized by arc length, s, so that it is moving at unit speed. Arc length is measured from the point P (both backward and forward). The curve is intended to continue indefinitely both forward and backward in s, with its forward motion curling more and more tightly around the indicated circle, B, and, backward, curling more and more tightly around the other circle, A. Near P the curve is parallel to the indicated line segment.



Sketch a graph of the curvature, κ, as a function of the arc length, s. What are $\lim_{s\to+\infty} \kappa(s)$ and $\lim_{s\to-\infty} \kappa(s)$? Use complete English sentences to briefly explain the numbers you give.

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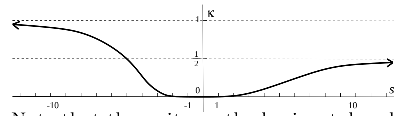
* I cheated, sort of. I computed this by hand, and got a horrible mess. I had `Maple` simplify the answer which I "reverse engineered" to get the displayed computation. Any correct mess submitted is certainly acceptable!

Answer As $s \to -\infty$, $\kappa \to 1$: the curve is getting closer to a circle of radius 1 with curvature $= 1$. There should be an interval of length ≈ 4 centered at $s = 0$ where the curve is "flat", $\kappa = 0$. As $s \to \infty$, $\kappa \to \frac{1}{2}$: the curve is getting closer to a circle of radius 2 with curvature $= \frac{1}{2}$. The graph should decrease and then increase. Also, $\lim\limits_{s\to-\infty} \kappa(s) = 1$ and $\lim\limits_{s\to+\infty} \kappa(s) = \frac{1}{2}$.



Note that the units on the horizontal and vertical axes differ in length.

(10) 6. Find the linear approximation of the function $f(x, y) = \sqrt{20 - x^2 - 7y^2}$ at $(2, 1)$ and use it to approximate $f(1.95, 1.08)$.

Answer $f(2, 1) = \sqrt{20 - 2^2 - 7 \cdot 1^2} = \sqrt{9} = 3$; $f_x = \frac{1}{2}(20-x^2-7y^2)^{-\frac{1}{2}}(-2x)$ so $f_x(2, 1) = \frac{1}{2} \cdot \frac{1}{3}(-2 \cdot 2) = -\frac{2}{3}$; $f_y = \frac{1}{2}(20 - x^2 - 7y^2)^{-\frac{1}{2}}(-14y)$ so $f_y(2, 1) = \frac{1}{2} \cdot \frac{1}{3}(-14 \cdot 1) = -\frac{7}{3}$. The linear approximation of $f(x, y)$ at $(2, 1)$ is $f(2 + \Delta x, 1 + \Delta y) \approx 3 - \frac{2}{3}\Delta x - \frac{7}{3}\Delta y$. To approximate $f(1.95, 1.08)$ take $\Delta x = -.05$ and $\Delta y = .08$. The result is $3 + \frac{2}{3}(.05) - \frac{7}{3}(.08)$**.

(12) 7. a) If $z = f(x^2 - 3y)$, show that $3\frac{\partial z}{\partial x} + 2x\frac{\partial z}{\partial y} = 0$.

Answer $z_x = f'(x^2 - 3y)(2x)$; $z_y = f'(x^2 - 3y)(-3)$. Therefore $3\frac{\partial z}{\partial x} + 2x\frac{\partial z}{\partial y} = 3f'(x^2 - 3y)(2x) + (2x)f'(x^2 - 3y)(-3) = (6x - 6x)f'(x^2 - 3y) = 0$.

b) If $z = f(x^2 - 3y)$, then $\frac{\partial^2 z}{\partial x^2} = A(x)f''(x^2 - 3y) + B(x)f'(x^2 - 3y)$ where $A(x)$ and $B(x)$ are simple functions of x alone or constants. What are $A(x)$ and $B(x)$?

Answer Let's $\frac{\partial}{\partial x}$ the equation $z_x = f'(x^2 - 3y)(2x)$. We need to apply the product rule and the chain rule. The result is $f''(x^2 - 3y)(2x)^2 + f'(x^2 - 3y)2$ so that $A(x) = (2x)^2$ or $4x^2$ and $B(x) = 2$.

(12) 8. Suppose that $F(x, y, z) = x^2 + 3yz$ and $p = (-3, 2, -1)$.

a) Find the maximum directional derivative of F at p and write a unit vector pointing in the direction this maximum value occurs.

Answer $\nabla F(x, y, z) = \langle 2x, 3z, 3y\rangle$ so that $\nabla F(-3, 2, -1) = \langle 2(-3), 3(-1), 3 \cdot (2)\rangle = \langle -6, -3, 6\rangle$. The maximum directional derivative is $|\nabla F(-3, 2, -1)| = \sqrt{(-6)^2 + (-3)^2 + 6^2} = \sqrt{81} = 9$. A unit vector in the desired direction is $\frac{1}{|\nabla F(-3,2,-1)|}\nabla F(-3, 2, -1)$ which is $\left\langle -\frac{6}{9}, -\frac{3}{9}, \frac{6}{9}\right\rangle$.

b) Suppose $C = F(-3, 2, -1)$. Compute C and write an equation for the plane tangent to the surface $F(x, y, z) = C$ at the point p.

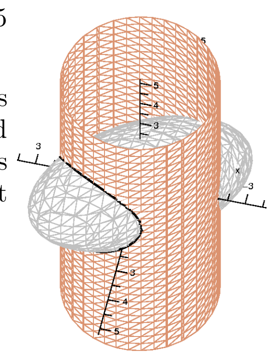
Answer $F(-3, 2, -1) = (-3)^2 - 3 \cdot 2(-1) = 3$, so $C = 6$. $\nabla F(-3, 2, -1)$ gives a normal vector, so an equation for a plane tangent to $F(x, y, z) = 3$ at p is $-6(x - 3) - 3(y - 2) + 6(z + 1) = 0$.

(12) 9. a) If $f(x, y, z) = x^2 + y^2$, compute $\nabla f(x, y, z)$. What are $f(2, 1, 2)$ and $\nabla f(2, 1, 2)$?

Answer $\nabla f(x, y, z) = \langle 2x, 2y, 0\rangle$. $f(2, 1, 2) = 2^2 + 1^1 = 5$ and $\nabla f(2, 1, 2) = \langle 4, 2, 0\rangle$.

b) If $g(x, y, z) = x^2 + y^2 + z^2 - xy - yz$, compute $\nabla g(x, y, z)$. What are $g(2, 1, 2)$ and $\nabla g(2, 1, 2)$?

Answer $\nabla g(x, y, z) = \langle 2x - y, 2y - x - z, 2z - y\rangle$. $g(2, 1, 2) = 2^2 + 1^1 + 2^2 - 2 - 2 = 5$ and $\nabla g(2, 1, 2) = \langle 3, -2, 3\rangle$.



c) The point $(2, 1, 2)$ is on both the surface $x^2 + y^2 = 5$, a circular cylinder whose axis of symmetry is the z-axis, and the surface $x^2 + y^2 + z^2 - xy - yz = 5$, an ellipsoid tilted with respect to the coordinate axes. The surfaces intersect in a curve. The surfaces and the curve are shown in the picture to the right. Find a vector tangent to that curve at $(2, 1, 2)$. Your answers to a) and b) can be used here.

Answer A vector tangent to the curve is perpendicular to both surface normals. $\nabla f(2, 1, 2) \times \nabla g(2, 1, 2)$ is such a vector, so det $\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 0 \\ 3 & -2 & 3 \end{pmatrix} =$ $6\mathbf{i} - 12\mathbf{j} - 14\mathbf{k}$, one valid answer.

** `Maple` reports the "true value" is about 2.834 and the approximation is about 2.847, if this matters.