Sources of the 403:02 final exam problems

These problems are mildly edited from the qualifying exams of the universities indicated.

Oklahoma 1. Let $u(x,y) = x^3 + x - 3xy^2$.
a) Show that $u(x,y)$ is harmonic on the complex plane.
b) Find all harmonic conjugates of $u(x,y)$.
c) Find an analytic function $f(z)$ so that $\operatorname{Re} f = u$ and find the Taylor series of $f(z)$ about the point 0.

Purdue 2. Construct a one-to-one analytic map from $Q = \{z : |z| < 1 \text{ and } \operatorname{Im} z > 0\}$ (the upper half of the unit disc) onto the unit disc, $U = \{z : |z| < 1\}$. Show how the boundary of Q is mapped to the boundary of U.

Temple 3. Use the Residue Theorem to compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+2x+2)^2}\, dx$.

Berkeley 4. Prove that for any fixed complex number ζ, $\displaystyle\frac{1}{2\pi}\int_0^{2\pi} e^{2\zeta\cos\theta}\, d\theta = \sum_{n=0}^{\infty}\left(\frac{\zeta^n}{n!}\right)^2$.
Hint Use the "dictionary" to convert this into a line integral and then use infinite series.

Temple 5. Let $\mathbb{R}^- = \{x \text{ is real and } x \le 0\}$. Suppose $f(z)$ is analytic in $\mathbb{C} \setminus R^-$, and $f(x) = x^x$ for real positive x. Find $f(i)$ and $f(-i)$.
Scoring 10 points for the values, and 10 points for explanation.

Temple 6. Show that if $f(z)$ is analytic at a and $g(z) = \dfrac{f(z) + af'(a) - zf'(a) - f(a)}{(z-a)^2}$ then $g(z)$ has a removable singularity at $z = a$. What value should be given to $g(a)$ so that the extended function is analytic at a?

Johns Hopkins 7. Find the number of zeros of the function $f(z) = 2z^5 + 8z - 1$ in the annulus $1 < |z| < 2$.

Missouri 8. Suppose $|f(z)| \le K$ on the circumference of a square whose side length is L, and let z_0 be the center of the square. If $f(z)$ is analytic in a domain containing the square, show that $|f'(z_0)| \le \dfrac{8K}{\pi L}$.
Hint Use an integral formula.

Penn State 9. Prove that if $f(z)$ is an entire function and if there is a positive number M so that $\operatorname{Re} f(z) \le M$ for all z, then $f(z)$ is constant.

Florida State 10. Suppose the *Bernoulli polynomials* are defined by the Taylor expansion $\dfrac{ze^{wz}}{e^z - 1} = \displaystyle\sum_{k=0}^{\infty} \frac{B_k(w)}{k!} z^k$. Find the first three Bernoulli polynomials, $B_0(w)$, $B_1(w)$, and $B_2(w)$.