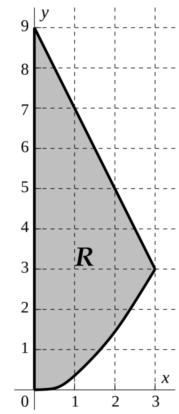
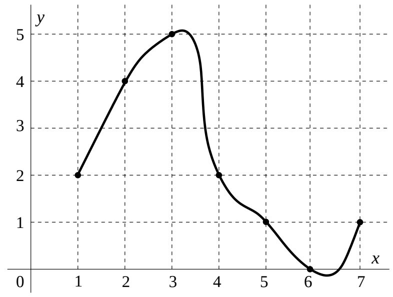
640:152: $\left\{\begin{matrix}\mathbf{1,2,3,}\\ \mathbf{6,7,8,9}\end{matrix}\right\}$ **Answers to version A of the First Exam** **2/21/2009**

Here are answers that would earn full credit. Other methods may also be valid.



(16) 1. A region R in the first quadrant of the plane is bounded by the y-axis, $y = \frac{1}{3}x^2$, and $y = 9 - 2x$.
a) Sketch R on the coordinate axes displayed to the right. Be sure to label the region R.
Answer Shown to the right.
b) The region R is revolved around the y-axis. Compute the volume of the resulting solid.
Answer Using shells, the volume is $\int_0^3 2\pi x\left(9 - 2x - \frac{1}{3}x^2\right)dx = 2\pi\int_0^3 9x - 2x - \frac{1}{3}x^3\,dx = 2\pi\left(\frac{9}{2}x^2 - \frac{2}{3}x^3 - \frac{1}{12}x^4\right)\Big|_0^3 = 2\pi\left(\frac{9}{2}3^2 - 2(3^2) - \frac{1}{12}3^4\right)$. The next page has an alternate answer.

(9) 2. The graph of a function $f(x)$ on the interval $[1,7]$ is shown below.
a) Write the Simpson's Rule approximation for $\int_1^7 f(x)\,dx$ with $n = 6$ subintervals. No arithmetic needs to be done, *but* all function evaluations should be performed (therefore an expression like $f(3)$ should not appear in the final answer).
Answer Δx is 1. The answer is $\frac{1}{3}\big(f(1) + 4f(2) + 2f(3) + 4f(4) + 2f(5) + 4f(6) + f(7)\big)$ which is $\frac{1}{3}\big(2 + 4\cdot 4 + 2\cdot 5 + 4\cdot 2 + 2\cdot 1 + 4\cdot 0 + 1\big)$.
b) Suppose that for all x in $[1,7]$, $|f''(x)| \le 25$ and $|f^{(4)}(x)| \le 50$. Find a bound for the possible error of the approximation written in part a). Again, the final answer need not be "simplified" arithmetically.
Answer The Simpson's Rule error bound uses K_4. Since $K_4 = 50$, $N = 6$, $b = 7$, and $a = 1$, the error is at most $\frac{50(7-1)^5}{180\cdot 6^4}$.



Graph of $y = f(x)$

(13) 3. Verify that $\int_0^1 \frac{5x^2-3x+4}{(x+1)(x^2+1)}\,dx = \frac{11}{2}\ln(2) - \frac{1}{2}\pi$.
Answer Use partial fractions.
$\frac{5x^2-3x+4}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)(x+1)}{x(x^2+1)}$ implies that $5x^2 - 3x + 4 = A(x^2+1) + (Bx+C)(x+1)$. Then $x = -1$ gives $A = \frac{12}{2} = 6$. Considering the x^2 coefficients shows that $5 = A+B$ so $B = -1$. Then the x coefficients tell us that $-3 = B+C$ and $C = -2$. Therefore we need to antidifferentiate $\frac{6}{x+1} + \frac{-x-2}{x^2+1}$. The first term is just $6\ln(x+1)$ and the second term is $-\frac{1}{2}\ln(x^2+1) - 2\arctan(x)$ (for $\int \frac{-x}{x^2+1}\,dx$, use the substitution $u = x^2+1$). Now the definite integral: $6\ln(x+1) - \frac{1}{2}\ln(x^2+1) - 2\arctan(x)\Big|_0^1 = 6\ln(2) - \frac{1}{2}\ln(2) - 2\cdot\frac{\pi}{4}$ given by the $x = 1$ terms. The $x = 0$ terms are all 0. This result equals the suggested answer.

(12) 4. Compute $\int_{e^2}^{e^3} (\ln x)^2\,dx$.
Answer First we compute $\int (\ln x)^2\,dx$ using integration by parts. Here $u = (\ln x)^2$ and $dv = dx$ so that $du = 2\ln x\left(\frac{1}{x}\right)dx$ and $v = x$. Then $uv - \int v\,du$ becomes $x(\ln x)^2 - 2\int \ln x\,dx$. Now we compute $\int \ln x\,dx$ again using integration by parts with $u = \ln x$ and $dv = dx$ so $du = \frac{1}{x}dx$ and $v = x$. The result, $uv - \int v\,du$, is $x\ln x - \int 1\,dx = x\ln x - x + C$. Now put things together, but be careful about signs and 2's. An antiderivative of $(\ln x)^2$ is $x(\ln x)^2 - 2x\ln x + 2x$. The value of the definite integral must be $x(\ln x)^2 - 2x\ln x + 2x\Big|_{e^2}^{e^3} = \left(e^3(3^2) - 2e^3(3) + 2e^3\right) - \left(e^2(2^2) - 2e^2(2) + 2e^2\right)$. If you must "simplify" this, the result is $5e^3 - 2e^2$.

(13) 5. Compute $\int_1^3 \frac{1}{3+e^x}\,dx$. Show in some way that the value of this integral is less than $\frac{2}{3}$.
Answer Try the substitution $w = 3 + e^x$. Then $dw = e^x\,dx$ so $dx = \frac{dw}{e^x} = \frac{dw}{w-3}$. Now $\int \frac{1}{3+e^x}\,dx$ becomes $\int \frac{1}{w(w-3)}\,dw$. Split up $\frac{1}{w(w-3)}$ using partial fractions: $\frac{1}{w(w-3)} = \frac{A}{w} + \frac{B}{w-3} = \frac{A(w-3)+Bw}{w(w-3)}$ so $1 = A(w-3) + Bw$. Then $w = 0$ shows that $A = -\frac{1}{3}$ and $w = 3$ shows that $B = \frac{1}{3}$. The antiderivative is $-\frac{1}{3}\ln(w) + \frac{1}{3}\ln(w-3) + C = -\frac{1}{3}\ln(3+e^x) + \frac{1}{3}\ln(3+e^x-3) + C$ Now $-\frac{1}{3}\ln(3+e^x) + \frac{1}{3}\ln(3+e^x-3)\Big|_1^3 = -\frac{1}{3}\ln(3+e^3) + \frac{1}{3}\ln(e^3) - \left(-\frac{1}{3}\ln(3+e^1) + \frac{1}{3}\ln(e^1)\right) = \frac{2}{3} + \frac{1}{3}\left(\ln(3+e) - \ln(3+e^3)\right)$. Since ln is increasing, what's inside the parentheses is negative, and the result is less than $\frac{2}{3}$. Another explanation is that $\frac{1}{3+e^x}$ is positive and less than $\frac{1}{3}$ always, and the interval has length 2, so the total integral will be less than $\frac{2}{3}$.

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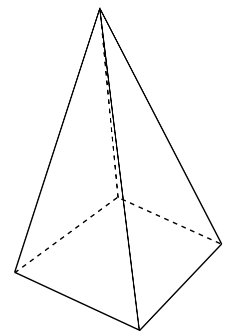
(13) 6. Compute the average value of $\left(\sec(x)\right)^3\tan(x)$ on the interval $\left[0,\frac{\pi}{3}\right]$. Is the answer larger than 2?

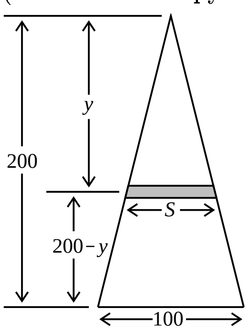
Answer We compute $\int_0^{\frac{\pi}{3}}\left(\sec(x)\right)^3\tan(x)\,dx$. First the antiderivative, using $w=\sec x$ so $dw=\sec x\tan x\,dx$ etc.: $\int\left(\sec(x)\right)^3\tan(x)\,dx=\int\left(\sec(x)\right)^2\cdot\left(\sec(x)\tan(x)\right)\,dx=\frac{1}{3}\left(\sec(x)\right)^3+C$. The integral is $\frac{8}{3}-\frac{1}{3}=\frac{7}{3}$ since $\sec(0)=1$ and $\sec\left(\frac{\pi}{3}\right)=2$. If we divide by $\frac{\pi}{3}$ the result is $\frac{7}{\pi}$, the average value. This *is* larger than 2 since $3.5>3.14159\ldots$. (The average value is ≈ 2.228, fairly close to 2. Evidence is needed!)

(12) 7. Compute $\int\frac{x^2}{\sqrt{1-x^2}}\,dx$.

Answer If $x=\sin\theta$, $\sqrt{1-x^2}=\cos\theta$ and $dx=\cos\theta\,d\theta$ so $\int\frac{x^2}{\sqrt{1-x^2}}\,dx=\int(\sin\theta)^2\,d\theta=\frac{1}{2}\int 1-\cos(2\theta)\,d\theta=\frac{1}{2}\left(\theta-\frac{\sin(2\theta)}{2}\right)+C=\frac{1}{2}\left(\theta-\frac{2\sin(\theta)\cos(\theta)}{2}\right)+C=\frac{1}{2}(\theta-\sin(\theta)\cos(\theta))+C=\frac{1}{2}\left(\arcsin(x)-x\sqrt{1-x^2}\right)+C$.

(12) 8. A pyramid, shown to the right, has a square base which has side length 100 feet. The height of the pyramid is 200 feet. The density of the construction material is 300 lbs per cubic foot. How much work must be done to raise the material from ground level (the base of the pyramid) and build the pyramid?





Answer Measure distance, y, from the top of the pyramid. A slice which is dy thick must be lifted $200-y$ feet. The volume, dV, is S^2dy (square cross-section). Analysis of similar triangles shows that $\frac{S}{100}=\frac{y}{200}$ so that $S=\frac{y}{2}$ and $dV=\frac{y^2}{4}\,dy$. The weight of this slice is $(300)\frac{y^2}{4}\,dy$ and it must be lifted $200-y$ feet. So the work to lift this slice is $\left(\frac{300}{4}\right)(200-y)y^2\,dy=\left(\frac{300}{4}\right)(200y^2-y^3)\,dy$ ft-lbs. Then the total work is $\int_0^{200}\left(\frac{300}{4}\right)(200y^2-y^3)\,dy=\left(\frac{300}{4}\right)\left(\frac{200}{3}y^3-\frac{y^4}{4}\right)\Big|_0^{200}$ which is $\left(\frac{300}{4}\right)\left(\left(\frac{200}{3}\right)200^3-\frac{200^4}{4}\right)$ ft-lbs (unnecessary simplification: 10^{10} ft-lbs, almost exciting).

Brief answers to other versions

1. In **A** the curve and line intersect at $(3,3)$, and the integral for the volume is $\int_0^3 2\pi x\left(9-2x-\frac{1}{3}x^2\right)\,dx$. For **B** the intersection is $(2,4)$ and the integral is $\int_0^2 2\pi x\left((8-x^2)-2x\right)\,dx$. For **C** the intersection is $(2,4)$ and the integral is $\int_0^2 2\pi x\left((8-2x)-\frac{1}{2}x^3\right)\,dx$. For **D** the intersection is $(3,6)$ and the integral is $\int_0^3 2\pi x\left((9-\frac{1}{9}x^3)-2x\right)\,dx$.
The volume can also be computed dy but this needs two integrals in all versions. For version **A** the volume is $\int_0^3\pi(3y)\,dy+\int_3^9\pi\left(\frac{9-y}{2}\right)^2\,dy$. The dy and dx methods for **A** both result in the answer $\frac{63}{2}\pi$, although I needed several attempts to get this agreement! Other "simplified" answers, if you must have them, are: $\frac{40}{3}\pi$ for **B**, $\frac{224}{15}\pi$ for **C**, and $\frac{171}{5}\pi$ for **D**. You *are* specifically advised that such simplification is <u>not required</u>!
2. In **A**, Δx is 1 (as it is in all versions). The approximation is $\frac{1}{3}\left(2+4\cdot 4+2\cdot 5+4\cdot 2+2\cdot 1+4\cdot 0+1\right)$ and the error bound, $\frac{K_4(b-a)^5}{180N^4}$, is $\frac{50(7-1)^5}{180\cdot 6^4}$. For **B**, these are $\frac{1}{3}\left(1+4\cdot 0+2\cdot 1+4\cdot 2+2\cdot 5+4\cdot 4+2\right)$ and $\frac{60(7-1)^5}{180\cdot 6^4}$, respectively. For **C**, these are $\frac{1}{3}\left(3+4\cdot 1+2\cdot 0+4\cdot 3+2\cdot 4+4\cdot 5+4\right)$ and $\frac{65(7-1)^5}{180\cdot 6^4}$, respectively. For **D**, these are $\frac{1}{3}\left(4+4\cdot 5+2\cdot 4+4\cdot 3+2\cdot 0+4\cdot 1+3\right)$ and $\frac{55(7-1)^5}{180\cdot 6^4}$, respectively.
3. In **A**, the partial fraction decomposition is $\frac{6}{x+1}+\frac{-x-2}{x^2+1}$. In **B** it is $\frac{2}{x+1}+\frac{3x+1}{x^2+1}$. In **C** it is $\frac{6}{x+1}+\frac{-x-3}{x^2+1}$. In **D** it is $\frac{2}{x+1}+\frac{x+3}{x^2+1}$.
4. The integrand is the same in all versions, and the evaluation is similar. **A**'s value is $5e^3-2e^2$, **B**'s value is $10e^4-5e^3$, **C**'s value is $10e^4-5e^3$, and **D**'s value is $5e^3-2e^2$.
5. The antiderivative is similar in all versions as is the logic for the requested inequality. Where **A**'s answer has $-\frac{1}{3}\ln(3+e^x)+\frac{1}{3}\ln(3+e^x-3)+C=-\frac{1}{3}\ln(3+e^x)+\frac{1}{3}\ln(e^x)+C$, **B** would have $-\frac{1}{5}\ln(5+e^x)+\frac{1}{5}\ln(e^x)+C$, **C** would have $-\frac{1}{5}\ln(5+e^x)+\frac{1}{5}\ln(e^x)+C$, and **D** would have $-\frac{1}{7}\ln(7+e^x)+\frac{1}{7}\ln(e^x)+C$,
6 and 7 are the same in all versions.
8. The answer for **A** is $\left(\frac{300}{4}\right)\left(\left(\frac{200}{3}\right)200^3-\frac{200^4}{4}\right)$ ft-lbs. The answer for **B** is $\left(\frac{350}{4}\right)\left(\left(\frac{240}{3}\right)240^3-\frac{240^4}{4}\right)$ ft-lbs. The answer for **C** is $\left(\frac{250}{4}\right)\left(\left(\frac{150}{3}\right)150^3-\frac{150^4}{4}\right)$ ft-lbs. The answer for **D** is $\left(\frac{275}{4}\right)\left(\left(\frac{180}{3}\right)180^3-\frac{180^4}{4}\right)$ ft-lbs.