Here are answers to Version A. Other answers with different methods may also be correct.

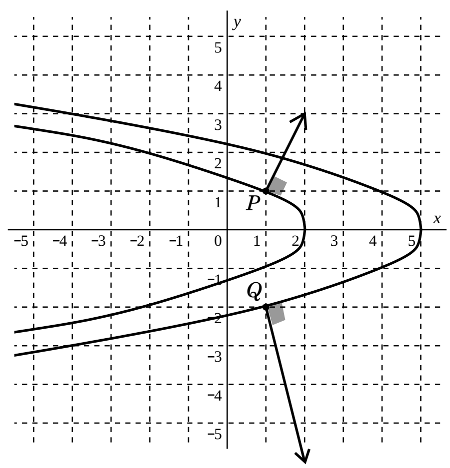
(16) 1. a) Find parametric equations for the line L containing $M = (1, -2, 3)$ and $N = (4, 0, 2)$.
Answer The vector from M to N is $3\vec{i} + 2\vec{j} - \vec{k}$. One correct answer is $\begin{cases} x = 3t + 1 \\ y = 2t - 2 \\ z = -t + 3 \end{cases}$

b) Find an equation for the plane P through $A = (0, 0, 1)$, $B = (2, 0, -1)$, and $C = (3, 3, 0)$.
Answer The vector from A to B is $\langle 2, 0, -2\rangle$ and the vector from A to C is $\langle 3, 3, -1\rangle$. A vector normal to the plane is the cross-product, so compute det $\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -2 \\ 3 & 3 & -1 \end{pmatrix} = (6)\mathbf{i} - (-2 + 6)\mathbf{j} + (6)\mathbf{k} = 6\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$. An answer is $6(x - 0) - 4(y - 0) + 6(z - 1) = 0$.

c) The line L and the plane P intersect. Find coordinates for the point of intersection.
Answer Put right-hand side of the parametric equations into the plane equation, and solve for t. Then insert that value of t into the parametric equations to get the intersection point. So: $6(x-0)-4(y-0)+6(z-1) = 0$ becomes $6(3t+1)-4(2t-2)+6(-t+3-1) = 0$ which is $(18-8-6)t+(6+8+12) = 0$ and so $t = -\frac{26}{4}$. The point is $\left(3\left(-\frac{26}{4}\right) + 1, 2\left(-\frac{26}{4}\right) - 2, -\left(-\frac{26}{4}\right) + 3\right)$. *A fine answer! Why mess with it?*

(14) 2. Suppose the position vector of a curve is given by $\mathbf{r}(t) = \langle e^{-2t}, e^t, t^2\rangle$. Find the unit tangent vector and the unit normal vector to this curve when $t = 0$. That is, find $\mathbf{T}(0)$ and $\mathbf{N}(0)$.
Answer The velocity vector is $\langle -2e^{-2t}, e^t, 2t\rangle$ and the acceleration vector is $\langle 4e^{-2t}, e^t, 2\rangle$. When $t = 0$, we get $\mathbf{v}(0) = \langle -2, 1, 0\rangle$ with speed equal to $\sqrt{(-2)^2 + 1^2} = \sqrt{5}$. So $\mathbf{T}(0) = \left\langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right\rangle$. To get the direction of $\mathbf{N}(0)$ we need the part of $\mathbf{a}(0) = \langle 4, 1, 2\rangle$ which is perpendicular to $\mathbf{v}(0)$ so we compute $\mathbf{a}(0) - \frac{\mathbf{a}(0)\cdot\mathbf{v}(0)}{\|\mathbf{v}(0)\|^2}\mathbf{v}(0)$. But $\mathbf{a}(0)\cdot\mathbf{v}(0) = -8 + 1 = -7$, and then $\langle 4, 1, 2\rangle - \left(\frac{-7}{5}\right)\langle -2, 1, 0\rangle = \left\langle \frac{6}{5}, \frac{12}{5}, \frac{10}{5}\right\rangle$ (check by dotting this with $\mathbf{v}(0)$: the result is 0, so they are perpendicular!). Therefore $\mathbf{N}(0) = \left\langle \frac{3}{\sqrt{70}}, \frac{6}{\sqrt{70}}, \frac{5}{\sqrt{70}}\right\rangle$.

(14) 3. Suppose $f(x, y) = x + y^2$.
a) Compute $\nabla f(x, y)$. **Answer** $\nabla f(x, y) = \langle 1, 2y\rangle$.
b) Graph the level curves of f which pass through the points $P = (1, 1)$ and $Q = (1, -2)$ on the axes given.
Answer Since $f(1, 1) = 1 + 1 = 2$, the level curve through P is $x + y^2 = 2$ or $x = 2 - y^2$. This is a parabola whose axis of symmetry is the x-axis and opens to the left. As for Q, we have $f(1, -2) = 1 + (-2)^2 = 5$. The level curve through Q is $x + y^2 = 5$ and $x = 5 - y^2$. This curve has the same shape as the first level but is translated 3 units right (positive). The curves are drawn on the axes to the right.
c) Compute $\nabla f(1, 1)$ and $\nabla f(1, -2)$. On the same axes as your answer to b), as well as you can, sketch $\nabla f(1, 1)$ starting at P and $\nabla f(1, -2)$ starting at Q. **Answer** $\nabla f(1, 1) = \langle 1, 2\rangle$, $\nabla f(1, -2) = \langle 1, -4\rangle$, and the sketch is to the right.



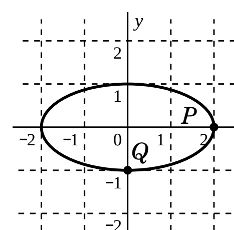
(16) 4. Suppose $f(t)$ is a differentiable function of one variable, and $f(1) = A$, $f'(1) = B$, and $f''(1) = C$. Define $F(x, y, z)$ using this equation: $F(x, y, z) = f(xz^2 - y^3)$.
a) Compute $F(1, 2, 3)$ in terms of the information supplied and any needed constants.
Answer $F(1, 2, 3) = f(1 \cdot 3^2 - 2^3) = f(1) = A$.
b) Compute $\frac{\partial F}{\partial x}(1, 2, 3)$ in terms of the information supplied and any needed constants.
Answer The Chain Rule implies $\frac{\partial F}{\partial x} = f'(xz^2 - y^3)z^2$ so at $(1, 2, 3)$ the value is $f'(1)3^2 = 9B$.
c) Compute $\frac{\partial F}{\partial z}(1, 2, 3)$ in terms of the information supplied and any needed constants.
Answer The Chain Rule implies $\frac{\partial F}{\partial z} = f'(xz^2 - y^3)2xz$ so at $(1, 2, 3)$ the value is $f'(1)2 \cdot 1 \cdot 3 = 6B$.
d) Compute $\frac{\partial^2 F}{\partial z^2}(1, 2, 3)$ in terms of the information supplied and any needed constants.
Answer Since $\frac{\partial F}{\partial z} = f'(xz^2 - y^3)2xz$ the Chain Rule and the Product Rule imply that $\frac{\partial^2 F}{\partial z^2} = f''(xz^2 - y^3)4x^2z^2 + f'(xz^2 - y^3)2x$. At $(1, 2, 3)$, the value is $f''(1)4 \cdot 1^2 \cdot 3^2 + f'(1)2 = 36C + 2B$.

e) Compute $\frac{\partial^2 F}{\partial x \partial z}(1,2,3)$ in terms of the information supplied and any needed constants.
Answer Since $\frac{\partial F}{\partial z} = f'(xz^2-y^3)2xz$ the Chain Rule <u>and</u> the Product Rule imply that $\frac{\partial^2 F}{\partial x \partial z} = f''(xz^2-y^3)2xz^3 + f'(xz^2-y^3)2z$. At $(1,2,3)$, the value is $f''(1)2\cdot 1\cdot 3^3 + f'(1)2\cdot 3 = 54C + 6B$.

(8) 5. Suppose z is implicitly defined as a function of x and y by the equation $x^2y - 7xz^3 + zy = 5$. Find a formula for $\frac{\partial z}{\partial x}$.
Answer We will $\frac{\partial}{\partial x}$ the equation. The result is $2xy - 7z^3 - 7x(3z^2)z_x + z_x y = 0$. Here z_x is an abbreviation for $\frac{\partial z}{\partial x}$. Now "solve" for z_x. We get $z_x = \frac{-2xy+7z^3}{-21xz^2+y}$.

(16) 6. In this problem, the function f is defined by $f(x,y,z) = \sin(2xy)z + yz^2$ and the point p is $(0,2,-1)$.
a) Compute $\nabla f(x,y,z)$ and find its value at p.
Answer We compute: $f_x = \cos(2xy)2yz$, $f_y = \cos(2xy)2xz + z^2$, and $f_z = \sin(2xy) + 2yz$. Therefore $\nabla f = \langle \cos(2xy)2yz, \cos(2xy)2xz + z^2, \sin(2xy) + 2yz\rangle$ so at p, the gradient is $\langle -4, 1, -4\rangle$.
b) Use your answer to a) to find
i) The value of the largest directional derivative of f at p and its direction (unit vector!).
Answer The largest directional derivative of f at p is $\sqrt{(-4)^2 + 1^2 + (-4)^2} = \sqrt{33}$. The direction of the largest directional derivative is $\left\langle \frac{-4}{\sqrt{33}}, \frac{1}{\sqrt{33}}, \frac{-4}{\sqrt{33}}\right\rangle$.
ii) The directional derivative for f at p in the direction from p to $q = (2,2,3)$.
Answer The vector from p to q is $\langle 2,0,4\rangle$ so a unit vector in that direction is $\left\langle \frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right\rangle$. The directional derivative is the dot product of that vector with $\nabla f(p) = \langle -4,1,-4\rangle$. The value is $-\frac{12}{\sqrt{5}}$.
iii) One unit vector $\vec{u}$ so that the directional derivative of f at p in the direction of u is 0. (The magnitude of $\vec{u}$ should be 1!) **Answer** If $\vec{u} = \langle a,b,c\rangle$ is such a vector, then $D_{\vec{u}}f(p)$ is $\nabla f(p)\cdot\langle a,b,c\rangle = \langle -4,1,-4\rangle \cdot \langle a,b,c\rangle = -4a + b - 4c$. So, for example, a vector in the direction of $\langle 1,4,0\rangle$ is such a vector. If we make it unit length, then $\left\langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}, 0\right\rangle$ is such a $\vec{u}$ (I *guessed* – there are *many* correct answers!).

(16) 7. An ellipse in $\mathbb{R}^2$ has parametric equations $\begin{cases} x = 2\cos t \\ y = \sin t \end{cases}$.



a) Sketch this ellipse on the axes to the right. [SHOWN TO THE RIGHT.]
b) Write an integral expression for the total length of the ellipse. Do *not* compute the value of this expression.
Answer The ellipse is traced once as t goes from 0 to 2π. Since $x'(t) = -2\sin t$ and $y'(t) = \cos t$, the length will be $\int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^{2\pi} \sqrt{4(\sin t)^2 + (\cos t)^2}\, dt$.
c) For such a curve, the curvature, κ, can be computed by $\kappa = \frac{|y''(t)x'(t) - x''(t)y'(t)|}{(x'(t)^2 + y'(t)^2)^{3/2}}$. Compute κ for this ellipse. Use this formula to find the curvature at the point $P = (2,0)$ and at the point $Q = (0,1)$.
Answer Since $x''(t) = -2\cos t$ and $y''(t) = -\sin t$, the top of the curvature formula, $|y''(t)x'(t) - x''(t)y'(t)|$, becomes $|(-\sin t)(-2\sin t) - (-2\cos t)(\cos t)| = |2(\sin t)^2 + 2(\cos t)^2| = 2$ (this simplification is *not* necessary). The bottom is $\left(4(\sin t)^2 + (\cos t)^2\right)^{3/2}$. A formula for κ is $2\left(4(\sin t)^2 + (\cos t)^2\right)^{-3/2}$ (note the minus power!). Since $\cos(0) = 1$ and $\sin(0) = 0$, the point $(x(0), y(0))$ is $P = (2,0)$. Then $\kappa = 2$ at P (just plug in!). Since $\cos\left(\frac{3\pi}{2}\right) = 0$ and $\sin\left(\frac{3\pi}{2}\right) = -1$, the point $\left(x\left(\frac{3\pi}{2}\right), y\left(\frac{3\pi}{2}\right)\right)$ is $Q = (0,-1)$. At Q, the formula gives $\kappa = \frac{1}{4}$. This ellipse *is* more curvey at the x-intercepts than at the y-intercepts.

Version B (bare answers)
1. $x = 2t-2$, $y = -t+3$, $z = 3t+1$; $-4(x-0) + 6(y-1) + 6(z-0) = 0$; $\left(2\left(-\frac{26}{4}\right) - 2, -\left(-\frac{26}{4}\right) + 3, 3\left(-\frac{26}{4}\right) + 1\right)$.
2. $\mathbf{T}(0) = \left\langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right\rangle$; $\mathbf{N}(0) = \left\langle \frac{6}{\sqrt{70}}, \frac{3}{\sqrt{70}}, \frac{5}{\sqrt{70}}\right\rangle$.
3. Similar – the parabolas open to the right. The gradient vectors are $\langle 1, -2y\rangle$, $\langle 1,-2\rangle$, and $\langle 1,4\rangle$,
4. The answers are the same.
5. $z_x = \frac{-3x^2y+5z^3}{-20xz^3+y}$.
6. The answers are the same.
7. The ellipse is rotated by $\frac{\pi}{2}$ with major axis vertical and minor axis horizontal. $\int_0^{2\pi} \sqrt{(\sin t)^2 + 4(\cos t)^2}\, dt$; $\kappa = 2\left((\sin t)^2 + 4(\cos t)^2\right)^{-3/2}$; κ at P (with $t = 0$) is $\frac{1}{4}$; κ at Q (with $t = \frac{3\pi}{2}$) is 2.