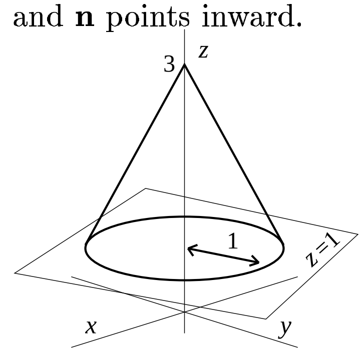
Review problems in vector calculus for sections 5–10 of Math 251, spring 2006

The course web page will have answers to these questions, a draft formula sheet, the schedule of some review sessions, and other information about the final exam.

1. Compute the following integrals. Use Green's Theorem, Stokes' Theorem or the Divergence Theorem wherever they are helpful.
a) $\iint_D xy\,dA$, where D is the triangle in the xy-plane with vertices (0,0), (2,0), and (0,2).
b) $\int_C \mathbf{F}\cdot d\mathbf{s}$, where $\mathbf{F}(x,y) = x^2\mathbf{i} - xy\mathbf{j}$ and C is the segment of the parabola $y = x^2$ beginning at $(-1,1)$ and ending at $(1,1)$.
c) $\iint_S (x^3\mathbf{i} + y^3\mathbf{j} + \cos(xy)\mathbf{k})\cdot\mathbf{n}\,dS$, where S is the unit sphere and $\mathbf{n}$ points inward.
d) $\iint_S z^2\,dS$, where S is the surface $z = \sqrt{x^2+y^2}$, $0 \le z \le 1$.
e) $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$, where $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz^2\mathbf{j} + z^3\mathbf{k}$ and S is the lateral surface of the cone as shown, with $\mathbf{n}$ pointing outward.
Note I didn't get the picture from the instructor who gave me this problem, so I drew what I hope is a suitable picture here. I think the problem can be done with a standard result of vector calculus (and maybe even by a heroic direct computation!).



2. Compute $\int_C e^x \sin z\,dx + y^2\,dy + e^x\cos z\,dz$, where C is the oriented curve $\mathbf{x}(t) = (\cos t)^3\,\mathbf{i} + (\sin t)^3\,\mathbf{j} + t\mathbf{k}$, $0 \le t \le \pi/2$. First find a potential function.

3. A fluid has density 1500 and velocity field $\mathbf{v} = -y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k}$. Find the flow outward through the sphere $x^2 + y^2 + z^2 = 25$.

4. Sketch the region E contained between the surfaces $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$ and let S be the boundary of E.
a) Find the volume of E.
b) Let $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Find $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ where $\mathbf{n}$ is the outer normal to S.

5. Find the total flux upward through the upper hemisphere ($z \ge 0$) of the sphere $x^2+y^2+z^2 = a^2$ of the vector field $\mathbf{T}(x,y,z) = \left(\frac{x^3}{3}\right)\mathbf{i} + \left(yz^2 + e^{\sqrt{zx}}\right)\mathbf{j} + \left(zy^2 + y + 2 + \sin\left(x^3\right)\right)\mathbf{k}$.
Note Don't compute this directly! Use the Divergence Theorem on some "simple" solid to change the desired computation to the computation of a triple integral and a much simpler flux integral. Evaluate those integrals, taking as much advantage of symmetry as possible.

6. Suppose $\mathbf{F} = -2xz\mathbf{i} + y^2\mathbf{k}$. **Note** There is *no* $\mathbf{j}$ component in $\mathbf{F}$.
a) Compute curl $\mathbf{F}$.
b) Compute the outward unit normal $\mathbf{n}$ for the sphere $x^2 + y^2 + z^2 = a^2$.
c) If R is any region on the sphere $x^2 + y^2 + z^2 = a^2$, verify that $\iint_R (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = 0$.
d) Suppose C is a simple closed curve on the sphere $x^2 + y^2 + z^2 = a^2$. Show that the line integral $\int_C -2xz\,dx + y^2\,dz = 0$. **Comment** Do *not* attempt a direct computation! Use c) and one of the big theorems.

Please also look at the previous exams and review material in our course.
If I thought it was important then, I probably still think it is important!