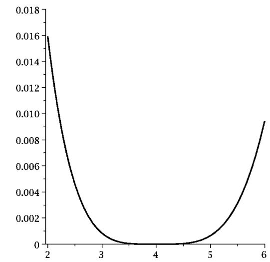
640:152: $\left\{\begin{matrix}\mathbf{1,2,3,}\\ \mathbf{6,7,8,9}\end{matrix}\right\}$ **Answers to version A of the Second Exam** **4/21/2009**

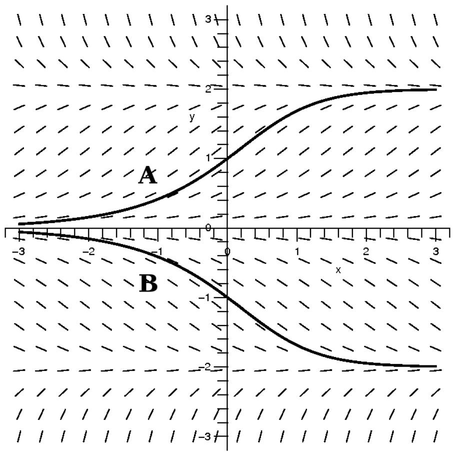
(12) 1. Suppose that $f(x) = x^{3/2}$.
a) Find the third degree Taylor polynomial, $T_3(x)$, centered at $c = 4$ for $f(x)$.
Answer If $f(x) = x^{3/2}$ then $f'(x) = \frac{3}{2}x^{1/2}$, $f''(x) = \frac{3}{4}x^{-1/2}$, and $f^{(3)}(x) = -\frac{3}{8}x^{-3/2}$, so $f(4) = 8$, $f'(4) = 3$, $f''(4) = \frac{3}{8}$, and $f^{(3)}(4) = -\frac{3}{64}$. Then $T_3(x) = 8+3(x-4)+\frac{3}{16}(x-4)^2 - \frac{1}{128}(x-4)^3$. (Don't forget factorials!)
b) Suppose $T_3(x)$ is the polynomial found in a). Use Taylor's inequality (the **Error Bound**) to find an overestimate for $|f(x) - T_3(x)|$ on the interval $[2,6]$. Your answer should be an explicit number valid for every x on this interval.



Answer Here $n = 3$ and $c = 4$. If x is in $[2,6]$, $|x - c|$ is at most 2. Also $|f^{(4)}(x)| = \left(\frac{9}{16}\right)x^{-5/2}$. This function is *decreasing* either because it is (positive)x^{negative} or because its derivative is negative so the maximum of $f^{(4)}(x)$ on $[2,6]$ is $f^{(4)}(2) = \left(\frac{9}{16}\right)2^{-5/2}$. Putting all this together, an overestimate is $\left(\frac{9}{16}\right)2^{-5/2}\left(\frac{2^4}{24}\right)$ (don't forget the factorial: $4! = 24$). This can be "simplified" (I wouldn't!) to $\frac{3}{64}\sqrt{2}$.
Comment This overestimate is ≈ 0.06629. The graph of $|f(x) - T_3(x)|$ on $[2,6]$ displayed to the right supports the overestimate but shows it is not "sharp".

(9) 2. a) Suppose that $f(x) = Ce^{(x^2)} + x$ (here C is an undetermined constant). Verify that $f(x)$ is a solution of the differential equation $y' = 2xy - 2x^2 + 1$.
Answer Since $f'(x) = Ce^{(x^2)}2x+1$ and $2xf(x)-2x^2+1 = 2x(Ce^{(x^2)}+x)-2x^2+1 = 2xCe^{(x^2)}+2x^2-2x^2+1 = 2xCe^{(x^2)} + 1$ is the same, $y = f(x)$ is a solution.
b) Find a solution of the differential equation $y' = 2xy - 2x^2 + 1$ which passes through the point $(2,3)$.
Answer Since $f(2) = 3$ we know $f(2) = Ce^4 + 2 = 3$ and $C = \frac{1}{e^4}$. The desired solution is $y = \left(\frac{1}{e^4}\right)e^{(x^2)} + x$.

(11) 3. This problem is about the differential equation $y' = y\left(1 - \frac{1}{4}y^2\right)$.
a) Find the equilibrium solutions (where y doesn't change) for this differential equation. **Answer** $y = 0$ and $y = 2$ and $y = -2$.
b) To the right is a direction field for this equation.
Sketch solution curves on this graph through the points below and find the indicated limits:

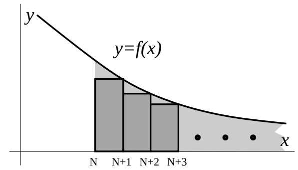


• $(0,1)$. <u>Label this curve **A**.</u> On curve **A**, $\lim_{x\to-\infty} y(x) = \underline{\ \ 0\ \ }$ and $\lim_{x\to+\infty} y(x) = \underline{\ \ 2\ \ }$.
• $(0,-1)$. <u>Label this curve **B**.</u> On curve **B**, $\lim_{x\to-\infty} y(x) = \underline{\ \ 0\ \ }$ and $\lim_{x\to+\infty} y(x) = \underline{\ -2\ }$.
Comment These curves were drawn with numerical approximation methods using `Maple`. Students do similar work in Math 244.

(12) 4. The infinite series $\sum_{n=1}^{\infty} \frac{3}{2\sqrt{n}+4^n}$ converges and its sum, to an accuracy of .001, is .719. Find a positive integer N so that the partial sum, $S_N = \sum_{n=1}^{N} \frac{3}{2\sqrt{n}+4^n}$, has a value within .001 of the sum of the whole series. Explain your reasoning.
Answer Certainly $\sum_{n=1}^{\infty} \frac{3}{2\sqrt{n}+4^n} = S_N + T_N$ and we overestimate T_N. Since $0 < \frac{3}{2\sqrt{n}+4^n} < \frac{3}{4^n}$, an overestimate is $\sum_{n=N+1}^{\infty} \frac{3}{4^n} = \frac{\frac{3}{4^{N+1}}}{1-\frac{1}{4}}$ (geometric series) which is $\frac{1}{4^N}$. A suitable value of N is 5 (from the table given).

(12) 5. The infinite series $\sum_{n=1}^{\infty} \frac{1}{5n^3+\sqrt{n}}$ converges and its sum, to an accuracy of .001, is .205. Find N so that the partial sum, $S_N = \sum_{n=1}^{N} \frac{1}{5n^3+\sqrt{n}}$, has a value within .001 of the sum of the whole series. Explain your reasoning.

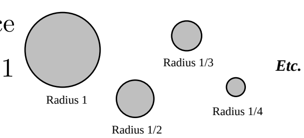
Answer Again $\sum_{n=1}^{\infty} \frac{1}{5n^3+\sqrt{n}} = S_N + T_N$. T_N can be overestimated by $\sum_{n=N+1}^{\infty} \frac{1}{5n^3}$, which is the area of a collection of rectangles. The n^{th} rectangle has lower side on the interval $[n, n+1]$ and sits under, touching at one point, the graph of $y = f(x)$ where $f(x) = \frac{1}{5x^3}$. So $T_N < \int_N^{\infty} \frac{1}{5x^3}\,dx$. Compute this improper integral: $\int_N^{\infty} \frac{1}{5x^3}\,dx = \lim_{B\to\infty} \int_N^B \frac{1}{5x^3}\,dx = \lim_{B\to\infty} -\frac{1}{10x^2}\Big|_{x=N}^{x=B} = \lim_{B\to\infty} -\frac{1}{10B^2} + \frac{1}{10N^2} = \frac{1}{10N^2}$. T_N is less than .001 if $N = 10$.



(12) 6. a) Suppose the sequence $\{a_n\}$ is defined by $a_n = (7n+3)^{5/n}$. Find the exact value of the limit of this sequence. **Answer** If $a_n = (7n+3)^{5/n}$ then $\ln(a_n) = \frac{5}{n}\ln(7n+3) = \frac{5\ln(7n+3)}{n}$. The top and bottom of this fraction both $\to\infty$ as $n\to\infty$, so this expression is eligible for L'H: $\lim_{n\to\infty} \frac{5\ln(7n+3)}{n} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{5\cdot\left(\frac{7}{7n+3}\right)}{1} = 0$. The limit of the original sequence is $e^0 = 1$.
Comment Convergence is not very fast. The sequence value is 1.04526 when n =1,000.

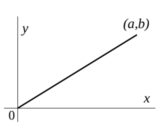
b) It is known that the sequence $\sqrt{3}$, $\sqrt{3+\sqrt{3}}$, $\sqrt{3+\sqrt{3+\sqrt{3}}}$, $\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3}}}}$, … converges. Find the exact value of the limit of this sequence. **Answer** If b_n is the n^{th} term of this sequence, then $b_{n+1} = \sqrt{3+b_n}$. Therefore $(b_{n+1})^2 = 3 + b_n$. Now take the limit as $n\to\infty$, and suppose L is the limit of the sequence. Then $L^2 = 3 + L$ or $L^2 - L - 3 = 0$ so (quadratic formula) $L = \frac{1\pm\sqrt{13}}{2}$. Since all of the terms of the sequence are positive, the limit will be non-negative, so it is $\frac{1+\sqrt{13}}{2}$.
Comment The limit is ≈ 2.30277. The 10^{th} term agrees with this to more than 5 decimal places.

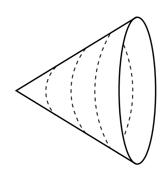
(10) 7. There is an infinite sequence of circles which do not overlap and which have radius 1, $\frac{1}{2}$, $\frac{1}{3}$, … as shown. (The n^{th} circle has radius $\frac{1}{n}$.) a) Is the total area inside all of the circles finite? **Answer** The n^{th} circle has area $\pi\left(\frac{1}{n}\right)^2$. The total area is $\pi\sum_{n=1}^{\infty}\frac{1}{n^2}$, a p-series with $p = 2 > 1$ which converges: the total area is finite.
b) Is the total circumference of all of the circles finite? **Answer** The circumference of the n^{th} circle is $2\pi\left(\frac{1}{n}\right)$ so the total area is $2\pi\sum_{n=1}^{\infty}\frac{1}{n}$. This is a p-series with $p = 1$ (actually, the harmonic series), so it diverges: the total circumference is infinite.



(10) 8. Does the infinite series $\sum_{n=1}^{\infty}(-1)^n\frac{(n!)^2}{(2n)!}$ converge or diverge? **Answer** First try absolute convergence. Consider a series whose n^{th} term c_n is $\frac{(n!)^2}{(2n)!}$ and use the Ratio Test: $\frac{c_{n+1}}{c_n} = \frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}} = \frac{((n+1)!)^2}{(n!)^2}\cdot\frac{(2n)!}{(2(n+1))!}$. Since $(n+1)! = (n+1)n!$ and $(2(n+1))! = (2n+2)! = (2n+2)(2n+1)(2n)!$, the fraction becomes $\frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}$ by dividing the top and bottom by n^2. Then $\lim_{n\to\infty}$ results in $\frac{1}{4} < 1$. The Ratio Test shows the original series converges absolutely, and therefore it must converge. (L'H could also be used.)

(12) 9. Suppose a and b are unknown positive numbers. When the line segment connecting the origin to the point (a,b) is revolved about the x-axis, the result is the slanted surface of a cone. The area of this surface is $\pi b\sqrt{a^2+b^2}$. Verify this formula using calculus. **Answer** The line segment is the graph of $y = \left(\frac{b}{a}\right)x$ over $[0,a]$. Therefore the surface area is $2\pi\int_0^a f(x)\sqrt{1+[f'(x)]^2}\,dx = 2\pi\int_0^a\left(\frac{b}{a}\right)x\sqrt{1+\left(\frac{b}{a}\right)^2}\,dx = 2\pi\left(\frac{b}{a}\right)\sqrt{1+\left(\frac{b}{a}\right)^2}\left(\frac{x^2}{2}\right)\Big|_{x=0}^{x=a} = 2\pi\left(\frac{b}{a}\right)\sqrt{1+\left(\frac{b}{a}\right)^2}\left(\frac{a^2}{2}\right) = \pi b\sqrt{a^2+b^2}$.





Brief answers to version B (for the questions which are different)
1. $T_3(x) = 243 + \frac{135}{2}(x-9) + \frac{45}{8}(x-9)^2 + \frac{5}{48}(x-9)^3$; the overestimate is $\left(\frac{15}{16}\right)7^{-3/2}\left(\frac{2^4}{24}\right)$ (this is $\frac{5}{392}\sqrt{7}$ but I hope no one writes this!).
2. a) is much the same. In b), $C = \frac{2}{e^8}$.
4. $N \geq 7$. The reasoning is similar.
5. $N \geq 10$ here also. The reasoning is similar.
6. a) The limit is 1. b) The limit is $\frac{1+\sqrt{21}}{2}$. The reasoning in both parts is similar.