Rudin Chapter 5, problems 22 and 23

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Suppose $f$ is a real function on $(-\infty, \infty)$. We say $x$ is a fixed point of $f$ if $f(x) = x$.

a) If $f$ is differentiable and $f'(t) \neq 1$ for every real $t$, prove that $f$ has at most one fixed point.

Proof: Suppose that $f$ satisfies the hypotheses, but, for distinct real numbers $a$ and $b$ with $a < b$, $f(a) = a$ and $f(b) = b$. Then, by the Mean Value Theorem, there exists $r \in (a, b)$ such that $f'(r) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1$, contradiction. Hence, $f$ has at most one fixed point.

b) Show that the function defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real $t$.

First, $f'(t) = 1 - e^t(1 + e^t)^{-2}$. To see that $f'(t)$ is bounded between 0 and 1, consider the following implications:

$$0 < e^t \Rightarrow 1 < 1 + e^t \Rightarrow e^t < (1 + e^t)^2 \Rightarrow -1 < -e^t(1 + e^t)^{-2} < 0 \Rightarrow 0 < f'(t) < 1.$$  

If $f$ did have a fixed point, then $f(t) = t \Rightarrow (1 + e^t)^{-1} = 0$, which is a contradiction since the left-hand side is positive for all $t$.

This shows that boundedness of $f'$ less than 1 does not guarantee a fixed point.

c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real $t$, then a fixed point $x$ of $f$ exists, and $x = \lim x_n$, where $x_1$ is an arbitrary real number and $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$.

Lemma: If, in addition to the above hypotheses, $f(t) > t$ for all $t$, then $f(t) \leq g(t) = At + f(0)$ for all $t \geq 0$. Similarly, if $f(t) < t$ for all $t$, then $f(t) \geq h(t) = At + f(0)$ for all $t \leq 0$.

Proof of lemma: Note that $f(0) = g(0)$, and that $g$ is a differentiable function with a constant derivative, namely $A$. Let $y \in (0, \infty)$ be arbitrary. Then, by
the Mean Value Theorem, there exists $r \in (0, y)$ such that $f'(r) = \frac{f(y) - f(0)}{y}$. Since $f'(r) \leq g'(r) = A$, we have $\frac{f(y) - f(0)}{y} \leq \frac{g(y) - g(0)}{y} \Rightarrow f(y) \leq g(y)$. Hence, $f(t) \leq g(t)$ for all $t \geq 0$.

Take an arbitrary $z \in (0, \infty)$. Then, by the Mean Value Theorem, there is a point $s \in (-z, 0)$ such that $f'(s) = \frac{f(0) - f(-z)}{z}$. As above, we find that $\frac{f(0) - f(-z)}{z} \leq \frac{h(0) - h(-z)}{z} \Rightarrow -f(-z) \leq -h(-z) \Rightarrow h(-z) \leq f(-z)$. Since $z$ was arbitrarily chosen, $h(t) \leq f(t)$ for all $t \leq 0$.

This is an image of the function $f(t) = \log(t + 8) + 5$, for which $f'(t) \leq 1/8$ for $t \geq 0$. Clearly, it cannot always be bounded between $t$ and $(1/8)t + f(0)$.

Proof of c): Recall that $f$ can have at most one fixed point. Suppose $f$ has none. Then, for all $t$, $f(t) > t$ or $f(t) < t$. If it is always the case that $f(t) > t$, then by the lemma we can conclude that $t < At + f(0)$ for all positive $t$; this is a contradiction, since $t = At + f(0)$ at $t = \frac{f(0)}{1-A}$. If $f(t) < t$ for all $t$, we have by the lemma that $At + f(0) < t$ for all negative $t$; again, this is a contradiction.

To see the following argument more clearly, we define $g(t) = f(t) - t$. We have assumed that $g(t) \neq 0$ for all $t$, and that neither $g(t) > 0$ or $g(t) < 0$ can hold for all $t$. Hence, there exist $a$ and $b$ with $a < b$ such that $g(a) > 0$ and $g(b) < 0$. By the Intermediate Value Theorem, there exists $r \in (a, b)$ such that $g(r) = 0$, contradicting that $g(t) \neq 0$ for all $t$. Hence, there is some point $x$ such that $f(x) = x$.

Finally, we show that $(x_n)$ converges to $x$. First, note that $|x - x_2| = |f(x) - f(x_1)|$. Then, by the Mean Value Theorem, there is some number $r$
between $x$ and $x_1$ such that $|f'(r)| = \frac{|f(x) - f(x_1)|}{|x - x_1|}$. As $|f'(r)| \leq A$, $|x - x_2| = |f(x) - f(x_1)| \leq A|x - x_1|$. Suppose that for some natural number $n \geq 1$ that $|x - x_{n+1}| = |f(x) - f(x_n)| \leq A^n|x - x_1|$. Then, applying the Mean Value Theorem as we did in the case $n = 1$, we find that $|f(x) - f(x_{n_1})| \leq A|x - x_{n+1}| \leq A \cdot A^n|x - x_1| = A^{n+1}|x - x_1|$. So, for every $n$, $|x - x_n| \leq A^{n-1}|x - x_1|$. By Bernoulli’s inequality, $0 \leq \lim_{n \to \infty} |x - x_n| \leq \lim_{n \to \infty} A^{n-1}|x - x_1| = 0$, and therefore $x_n \to x$.

**d)** Here is a picture of the algorithm converging. The path $(x_1, x_2) \to (x_2, x_3) \to (x_3, x_4) \to \ldots$ is represented by the zig-zag lines. The function whose fixed point is being found is $-\sqrt{x} + 2$, and I chose $x_1 = 3$.

![Graph of $f(x) = \frac{x^3 + 1}{3}$](image)

As an example of how this can be applied, consider the function

$$f(x) = \frac{x^3 + 1}{3}$$

which has three fixed points. The fixed points $\alpha$, $\beta$, and $\gamma$ satisfy

$$-2 < \alpha < -1, \quad 0 < \beta < 1, \quad 1 < \gamma < 2$$

Suppose that we have a sequence defined as in 22c.

**a)** If $x_1 < \alpha$, then $x_n \to -\infty$ as $n \to \infty$.

If $x_n < \alpha$, then $x_{n+1} = \frac{x_n^3 + 1}{3} < \alpha$. Hence, $x_n \in (-\infty, \alpha)$ for all $n$. It follows that $f'(x_n) > f'(\alpha) = \alpha^2 > 1$ for each $n$. By the Mean Value Theorem, for each $n$ there exists a point $c_n \in (x_n, \alpha)$ such that $f'(c_n) = \frac{f(\alpha) - f(x_n)}{\alpha - x_n}$.
Hence, \( |\alpha - x_{n+1}| = |f(\alpha) - f(x_n)| = f'(c_n) |\alpha - x_n| > \alpha^2 |\alpha - x_n| \). Since \( |\alpha - x_2| > \alpha^2 |\alpha - x_1| \), the relation \( |\alpha - x_n| > \alpha^{2(n-1)} |\alpha - x_1| \) holds for all \( n \geq 2 \).

Since \( \lim_{n \to \infty} \alpha^{2(n-1)} |\alpha - x_1| = \infty \), \( \lim_{n \to \infty} |\alpha - x_n| = \infty \). Thus, \( x_n \to -\infty \).

b) If \( \alpha < x_1 < \gamma \), then \( x_n \to \beta \) as \( n \to \infty \). [Note: \( f'(x) = x^2 \).]

The cases we consider are \( x_1 \in (\alpha, -1) \), \( x_1 \in [-1, -1/2) \), \( x_1 \in [-1/2, 1/2] \), \( x_1 \in (1/2, 1) \), and \( x_1 \in (1, \gamma) \).

Case 1: First, suppose \( x_1 \in [-1, -1/2) \). Take \( t \in [-1/2, 1/2] \), \( f'(t) \leq 1/4 < 1 \). Then, \( |\beta - x_2| = |f(\beta) - f(x_1)| \leq (1/4)|\beta - x_1| \). Since \( x_2 \) is closer to \( \beta \) than \( x_1 \), \( x_2 \in [-1/2, 1/2] \). It can be shown by an argument similar to the one given in 22c that \( x_n \to \beta \).

Case 2: Next, suppose that \( x_1 \in [-1, -1/2) \). Since \( -1 \leq x_1 < -1/2 \), \( -1 \leq x_1^3 < (1/2)^3 \) \( \Rightarrow 0 \leq \frac{2x_1^3}{3} = x_2 < 7/24 < 1/2 \). Thus, \( x_2 \in [-1/2, 1/2] \), and it follows from case 1 that \( x_n \to \beta \).

Case 3: It is easy to see that if \( x_n \in (\alpha, -1) \) then \( x_{n+1} \leq 0 \), and therefore \( x_{n+1} \not\in (1/2, \gamma) \). Thus, \( x_{n+1} \in [-1, 1/2] \), in which case convergence to \( \beta \) follows, or \( x_{n+1} \in (\alpha, -1) \).

So, suppose that \( x_n \in (\alpha, -1) \) for all \( n \). If \( x_1 \in (\alpha, -1) \), then \( \alpha < x_1 \Rightarrow \alpha < \frac{x_1^3 + 1}{3} = x_2 \). An inductive argument shows that \( x_n > \alpha \) for all \( n \).

Notice that for \( t \in (\alpha, -1) \) we have that \( f'(t) > 1 \). Consider \( |\alpha - x_{n+1}| = |f(\alpha) - f(x_n)| \). By the Mean Value Theorem, there exists \( c_n \in (\alpha, x_n) \) such that \( f'(c_n) = \frac{f(\alpha) - f(x_n)}{\alpha - x_n} \). Thus, \( |\alpha - x_{n+1}| > |\alpha - x_n| \) for all \( n \). Therefore, \( (x_n) \) is monotonically increasing, and is bounded above by \( -1 \). Hence, \( (x_n) \) must converge to some number \( x \in (\alpha, -1) \). From continuity of \( f \) and the fact that \( (x_n) \) and \( (f(x_n)) \) have the same limit, it follows that \( f(x) = x \), so \( x \) is a fixed point. But this is a contradiction, since there is no fixed point in \( (\alpha, -1) \). Hence, the sequence is not bounded above by \( -1 \), and therefore \( x_n \to \beta \).

Case 4: The argument for when \( x_1 \in (1/2, 1] \) is similar to case 2.

Case 5: The argument for when \( x_1 \in (1, \gamma) \) is similar to case 3.

c) If \( \gamma < x_1 \), then \( x_n \to \infty \) as \( n \to \infty \).

The result follows from the same argument used in part a).

Here is an image of the fixed point iteration algorithm converging to \( \beta \). I chose \( x_1 = -3/2 \).
\( \frac{x^3 + 1}{3} \)