640:251:1, 2 & 3 **Answers to the Second Exam** **4/16/2010**

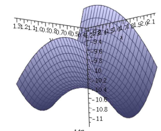
Here are answers to Version A. Other answers with different methods may also be correct.

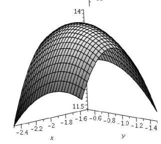
(12) 1. Find and classify using the Second Derivative Test all critical points of $f(x,y)=x^3+2xy-2y^2-10x$.

Answer $f_x=3x^2+2y-10$ and $f_y=2x-4y$. If $f_y=0$ then $x=2y$ and $f_x=0$ becomes $12y^2+2y-10=0$ or $6y^2+y-5=0$. A little thinking reveals that this quadratic polynomial factors with integer coefficients (it *is* a textbook problem!) so we have $(6y-5)(y+1)=0$. The roots are $y=\frac{5}{6}$ and $y=-1$ so the critical points of f are $\left(\frac{10}{6},\frac{5}{6}\right)$ and $(-2,-1)$.

Continuing: $f_{xx}=6x$, $f_{xy}=2$, $f_{yx}=2$, and $f_{yy}=-4$. The Hessian is $-24x-4$. At $\left(\frac{10}{6},\frac{5}{6}\right)$, this is negative (-44), so the point is a *saddle*. At $(-2,-1)$, the Hessian is 44 and $f_{xx}=-12$, so this critical point is a local maximum.

Comment To the right are "closeup" `Maple` graphs near the c.p.'s.



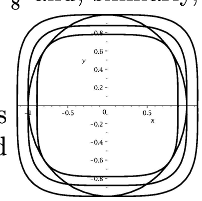


(12) 2. Use the Lagrange multiplier method to find the maximum and minimum values of $f(x,y)=3x^4+5y^4$ subject to the constraint $x^2+y^2=1$. Be sure to check *all* solutions to the Lagrange multiplier equations!

Answer Since $f_x=12x^3$ and $f_y=20y^3$, the Lagrange multiplier equations are $12x^3=\lambda(2x)$, $20y^3=\lambda(2y)$, and $x^2+y^2=1$. If $\lambda=0$ then both x and y must be 0 using the first two equations. That's not a solution of the third equation since x and y can't both be 0. But $x=0$ means that $y=\pm1$ from the constraint equation so that the value of the objective function is 5. If $y=0$ then similarly $x=\pm1$ and the objective function's value is 3.

Now assume none of the variables are 0, The first two equations become $\lambda=6x^2$ and also $\lambda=10y^2$. Then the constraint equation becomes $\frac{1}{6}+\frac{1}{10}\lambda=1$ so that $\lambda=\frac{15}{4}$. Then $6x^2=\frac{15}{4}$ so $x=\pm\sqrt{\frac{5}{8}}$ and, similarly, $10y^2=\frac{15}{4}$ so $y=\pm\sqrt{\frac{3}{8}}$. The value of $f(x,y)=3x^4+5y^4$ at these four points is $\frac{120}{64}=\frac{15}{8}$. Since $\frac{15}{8}<3<5$, the minimum value of f is $\frac{15}{8}$ and the maximum value is 5.

Comment To the right are `Maple` graphs of the constraint (the unit circle) and level curves of the objective for $\frac{15}{8}$, 3, and 5. The big "boxy circle" outside the constraint is associated with 5. The curve inside the constraint comes from $\frac{15}{8}$. The intermediate one is 3's curve.

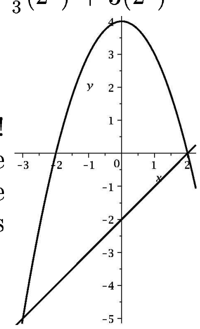


(16) 3. This problem investigates the iterated integral $I=\int_{-3}^{2}\int_{x-2}^{4-x^2} x\,dy\,dx$.

a) Compute I. **Answer** Inside: $\int_{x-2}^{4-x^2} x\,dy = xy\Big|_{y=x-2}^{y=4-x^2} = x(4-x^2)-x(x-2)=4x-x^3-(x^2-2x)=-x^3-x^2+6x$. Outside: $\int_{-3}^{2}-x^3-x^2+6x\,dx=-\frac{1}{4}x^4-\frac{1}{3}x^3+3x^2\Big|_{-3}^{2}=-\frac{1}{4}(2^4)-\frac{1}{3}(2^3)+3(2^2)-\left(-\frac{1}{4}(-3)^4-\frac{1}{3}(-3)^3+3(-3)^2\right)$: fine! I hope you *don't* "simplify", but this is also $-\frac{125}{12}$.

b) Use the axes to the right to sketch the region of integration for I.

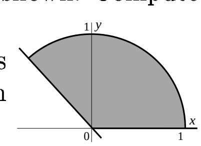
Answer An answer is shown to the right.



c) Write I as a sum of one or more $dx\,dy$ integrals. You do not need to compute the result!

Answer The curves $y=4-x^2$ and $y=x-2$ intersect at $(2,0)$ and $(-3,-5)$ (since $4-x^2=x-2$ is the same as $x^2+x-2=0$ or $(x+2)(x-1)=0$). Two integrals are needed, one from $y=-5$ to $y=0$ and the other from $y=0$ to $y=4$. The result is $\int_{-5}^{0}\int_{-\sqrt{4-y}}^{2+y} x\,dx\,dy+\int_{0}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}} x\,dx\,dy$.

(12) 4. The average value of a function f defined in a region R of $\mathbb{R}^2$ is $\dfrac{\iint_R f\,dA}{\iint_R dA}$. Suppose the region R is bounded by an arc of the unit circle, $x^2+y^2=1$, a part of $y=-x$, and a part of $y=0$ as shown. Compute the average value of <u>the distance to the origin</u> over this region.

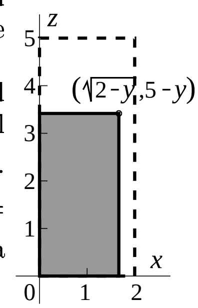


Answer This computation is easiest in *polar coordinates*. Then the distance to the origin is just r. The region is easy to describe: the r limits go from 0 to 1, and the θ limits go from 0 to $\frac{3\pi}{4}$. Remember that $dA=r\,dr\,d\theta$. Then $\int_0^{\frac{3\pi}{4}}\int_0^1 r^2\,dr\,d\theta=\left(\frac{1}{3}\right)\left(\frac{3\pi}{4}\right)=\frac{\pi}{4}$. Then divide by the area of the region, which is $\frac{3}{8}\pi$ (computed either as a fraction of the circle's area or by evaluating $\int_0^{\frac{3\pi}{4}}\int_0^1 r\,dr\,d\theta$). The result is $\frac{2}{3}$.

(16) 5. A bounded solid object A in $\mathbb{R}^3$ is located in the first octant, where $x \geq 0$, $y \geq 0$, and $z \geq 0$. One side of the object is given by $y = 4 - x^2$ and another side by $z = 5 - y$. Compute the triple integral of $2x$ over the object A by writing a triple iterated integral for $2x$ over the object A and then computing the value of this integral. **Remark** Four pictures of the object were given.

Answer If dz is last, the setup will be difficult since the shapes of the slices with z fixed change (they have one curvy side and two straight sides when $0 \leq z \leq 1$ and a curvy side and *three* straight sides for $1 < z < 5$). Here we use $dz\,dx\,dy$.

y goes from 0 to 4. A y slice is simple: a rectangle. x is in the interval $[0, \sqrt{4-y}]$ (obtained by solving for positive x in the equation $y = 4 - x^2$) and z, in the interval $[0, 5-y]$. A "typical slice" is shown to the right. One correct triple iterated integral is $\int_0^4 \int_0^{\sqrt{4-y}} \int_0^{5-y} 2x\,dz\,dx\,dy$. The innermost integral is just $2x(5-y)$. The next antidifferentiation gives $x^2(5-y)\Big|_0^{\sqrt{4-y}} = (4-y)(5-y) = 20 - 9y + y^2$. We end with $20y - \frac{9}{2}y^2 + \frac{1}{3}y^3\Big|_0^4 = 20(4) - \frac{9}{2}(4^2) + \frac{1}{3}(4^3)$, a fine answer. It is also $\frac{88}{3}$. $dx\,dz\,dy$ is simple, too.



The answer in another popular (?) order is $\int_0^2 \int_0^{4-x^2} \int_0^{5-y} 2x\,dz\,dy\,dx$. The x slices are trapezoids.

The other three orders all need more than one iterated triple integral and they should be eschewed*.

(16) 6. Calculate the volume of the sphere $x^2 + y^2 + z^2 = a^2$, using <u>both</u> spherical and cylindrical coordinates.

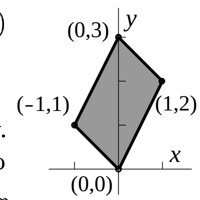
Answer Spherical The sphere is exactly described by $0 \leq \rho \leq a$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi$. The volume is therefore $\int_0^a \int_0^{2\pi} \int_0^{\pi} 1\,(\rho^2 \sin\phi\,d\phi\,d\theta\,d\rho)$. The inner integral is $-\rho^2\cos\phi\Big|_0^{\pi} = -\rho^2(-1) - \left(-\rho^2(-1)\right) = 2\rho^2$. The next antidifferentiation multiplies by 2π. Finally we have $\int_0^a 4\pi\rho^2\,d\rho = \frac{4\pi}{3}\Big|_0^a = \frac{4\pi}{3}a^3$.

Cylindrical Since $x^2 + y^2 + z^2 = a^2$ we know that $r^2 = a^2 - z^2$. Each z slice as z goes from $-a$ to a is a circle, and the radius of the circle is $\sqrt{a^2 - z^2}$. The volume is $\int_{-a}^{a} \int_0^{2\pi} \int_0^{\sqrt{a^2-z^2}} 1\,(r\,dr\,d\theta\,dz)$. The first integral gives $\frac{1}{2}r^2\Big|_0^{\sqrt{a^2-z^2}} = \frac{1}{2}(a^2 - z^2)$. The second integral multiplies this by 2π, and so we finish with $\int_{-a}^{a} \pi(a^2 - z^2)\,dz = \pi\left(a^2 z - \frac{1}{3}z^3\right)\Big|_{-a}^{a} = \pi\left(a^3 - \frac{1}{3}a^3\right) - \pi\left(-a^3 - \frac{1}{3}(-a)^3\right) = \frac{4}{3}\pi a^3$.

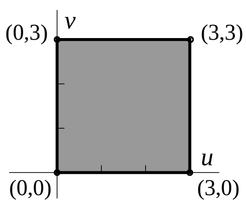
Another valid description is $\int_0^a \int_0^{2\pi} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} 1\,(r\,dz\,d\theta\,dr)$.

(16) 7. The region R in $\mathbb{R}^2$ is the parallelogram shown to the right which has vertices (corners) at $(1,2)$, $(0,3)$, $(-1,1)$, and $(0,0)$. Verify that $\iint_R (y+x)^6 \cos(y-2x)\,dA_{xy} = \frac{3^6 \sin(3)}{7}$.

$\iint_{R_{uv}}$(Func described with u and v) **|JAC|** $dA_{uv} = \iint_{R_{xy}}$(Func described with x and y) dA_{xy}



Answer We use change of variables. The problem *can* be done in other ways, but not easily. The variables will be $u = y + x$ and $v = y - 2x$. One opposite pair of edges corresponds to



$u = 3$ and $u = 0$ and the other, to $v = 3$ and $v = 0$. If $\begin{cases} u = y + x \\ v = y - 2x \end{cases}$ then subtract the equations and divide by 3 to get $x = \frac{1}{3}u - \frac{1}{3}v$. Then double the first equation, add, and divide by 3, the result is $y = \frac{2}{3}u + \frac{1}{3}v$. The Jacobian is the absolute value of $\det\begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix} = \det\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} = \frac{1}{3}$. The Change of Variables formula states that the value of $\iint_R (y+x)^6 \cos(y-2x)\,dA_{xy}$ is $\int_{u=0}^{u=3} \int_{v=0}^{v=3} u^6 \cos(v)\,\frac{1}{3}\,dv\,du$. Each integral is easy, and the result is $\left(\frac{1}{3}\right)\left(\frac{3^7}{7}\right)\sin(3) = \frac{3^6 \sin(3)}{7}$.

Version B (bare answers)

1. The same.
2. 5 and 3 are flipped. The max and min values occur at different points but they are the same.
3. The same.
4. The average value is the same, but the integrals are now $\int_{\frac{\pi}{4}}^{\pi} \int_0^1 (r \text{ or } 1)\,r\,dr\,d\theta$.
5. The same.
6. The same.
7. Here $u = x + y$ and $v = 2y - x$ so that $x = \frac{2}{3}u - \frac{1}{3}v$ and $y = \frac{1}{3}u + \frac{1}{3}v$. The Jacobian is $\frac{1}{3}$. The region in the (u,v) plane has $0 \leq u \leq 3$ and $0 \leq v \leq 6$. The computations are similar to the solution for version A.

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* eschew: shun; avoid and stay away from deliberately; stay clear of.