Math 251:1-2-3

Workshop Problems #2

March 1, 2010

1. Suppose that the line L_1 is $\begin{cases} x = 2t+1 \\ y = -t-1 \\ z = 3t \end{cases}$ and the line L_2 is $\begin{cases} x = 3s+2 \\ y = 5s-2 \\ z = -4 \end{cases}$. Define the function $f(s,t)$ to be the distance between the point on line L_1 with parameter value s and the point on the line L_2 with parameter value t.

a) Find and classify (max/min/saddle) all critical points of $f(s,t)$. (There is exactly one!)

b) The line segment which has endpoints characterized by the values of s and t discovered in a) has an interesting geometric property related to L_1 and L_2. What is this property? Use a drawing to help your explanation.

2. a) A rectangle with length L and width W is cut into four smaller rectangles by two lines which are parallel to the sides. Find the minimum value of the sum of the squares of the areas of the smaller rectangles.

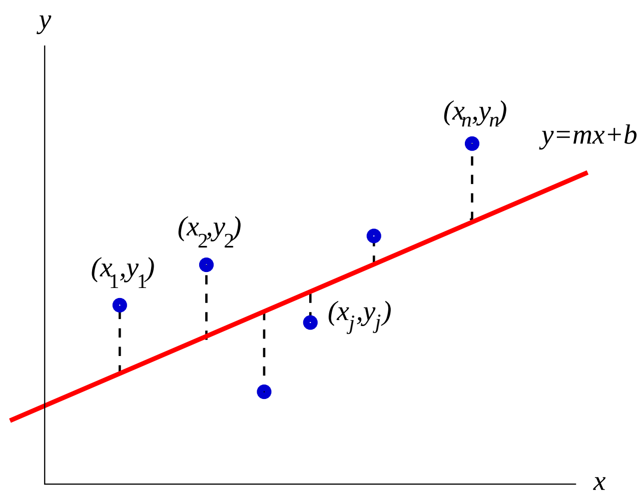
b) Show that the maximum of the sums of the squares of the areas occurs when the cutting lines correspond to sides of the rectangle (so that there is only one rectangle).

3. Given n data points $(x_1, y_1), \ldots, (x_n, y_n)$, we may seek a linear function $y = mx + b$ that best fits the data. The **linear least-squares fit** is the linear function $f(x) = mx + b$ that minimizes the sum of the squares (see the Figure) $E(m,b) = \sum_{nj=1}^{n} (y_j - f(x_j))^2$.

Show that E is minimized for m and b satisfying

$$m\sum_{j=1}^{n} x_j + bn = \sum_{j=1}^{n} y_j \text{ and } m\sum_{j=1}^{n} x_j^2 + b\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} x_j y_j$$

Comment This is problem 44 in the textbook's section 14.7. The result is really important in practical computation. Several assertions must be verified: that E has one critical point which is a local minimum, and that this local minimum is actually an *absolute* minimum.



4. A rectangular box with an open top has a square base. The sides are made of cardboard, costing 3 cents per square foot. The base is made of plywood, costing a half dollar per

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square foot. The box should have a capacity of no more than 10 cubic feet and no less than 2 cubic feet. At the same time, due to limitations of construction, no edge of the box should be shorter than 3 inches or longer than 36 inches. Find a plausible domain for the dimensions of the box based on these specifications and describe the domain carefully, algebraically. Sketch the domain in $\mathbb{R}^2$. (You *must* give a complete algebraic description of the domain, however. The picture is *not* a substitute for this description.) Write a formula for a function which calculates the cost of the materials in each possible box.

5. It is certainly possible for the set of critical points of a function defined in $\mathbb{R}^3$ to be a point (e.g., $x^2 + y^2 + z^2$) or a line (e.g., $x^2 + y^2$) or a plane (e.g., x^2). Can you create a function $F : \mathbb{R}^3 \to \mathbb{R}$ whose set of critical points is the twisted cubic, $\mathbf{c}(t) = (t, t^2, t^3)$?