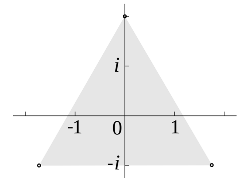
(10) 1. Describe all solutions of $z^3 = -8i$ algebraically in rectangular form. Sketch the solutions on the axes provided.

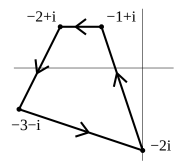


Answer Since $-8i = 8e^{i3\pi/2}$, one cube root is $8^{1/3}e^{i\frac{\pi}{2}}$ which is $2i$. The other cube roots are rotated by an angle of $\frac{2\pi}{3}$. That's multiplication by $w = e^{i2\pi/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. So another root is $2iw = -\sqrt{3} - i$ and the third root is $w\left(-\sqrt{3} - i\right) = \sqrt{3} - i$. The roots are shown on the graph and form an equilateral triangle.

(10) 2. Verify that if t is real, then $\frac{1}{t+i}$ is on the circle of radius $\frac{1}{2}$ centered at $-\frac{i}{2}$.
Answer I'll follow the hint and compute $\left|\frac{1}{t+i} - \left(-\frac{i}{2}\right)\right|$. In fact, since I hope this is $\frac{1}{2}$, I'll double it and try to see why the result is 1. So: $\left|\frac{2}{t+i} + i\right| = \left|\frac{2+i(t+i)}{t+i}\right| = \left|\frac{1+it}{t+i}\right| = \frac{|1+it|}{|t+i|}$. But $|1+it| = |\overline{1+it}| = |1-it| = |-i|\,|i+t| = |i+t|$ since $|i| = 1$. And the result is 1 as desired.

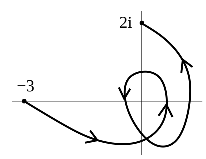
(10) 3. Suppose f is an entire function and that $\mathrm{Re}\big(f(z)\big) = \mathrm{Im}\big(f(z)\big)$ for all $z \in \mathbb{C}$. Show that f is constant.
Answer We know that $u(x,y) = v(x,y)$ always where u and v are, respectively, the real and imaginary parts of f. Then (1) $u_x = v_x$ and (2) $u_y = v_y$. One Cauchy-Riemann equation is (3) $u_x = v_y$. Then (1) and (2) imply $u_x = u_y$. The other Cauchy-Riemann equation is (4) $u_y = -v_x$. This and (1) implies $u_y = -u_x$. Together $u_x = u_y$ and $u_y = -u_x$ imply that $u_x = 0$ and $u_y = 0$ always, so u is constant. Similarly, (3) and (4) show that v's partial derivatives are both always 0, so v is constant. Therefore f must be constant.

(10) 4. Compute $\int_C e^{(z^2+5z-i)}dz$ where C is the indicated closed curve.



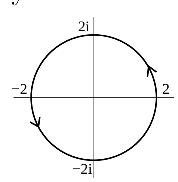
Answer I don't "know" an antiderivative of this integrand and I don't want to parameterize this curve. But $f(z) = e^{(z^2+5z-i)}$ is entire, so Cauchy's Theorem implies that the integral is $\mathbf{0}$. We are done!

(10) 5. Compute $\int_C z^2 - \frac{1}{z^2}\,dz$ where C is the indicated curve.

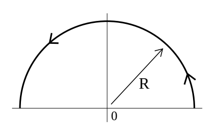


Answer I can't parameterize this curve (not enough information is given!) and the curve and function do not satisfy the hypotheses of Cauchy's Theorem. But I know that the derivative of $F(z) = \frac{z^3}{3} + \frac{1}{z}$ is the integrand. So the value of the integral is just $F(\mathtt{END}) - F(\mathtt{START})$ and this is $\left(\frac{(2i)^3}{3} + \frac{1}{2i}\right) - \left(\frac{(-3)^3}{3} + \frac{1}{-3}\right)$ which is a *fine answer* – you can leave it that way. If you insist, this works out to $\frac{28}{3} - \frac{19}{6}i$ (I hope!).

(10) 6. Compute $\int_C z^3 - \frac{1}{z}\,dz$ where C is the indicated closed curve (a circle oriented counterclockwise centered at 0 with radius 2). **Answer** Cauchy's Theorem doesn't apply since the integrand is *not* analytic inside the simple closed curve. I also don't know an analytic antiderivative valid for the entire contour (there's no "branch" analytic entirely around the origin!). So I'll parameterize. Here $z = 2e^{it}$ with t in $[0, 2\pi]$. Then $z^3 - \frac{1}{z}$ becomes $8e^{3it} - \frac{e^{-it}}{2}$ and $dz = 2i\,e^{it}dt$ so we compute $\int_0^{2\pi}(8e^{3it} - \frac{e^{-it}}{2})2i\,e^{it}dt = \int_0^{2\pi} 4ie^{4it} - i\,dt = e^{4it} - it\Big]_0^{2\pi} = e^{8\pi i} - 2\pi i - (e^0 - 0) = -2\pi i$ since exponential is $2\pi i$ periodic.



(10) 7. Suppose S_R is the upper semicircle of radius $R > 0$ centered at the origin as shown, A, B, and C are <u>positive</u> numbers, and $f(z) = \frac{e^{iAz}}{z^2+Bz+C}$. Prove that $\lim\limits_{R\to\infty} \int_{S_R} f(z)\ dz = 0$.



Answer We will use ML to get an overestimate on the modulus of the integral. Certainly L is just πR, the length of the semicircle S_R. Now we need to estimate $\left|\frac{e^{iAz}}{z^2+Bz+C}\right|$ when z is on S_R. This means both that $|z| = R$ <u>and</u> $\mathrm{Im}(z) \geq 0$. The second restriction is helpful because $|e^{iAz}| = e^{\mathrm{Re}(iAz)}$. But since z is in the upper halfplane, $iAz = iA(x+iy) = -y + iAx$. The real part, $-y$, is at most 0, and $|e^{iAz}| \leq 1$. So $|z^2 + Bz + C| \overset{\nabla}{\geq} |z^2| - |Bz + C| = R^2 - |Bz + C|$ and certainly $|Bz + C| \overset{\Delta}{\leq} B|z| + C = BR + C$. We combine and, on S_R, $|z^2 + Bz + C| \geq R^2 - (BR + C)$. Now put the top and bottom of $|f(z)|$ together: on S_R, $|f(z)| \leq \frac{1}{R^2-(BR+C)}$ and this last term is M. So $\left|\int_{S_R} f(z)\,dz\right| \leq \frac{\pi R}{R^2-(BR+C)}$. As $R \to \infty$, the quotient $\to 0$ because the degree of the bottom is larger than the degree of the top. Since the modulus of the integral $\to 0$, the integral itself must $\to 0$.

(10) 8. Consider the sum $\sum_{j=1}^{\infty} \frac{z^j}{3^j j^2}$. a) Find the radius of convergence of this series.

Answer I'll use the Ratio Test. So if $A_j = \frac{z^j}{3^j j^2}$, we consider $\frac{|A_{j+1}|}{|A_j|} = \frac{\left|\frac{z^j}{3^{j+1}(j+1)^2}\right|}{\left|\frac{z^j}{3^j j^2}\right|} = \frac{|z|^{j+1}3^j j^2}{|z|^j 3^{j+1}(j+1)^2} = \frac{|z|}{3}\left(\frac{j}{j+1}\right)^2$. Now take the limit as $j \to \infty$. The quotient $\frac{j}{j+1} \to 1$ (either L'H or divide top and bottom by j or … rather clearly). The limit is $\frac{|z|}{3}$. This is < 1 when $|z| < 3$. The radius of convergence of this series is 3.

b) For which z's does this series converge absolutely? For which z's must the series diverge? (Please *omit* behavior on the boundary of the circle of convergence.)

Answer The series converges absolutely when $|z| < 3$. The series diverges when $|z| > 3$. These results follow from the Ratio Test.

c) For which z's does the series converge? (Again, please *omit* behavior on the boundary of the circle of convergence.) Briefly explain why the answer to b) implies an answer to this question.

Answer The series must converge when $|z| < 3$. This follows from b) since "Absolute convergence of a series implies that the series converges."

Comment This series is $\operatorname{dilog}\left(\frac{x}{3}\right)$, where dilog is a known "special function", called the dilogarithm (really!).

(10) 9. a) Find all real numbers A and B so that the function $x^4 + Ax^2y^2 + By^4$ is harmonic.

Answer One x derivative gives $4x^3 + 2Axy^2$ and another x derivative results in $12x^2 + 2Ay^2$. Two y derivatives have the result $2Ax^2 + 12By^2$, and this means that the Laplacian is $12x^2 + 2Ay^2 + 2Ax^2 + 12By^2$. This will be 0 (as a function, not as a value at a specific (x, y)) when $12 + 2A = 0$ and $2A + 12B = 0$. The first equation implies that $A = -6$ and the second implies that $B = 1$. So the only numbers which result in a harmonic function are those. **Comment** $x^4 - 6x^2y^2 + y^4$ is $\text{Re}(z^4)$.

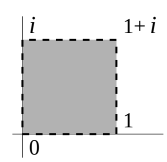
b) If $f(x, y)$ and $g(x, y)$ are both harmonic, must the sum $f(x, y) + g(x, y)$ always be harmonic also? If yes, briefly give a reason. If no, give an example so that $f(x, y) + g(x, y)$ is *not* harmonic.

Answer The sum of derivatives is the derivative of the sums, and so $\Delta(f + g) = \Delta(f) + \Delta(g)$. If both f and g are harmonic, the right-hand side is 0. Their sum will also be harmonic.

c) If $f(x, y)$ and $g(x, y)$ are both harmonic, must the product $f(x, y)g(x, y)$ always be harmonic also? If yes, briefly give a reason. If no, give an example so that $f(x, y)g(x, y)$ is *not* harmonic.

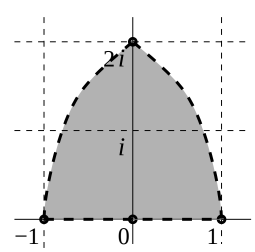
Answer No, products of derivatives don't work together simply. Indeed, $x^2 - y^2 = \text{Re}(z^2)$ is harmonic but $(x^2 - y^2)^2 = x^4 - 2x^2y^2 + y^4$ is *not* harmonic (see part a) where this is shown).

(10) 10. Suppose S is the open square with corners at 0, 1, $1 + i$, and i, shown to the right.



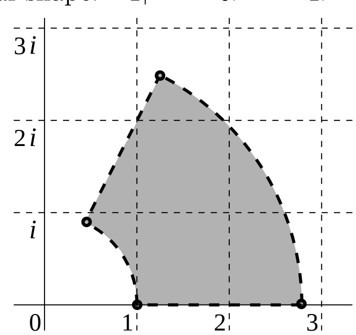
a) Sketch the image of S on the axes to the right under the mapping $z \to z^2$. But sure to indicate as well as you can what happens to the corners, the sides, and the interior of the square.

Answer $0^2 = 0$, $1^2 = 1$, $(1 + i)^2 = 2i$, and $i^2 = -1$. The •'s indicate the images of these points. The edge from 0 to 1 has image itself. The image of the edge from i to 0 is the line segment from -1 to 0. The vertical line $1 + ti$ for $0 \le t \le 1$ is squared and becomes $(1 - t^2) + 2ti$. This is part of the parabola $x = 1 - \frac{y^2}{4}$ from $(1, 0)$ to $(0, 2)$ (or from 1 to $2i$). The top edge, $s + i$ for $0 \le s \le 1$, changes to $(s^2 - 1) + 2si$, another parabola arc (part of $x = \frac{y^2}{4} - 1$ from $2i$ to -1, the reflection in the imaginary axis of the first arc). The inside of the square is changed to the inside of the triangular shape.



b) Sketch the image of S (the original square!) on the axes below under the mapping $z \to \exp(z) = e^z$. But sure to indicate as well as you can what happens to the corners, the sides, and the interior of the square.

Answer $e^0 = 1$, $e^1 = 2 \approx 2.7$, $e^{1+i} = e^1 e^i = e^1 \cos(1) + ie^1 \sin(1)$, and $e^i = \cos(1) + i \sin(1)$. Since $\frac{\pi}{4} < 1 < \frac{\pi}{2}$, I know $\cos(1) + i \sin(1)$ is a number of modulus 1 with argument between $\frac{\pi}{4}$ and $\frac{\pi}{2}$ (actually close to the lower number). e^{1+i} has the same argument, but has distance e from 0. The horizontal sides of the square get ransformed to radial line segments, and vertical sides of the square get changed to arcs of circles centered at 0. The interior of the square becomes the interior of this annular chunk.



Huh? If you *insist*, 1 radian is about 57.2957795 degrees. Does that help?