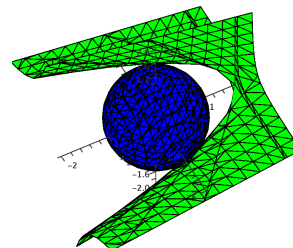


Here are detailed answers to version A. Brief answers to version B are at the end.

- (12) 1. Theoretical results imply that $x + 3yz$ has a maximum and a minimum on the sphere $x^2 + y^2 + z^2 = 1$. Use Lagrange multipliers to find these maximum and minimum values.

Answer Suppose $f(x, y, z) = x + 3yz$ and $g(x, y, z) = x^2 + y^2 + z^2$. Then $\nabla f = \langle 1, 3z, 3y \rangle$ and $\nabla g = \langle 2x, 2y, 2z \rangle$ so that the Lagrange multiplier equations are (including the constraint equation) $1 = \lambda(2x)$, $3z = \lambda(2y)$, $3y = \lambda(2z)$, and $1 = x^2 + y^2 + z^2$. Now we solve these equations. The first equation immediately tells us that neither x nor λ can be 0. The second and third equations imply if $y = 0$ then $z = 0$ (and *vice versa*). If both y and z are 0 then $1 = x^2 + y^2 + z^2$ shows that $x = \pm 1$ so the objective function $x + 3yz$ is ± 1 . Can we do better (get larger and smaller values of the objective function)? If no variable is 0, the second and third equations can be rewritten as $\frac{3z}{2y} = \lambda$ and $\frac{3y}{2z} = \lambda$ so $\frac{3z}{2y} = \frac{3y}{2z}$ and $z^2 = y^2$. Then λ must be $\pm \frac{3}{2}$ itself since $y = \pm z$. So $1 = \lambda(2x)$ implies that $x = \pm \frac{1}{\lambda}$ and $1 = x^2 + y^2 + z^2$ gives $1 = (\frac{1}{\lambda})^2 + 2z^2$ and $z = \pm \sqrt{\frac{4}{9}}$ and $y = \pm \sqrt{\frac{4}{9}}$. If $\lambda > 0$, then $x > 0$, and y and z have the same sign. If $\lambda < 0$, then $x < 0$, and y and z have opposite signs. With all + signs, $x + 3yz$ becomes $\frac{1}{3} + 3(\frac{4}{9}) = \frac{5}{3}$. With - signs for x and y and + for z we get $-\frac{5}{3}$. These are the actual minimum and maximum values. To the right is a picture of the ball together with the surface $x + 3yz = \frac{5}{3}$. They do indeed seem to be tangent (and at two points, the other corresponding to minus signs on y and z) just as the Lagrange multiplier method suggests.



- (12) 2. Suppose $I = \int_0^2 \int_{x^2}^5 xy \, dy \, dx$.

a) Compute I .

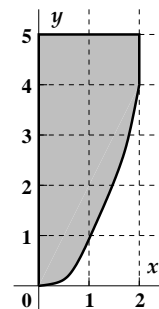
Answer $\int_0^2 \int_{x^2}^5 xy \, dy \, dx = \int_0^2 \frac{xy^2}{2} \Big|_{y=x^2}^{y=5} dx = \int_0^2 \frac{25}{2}x - \frac{x^5}{2} dx = \frac{25}{4}x^2 - \frac{x^6}{12} \Big|_{x=0}^{x=2} = 25 - \frac{2^6}{12} = 25 - \frac{16}{3} = \frac{59}{3}$.

b) Use the axes to the right to sketch the region of integration for I .

Answer Shown to the right.

c) Write I as a sum of one or more $dx \, dy$ integrals. You do not need to compute the result!

Answer $\int_4^5 \int_0^2 xy \, dx \, dy + \int_0^4 \int_0^{\sqrt{y}} xy \, dx \, dy$.

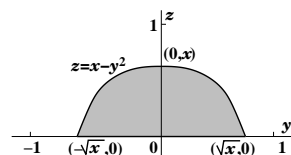
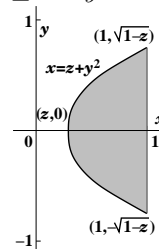


- (12) 3. The coordinates (x, y, z) of points in a solid object A in \mathbb{R}^3 satisfy the inequalities $0 \leq z \leq x - y^2$ and $0 \leq x \leq 1$. Compute the triple integral of 1 over the object A . (This is the volume of A .)

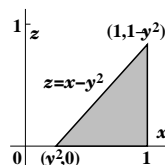
Note Four views of the object were given.

Answer A picture of a slice with z fixed is to the right, where $0 < z < 1$. The volume is

$$\int_0^1 \int_z^1 \int_{-\sqrt{x-z}}^{\sqrt{x-z}} 1 \, dy \, dx \, dz = \int_0^1 \int_z^1 y \Big|_{y=-\sqrt{x-z}}^{y=\sqrt{x-z}} dx \, dz = \int_0^1 \int_z^1 2\sqrt{x-z} \, dx \, dz = \int_0^1 \frac{4}{3}(x-z)^{3/2} \Big|_{x=z}^{x=1} dz = \int_0^1 \frac{4}{3}(1-z)^{3/2} dz = -\frac{8}{15}(1-z)^{5/2} \Big|_{z=0}^{z=1} = \frac{8}{15}.$$



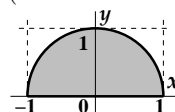
A picture of a slice with x fixed is to the left, where $0 < x < 1$ so another answer is $\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{x-y^2} 1 \, dz \, dy \, dx$.



$\int_{-1}^1 \int_0^{1-y^2} \int_{z+y^2}^1 1 \, dx \, dz \, dy$ is another answer. A slice with y fixed is shown to the left, where $-1 < y < 1$.

- (12) 4. Compute $\iint_D e^{-x^2-y^2} dA$ where D is the region in the plane which is inside the unit circle (the circle with center at $(0, 0)$ and radius 1) and also inside the upper half plane (where $y \geq 0$).

Answer A picture of the region is to the right. It is *friendly* to polar coordinates. The integral is $\int_0^\pi \int_0^1 e^{-r^2} r \, dr \, d\theta = \int_0^\pi -\frac{1}{2}e^{-r^2} \Big|_{r=0}^{r=1} d\theta = \int_0^\pi (-\frac{1}{2}e^{-1} + \frac{1}{2}) d\theta = \frac{\pi}{2}(1 - \frac{1}{e})$.



(12) 5. Express in cylindrical coordinates and evaluate: $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} z \, dz \, dy \, dx$.

Answer $z \, dz \, dy \, dx$ becomes $zr \, dr \, d\theta \, dz$. The boundary $z = \sqrt{x^2+y^2}$ becomes $z = r$. The boundary $y = \sqrt{1-x^2}$ along with the knowledge that x goes from 0 to 1 describes the part of the unit disc in the first quadrant (similar to the setup of the previous problem) because $y = \sqrt{1-x^2}$ is part of $x^2+y^2=1$, and $\sqrt{\quad}$ is always *non-negative* square root. Since the z boundary description involves r , I will change the order from $zr \, dr \, d\theta \, dz$ to $zr \, dz \, d\theta \, dr$. The triple integral becomes $\int_0^1 \int_0^{\pi/2} \int_0^r zr \, dz \, d\theta \, dr = \int_0^1 \int_0^{\pi/2} \frac{r^2}{2} \Big|_{z=0}^{z=r} d\theta \, dr = \int_0^1 \int_0^{\pi/2} \frac{r^3}{2} d\theta \, dr = \int_0^1 \frac{r^3}{2} \theta \Big|_{\theta=0}^{\theta=\pi/2} dr = \int_0^1 \frac{\pi r^3}{4} dr = \frac{\pi r^4}{16} \Big|_{r=0}^{r=1} = \frac{\pi}{16}$

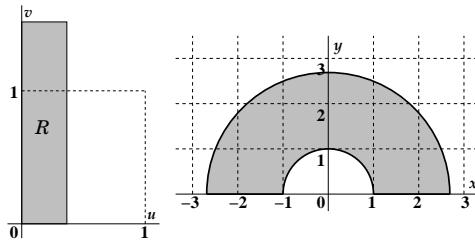
(12) 6. Use spherical coordinates to calculate the triple integral of $f(x,y,z) = x^2 + y^2 + z^2$ over the region $1 \leq x^2 + y^2 + z^2 \leq 4$.

Answer $\rho^2 = x^2 + y^2 + z^2$ so the region is just $1 \leq \rho \leq 2$ with all θ 's ($0 \leq \theta \leq 2\pi$) and all ϕ 's ($0 \leq \phi \leq \pi$). The integrand in spherical coordinates is ρ^2 . So the desired triple integral is $\int_0^{2\pi} \int_0^\pi \int_1^2 \rho^2 (\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi \int_1^2 \rho^4 (\sin \phi) \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi \frac{\rho^5}{5} (\sin \phi) \Big|_{\rho=1}^{\rho=2} d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi \frac{31}{5} (\sin \phi) \, d\phi \, d\theta = \int_0^{2\pi} \frac{31}{5} (-\cos \phi) \Big|_{\phi=0}^{\phi=\pi} d\theta = \int_0^{2\pi} \frac{31}{5} (-\cos \pi - (-\cos 0)) \Big|_{\phi=0}^{\phi=\pi} d\theta = \int_0^{2\pi} 2 \left(\frac{31}{5}\right) d\theta = \frac{62}{5} \theta \Big|_{\theta=0}^{\theta=2\pi} = \frac{124\pi}{5}$, although $\frac{(2^5-1)4\pi}{5}$ is simpler.

(12) 7. This problem is about the transformation $\begin{cases} x = e^{3u} \cos(2v) \\ y = e^{3u} \sin(2v) \end{cases}$.

a) Compute the Jacobian of this transformation. The result should be $6e^{6u}$ but you must show the details of the computation. **Answer** We need $\det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \det \begin{pmatrix} 3e^{3u} \cos(2v) & -2e^{3u} \sin(2v) \\ 3e^{3u} \sin(2v) & 2e^{3u} \cos(2v) \end{pmatrix} = (3e^{3u} \cos(2v))(2e^{3u} \cos(2v)) - (-2e^{3u} \sin(2v))(3e^{3u} \sin(2v)) = 6e^{6u}(\cos(2v))^2 + 6e^{6u}(\sin(2v))^2 = 6e^{6u}((\cos(2v))^2 + (\sin(2v))^2)$ and this is $6e^{6u}$.

b) Suppose R is the region in the uv -plane determined by $u = 0$, $u = \frac{1}{3}$, $v = 0$, and $v = \frac{\pi}{2}$ as shown on the coordinate axes below and to the left. Sketch the image region using this transformation in the xy -plane below and to the right.



(16) 8. a) Compute $\int_C x \, dx + y^2 \, dy$ if C is a quarter circle centered at $(0,0)$ from $(1,0)$ to $(0,1)$ followed by a line segment from $(0,1)$ to $(3,1)$. C is shown in a diagram to the right. You may need more than one integral!

Answer From $(0,0)$ to $(0,1)$ use $x = \cos t$ and $y = \sin t$ so $dx = -\sin t \, dt$, $dy = \cos t \, dt$, and $0 \leq t \leq \frac{\pi}{2}$. The integral over that portion of the curve is $\int_0^{\pi/2} -(\cos t)(\sin t) + (\sin t)^2(\cos t) \, dt = -\frac{(\sin t)^2}{2} + \frac{(\sin t)^3}{3} \Big|_0^{\pi/2} = -\frac{1}{2} + \frac{1}{3} = -\frac{1}{6}$. For the line segment, $x = t$ and $y = 1$ so $dx = dt$ and $dy = 0 \, dt$, and $0 \leq t \leq 3$. So this integral is $\int_0^3 t \, dt = \frac{9}{2}$. The total integral is therefore $\frac{9}{2} - \frac{1}{6} = \frac{13}{3}$.

Another method $\varphi(x,y) = \frac{x^2}{2} + \frac{y^3}{3}$ is a potential for $x\mathbf{i} + y^2\mathbf{j}$ (verify this by checking φ using partial differentiation). Then the integral is $\varphi(\text{The end}) - \varphi(\text{The start}) = \varphi(3,1) - \varphi(1,0) = \left(\frac{3^2}{2} + \frac{1}{3}\right) - \frac{1}{2} = \frac{13}{3}$.

b) Suppose \mathbf{F} is the vector field $(x + 5y^2)\mathbf{i} + (Axy)\mathbf{j}$, where A is a constant. There is one value of A for which this vector field is a gradient vector field. Find that value of A . Then find all potentials of \mathbf{F} , using that value of A . **Answer** $\frac{\partial}{\partial y}$ of $x + 5y^2$ is $10y$, and $\frac{\partial}{\partial x}$ of Axy is Ay , so the desired value of A is 10 . Now $\int x + 5y^2 \, dx = \frac{x^2}{2} + 5xy^2 + C_1(y)$ and $\int 10xy \, dy = 5xy^2 + C_2(x)$ where $C_1(y)$ and $C_2(x)$ are unknown functions. But inspection of the two descriptions of the potential tells me that the most general potential of \mathbf{F} is $\frac{x^2}{2} + 5xy^2 + C$ for any constant C .

Brief answers to version B

1. $x+5yz$ has answer $\pm \frac{13}{5}$. 2. $I = \int_0^2 \int_{x^3}^9 xy \, dy \, dx = 65$. The first partial integration has answer $\frac{81x}{2} - \frac{x^7}{2}$. The graph is similar, and c)'s answer is $\int_8^9 \int_0^2 xy \, dx \, dy + \int_0^8 \int_0^{y^{1/3}} xy \, dx \, dy$. 3. The same. 4. $\int_\pi^{2\pi} \int_0^1 e^{-r^2} r \, dr \, d\theta$ with the same answer. 5. The same. 6. The answer is $\frac{(3^5-1)4\pi}{5} = \frac{968\pi}{5}$. 7. a) A similar computation gives the stated answer. The graph in b) is the same. 8. a) Much the same parameterizations can be used. The answer is $\frac{16}{2} - \frac{1}{6} = 8 - \frac{1}{6} = \frac{47}{6}$. In b), $A=6$ and the potential is $\frac{x^2}{2} + 3xy^2 + C$ for any constant C .