640:153 **Answers to version A of the First Exam** **10/5/2009**

These answers would earn full credit. Other methods may be valid. *Only version A answers are available.*

(12) 1. Suppose $f(x) = \frac{3x+e^x}{x^2+2e^x}$.

a) Compute $f'(x)$ (here you may write only the answer to earn full credit).

Answer $\frac{(3+e^x)(x^2+2e^x)-(3x+e^x)(2x+2e^x)}{(x^2+2e^x)^2}$.

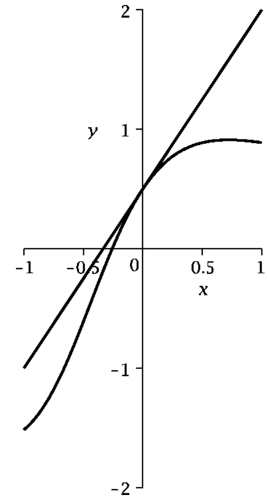
b) Find $f(0)$ and $f'(0)$.

Answer $f(0) = \underline{\frac{1}{2}}$ and $f'(0) = \underline{\frac{3}{2}}$

c) Write an equation for the line tangent to $y = f(x)$ when $x = 0$.

Answer $y - \frac{1}{2} = \frac{3}{2}(x - 0)$

d) Sketch the line you have found in c) on the axes to the right. These axes already have a graph of $y = f(x)$.



(8) 2. Suppose f is a differentiable function, $g(x) = \big(f(x)\big)^3$, $f(1) = 2$, $f'(1) = -1$, and $f''(1) = 4$. Compute the exact values of $g(1)$, $g'(1)$, and $g''(1)$. (You must show work in this problem to get full credit.)

Answer Since $g(x) = \big(f(x)\big)^3$, $g(1) = \big(f(1)\big)^3 = 2^3 = 8$. The Chain Rule implies $g'(x) = 3\big(f(x)\big)^2 f'(x)$, so $g'(x) = 3\big(f(1)\big)^2 f'(1) = 3(2^2)(-1) = -12$. The Chain Rule and the Product Rule then imply $g''(x) = 6f(x)\big(f'(x)\big)^2 + 3\big(f(x)\big)^2 f''(x)$ so $g''(1) = 6f(1)\big(f'(1)\big)^2 + 3\big(f(1)\big)^2 f''(1) = 6(2)(-1)^2 + 3(2)^2(4) = 60$. So $g(1) = \underline{\ 8\ }$, $g'(1) = \underline{\ -12\ }$, and $g''(1) = \underline{\ 60\ }$.

(10) 3. Use the definition of derivative combined with algebraic manipulation and standard properties of limits stated in this course to compute the derivative of $f(x) = \frac{1}{x+3}$.

Answer $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Since $f(x) = \frac{1}{x+3}$, we know $f(x+h) = \frac{1}{(x+h)+3}$. Therefore $f(x+h) - f(x) = \frac{1}{(x+h)+3} - \frac{1}{x+3} = \frac{(x+3)-\big((x+h)+3\big)}{(x+3)\big((x+h)+3\big)} = \frac{-h}{(x+3)(x+h+3)}$ and $\frac{f(x+h)-f(x)}{h} = \frac{\frac{-h}{(x+3)(x+h+3)}}{h} = \frac{-h}{h(x+3)(x+h+3)} = \frac{-1}{(x+3)(x+h+3)}$. The limit as $h \to 0$ of the last expression is $\frac{-1}{(x+3)^2}$ which is $f'(x)$.

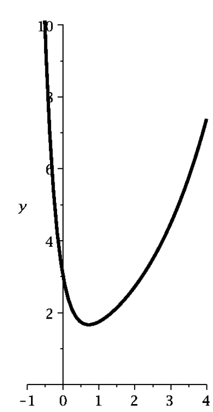
(12) 4. Compute the derivatives of the functions shown. In this problem, you may write only the answers and get full credit. *Please* do not "simplify" the answers!

a) $5 + \frac{1}{x+3} - 17x^6$ **Answer** $0 + \frac{-1}{(x+3)^2} - 17 \cdot 6x^5$

b) $e^{(x^2 \cos x)}$ **Answer** $e^{(x^2 \cos x)}\big(2x(\cos x) + x^2(-\sin x)\big)$.

c) $\sin(5x+7)\left(x^3-4\right)^2$ **Answer** $\cos(5x+7)(5)\left(x^3-4\right)^2 + \sin(5x+7)2(x^3-4)^1(3x^2-0)$

(12) 5. Suppose $f(x) = 2e^{-3x} + e^{\left(\frac{1}{2}\right)x}$. A portion of a computer-drawn graph of $y = f(x)$ is shown to the right. Find the exact value of the *first coordinate*, x, of the lowest point on this graph.

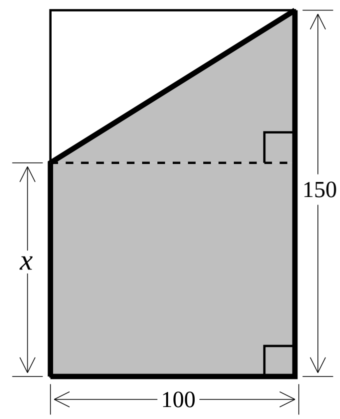


Answer Since $f(x) = 2e^{-3x} + e^{\left(\frac{1}{2}\right)x}$, we know that $f'(x) = 2e^{-3x}(-3) + e^{\left(\frac{1}{2}\right)x}\frac{1}{2}$. The tangent line at the lowest point is horizontal, so the slope of that tangent line must be 0. Since the derivative computes the slope of the tangent line, we must solve $2e^{-3x}(-3) + e^{\left(\frac{1}{2}\right)x}\frac{1}{2} = 0$.

This is $-12e^{-3x} + e^{\frac{x}{2}} = 0$ or $e^{\frac{x}{2}} = 12e^{-3x}$ or (multiplying by e^{3x} and combining exponentials) $e^{\left(\frac{7}{2}\right)x} = 12$. Taking logs we get $\left(\frac{7}{2}\right)x = \ln(12)$ so that $x = \left(\frac{2}{7}\right)\ln(12)$.

The lowest point has $x = \underline{\left(\frac{2}{7}\right)\ln(12)}$.

(12) 6. A rectangular field has dimensions 100 feet by 150 feet. A point x feet from the corner on one of the long sides is connected by a line segment to the opposite corner resulting in a four-sided region, as shown. A fence is constructed to enclose this four-sided region. The fence costs \$8 per foot for the portion on the sides of the rectangle and \$12 per foot for the portion across the inside of the rectangle. Write a formula for $f(x)$, the <u>total cost of the fence</u>, as a function of x.
What is the domain of this function when used to describe this problem? (The domain should be related to the problem statement.)



Answer The part of the fence which is on the boundary of the rectangle has length $150 + 100 + x = 250 + x$ feet. It contributes $(250 + x)8$ dollars to the cost of the fence. The "diagonal" part of the fence is the hypotenuse of a right triangle. The side of the triangle which is horizontal in the diagram supplied has length 100, and the other leg, vertical in the diagram, has length $150 - x$. Therefore (Pythagoras) the hypotenuse has length $\sqrt{(100)^2 + (150 - x)^2}$. This length costs \$12 per foot so it contributes $12\sqrt{(100)^2 + (150 - x)^2}$ dollars to the cost of the fence. The sum of the two dollar amounts is $f(x)$, the total cost, which is $(250+x)8+12\sqrt{(100)^2 + (150 - x)^2}$ dollars. In this problem, x is at least 0 and at most 150, the length of the longer side.
Therefore $f(x) =$ $\underline{(250 + x)8 + 12\sqrt{(100)^2 + (150 - x)^2}}$ and the Domain of $f =$ $\underline{[0, 150]}$.

(8) 7. Find some closed interval which contains the number $(20)^{1/5}$. You must give evidence supporting your assertion – the interval given can be *any* valid interval. The evidence should include *specific vocabulary and results* from this course. **Hint** You may want to consider $f(x) = x^5 - 20$.
Answer The function $f(x) = x^5 - 20$ is continuous and the Intermediate Value Theorem applies to it. Since $f(0) = -20 < 0$ and $f(2) = 2^5 - 20 = 32 - 20 = 12 > 0$, the Intermediate Value Theorem implies there is some x in $[0, 2]$ with $f(x) = 0$, so for that x, we know $x^5 = 20$.
One answer for such an interval is $\underline{[0, 2]}$.

(12) 8. Compute these limits. Supporting work using methods of this course for each answer *must* be given to earn full credit. **Comment** Your results should be a specific number, $+\infty$, $-\infty$, or Does Not Exist.

a) $\lim\limits_{x\to 3} \frac{x^2-9}{x^2-2x-3}$ **Answer** $x^2 - 2x - 3 = (x - 3)(x + 1)$ and $x^2 - 9 = (x - 3)(x + 3)$, so the given quotient is $\frac{x+3}{x+1}$ for $x \neq 3$. The limit can be found by "plugging in" since the result is continuous at 3. The answer is $\frac{6}{4}$.

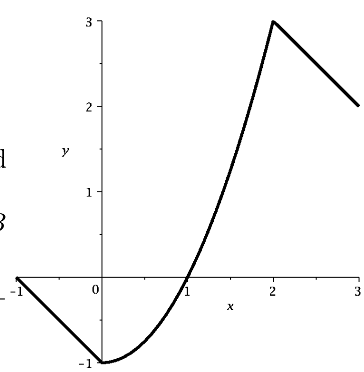
b) $\lim\limits_{x\to 2} \frac{x-\cos x}{x^2}$ **Answer** The expression is continuous for $x \neq 0$ and the limit can be found by "plugging in". Its value is $\frac{2-\cos 2}{4}$.

c) $\lim\limits_{x\to 1^+} \frac{x^2-1}{|x-1|}$ **Answer** For $x > 1$, $x - 1 > 0$ so $|x - 1| = x - 1$ and the expression $\frac{x^2-1}{x-1}$ is just $x + 1$, continuous everywhere. So, again, "plug in" to get the value 2.

(14) 9. Suppose $f(x) = \begin{cases} -x-1 & \text{for } x < 0. \\ Ax^2 + B & \text{for } 0 \le x \le 2. \\ -x+5 & \text{for } x > 2. \end{cases}$

a) Find values of A and B so that f will be continuous for all x.

Answer Since f is given by familiar functions away from 0 and 2, we need only check what happens at those two numbers.
At <u>$x = 0$</u>: $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} -x-1 = -1$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} Ax^2+B = B$ so that B must be -1.
At <u>$x = 2$</u>: $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} Ax^2 + B = 4A + B$ and $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} -x + 5 = 3$ so that $4A + B = 3$ and since $B = -1$ we see that A must be 1.
So $A =$ $\underline{4}$ and $B =$ $\underline{-1}$.



b) Sketch $y = f(x)$ on the axes given using the values of A and B found in a).