640:403:01 **Answers to the Second Exam** **4/20/2010**

(16) 1. If $f(z) = \frac{\sin(z^2)}{(\sin z)^2}$, find and classify *all* isolated singularities of f. If the isolated singularity is a pole, tell the order of the pole and the residue of f at the pole.

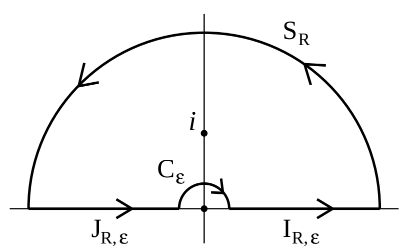
Answer If $z = x + iy$, $\sin z = (\sin x)(\cosh y) + i(\cos x)(\sinh y)$ and $\sin z = 0$ when (real part) $\sin x = 0$ and so (imaginary part) $\cos x \neq 0$ and $\sinh y = 0$. $\sin x = 0$ and $\sinh y = 0$ imply $z = n\pi$ where n is an integer.

In this problem, $H.O.T.$ stands for "higher order terms" in various power series.

$\underline{n = 0}$ Here $\sin(z^2) = z^2 - \frac{1}{3!}z^6 + H.O.T. = z^2(1 - \frac{1}{3!}z^2 + H.O.T.) = z^2 h_1(z)$ and $(\sin z)^2 = (z - \frac{1}{3!}z^3 + H.O.T.)^2 = z^2(1 - \frac{2}{3!}z^4 + H.O.T.) = z^2 h_2(z)$ where both h_1 and h_2 represent sums of power series with radius of convergence $= \infty$ so they are entire functions. Also, $h_1(0) = 1$ and $h_2(0) = 1$. Therefore, if $z \neq 0$ and $|z|$ is small (away from other 0's of h_2), $f(z) = 1 + H.O.T.$ and $f(z)$ has a removable singularity at 0.

$\underline{n \neq 0}$ $\sin(n^2\pi^2) \neq 0$ and $\sin(z^2) = \sin(n^2\pi^2) + 2n\pi\cos(n^2\pi^2)(z - n\pi) + H.O.T.$ (the second term comes from the Taylor series formula). And $\sin z = (-1)^n(z - n\pi) + \frac{(-1)^{n+1}}{3!}(z - n\pi)^3 + H.O.T.$ (again, Taylor) so $(\sin z)^2 = (z - n\pi)^2 - \frac{1}{2}(z - n\pi)^4 + H.O.T. = (z - n\pi)^2 h(z)$ where $h(z)$ is analytic near $n\pi$, $h(n\pi) = 1$, and $h'(n\pi) = 0$ (there's no degree 1 term in the series for h centered at $n\pi$). Thus we conclude that $f(z)$ has a pole of order 2 at $n\pi$ ($n \neq 0$ here!). Since $f(z) = \left(\frac{1}{(z-n\pi)^2}\right)\frac{\sin(z^2)}{h(z)}$ the residue will be the coefficient of the first order term in the Taylor expansion centered at $n\pi$ of $q(z) = \frac{\sin(z^2)}{h(z)}$. We need $q'(n\pi)$. We get $q'(z) = \frac{h(z)\cos(z^2)2z - h'(z)\sin(z^2)}{(h(z))^2}$ and then $q'(n\pi) = 2n\pi\cos(n^2\pi^2)$, the residue requested.

(16) 2. Use complex analysis to show that $\int_0^\infty \frac{\sqrt{x}}{(1+x^2)^2}\,dx = \frac{\pi\sqrt{2}}{8}$. Show clearly any contour of integration and any residue computation. Explain why the limiting value of certain integrals is 0.



Answer We use the indented contour described in the second example of Lecture #21 (4/12/2010). The contour has two parameters: R is a "large" positive real number ($R > 1$) and ε is a "small" positive real number ($0 < \varepsilon < 1$). The curve is a simple closed curve, the sum of four curves: $I_{R,\varepsilon}$, an interval on the real axis from ε to R; S_R, the upper semicircle of radius R centered at 0; $J_{R,\varepsilon}$, an interval on the real axis from $-R$ to $-\varepsilon$; and (backwards!) of radius ε centered at 0. We apply the Residue Theorem to this curve with $f(z) = \frac{\sqrt{z}}{(1+z^2)^2}$. Here $\sqrt{z}$ will be defined and analytic, the "principal branch", in the closed upper half plane except for 0. $\sqrt{z}$ is *neither* analytic *nor* does it have an isolated singularity at 0. The indentation takes care of this problem which *can't* be ignored!

<u>The residue computation</u> Since $f(z) = \left(\frac{1}{(z-i)^2}\right)\frac{\sqrt{z}}{(z+i)^2}$, we know that $f(z)$ has a pole of order 2 at i, and (just as in the previous problem) the residue of f at i will be the first derivative of $\frac{\sqrt{z}}{(z+i)^2}$ at i. The first derivative is $\frac{\frac{1}{2\sqrt{z}}(z+i)^2 - \sqrt{z}2(z+i)}{(z+i)^4} = \frac{\frac{1}{2}(z+i) - 2z}{\sqrt{z}(z+i)^3}$. When $z = i$, $\sqrt{z} = \frac{1+i}{\sqrt{2}}$ (principal branch) and the expression is $\frac{\frac{1}{2}(2i) - 2i}{\left(\frac{1+i}{\sqrt{2}}\right)(2i)^3}$ (wow!) or just $\frac{\sqrt{2}(1-i)}{16}$.

The S_R integral The modulus of $f(z)$ on S_R is overestimated by $\frac{\sqrt{R}}{(R^2-1)^2}$ and the length of S_R is πR. Therefore ML implies $\left|\int_{S_R} f(z)\,dz\right| \le \frac{\pi R^{3/2}}{(R^2-1)^2}$, so this integral $\to 0$ as $R\to\infty$.

The C_ε integral The modulus of $f(z)$ on C_ε is overestimated by $\frac{\sqrt{\varepsilon}}{(1-\varepsilon^2)^2}$ and the length of C_ε is $\pi\varepsilon$. Here ML implies $\left|\int_{C_\varepsilon} f(z)\,dz\right| \le \frac{\pi \varepsilon^{3/2}}{(1-\varepsilon^2)^2}$, so this integral $\to 0$ as $\varepsilon \to 0^+$.

The $I_{R,\varepsilon}$ integral If $z = x + i0$, then $\int_{I_{R,\varepsilon}} f(z)\,dz = \int_\epsilon^R \frac{\sqrt{x}}{(1+x^2)^2}\,dx$ and, as $R\to\infty$ and $\varepsilon\to 0^+$, this integral $\to \int_0^\infty \frac{\sqrt{x}}{(1+x^2)^2}\,dx$, the "target" integral.

The $J_{R,\varepsilon}$ integral If $z = x + i0$ with $x < 0$, $\sqrt{z}$ (principal branch!) is $i\sqrt{-x}$. So $\int_{J_{R,\varepsilon}} f(z)\,dz = \int_{-R}^{-\epsilon} \frac{i\sqrt{-x}}{(1+x^2)^2}\,dx = -\int_R^\epsilon \frac{i\sqrt{w}}{(1+w^2)^2}\,dw$ (the last equality results from the substitution $x \mapsto -w$). But this means that as $R\to\infty$ and $\varepsilon\to 0^+$, the $J_{R,\varepsilon}$ integral $\to i$ times the target integral, which I'll call I.

The Residue Theorem now asserts that $\int_{\text{indented contour}} f(z)\,dz = (2\pi i)(\text{the residue at } i) = (2\pi i)\left(\frac{\sqrt{2}(1-i)}{16}\right) = \frac{\sqrt{2}\pi(1+i)}{8}$. But as $R\to\infty$ and $\varepsilon\to 0^+$, the limit of the integral is $(1+i)I$ and therefore I is indeed $\frac{\sqrt{2}\pi}{8}$.

(16) 3. Suppose $f(z)$ is entire and $\sum_{n=0}^{\infty} a_n z^n$ is its Taylor series centered at 0. Let $g(z) = f\left(\frac{1}{z}\right)$.

a) Write $g(z)$'s Laurent series in the annulus $0 < |z| < \infty$ using what's given about $f(z)$.

Answer Since $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^{-n}$. This is a Laurent series (powers of z and $\frac{1}{z}$, and since such series are unique, this is the desired Laurent series.

b) Suppose we know that if $\{z_n\}$ is *any* sequence in $\mathbb{C}$ with $\lim_{n\to\infty} z_n = \infty$, then $\lim_{n\to\infty} f(z_n) = \infty$. Classify the isolated singularity of $g(z)$ at 0 with some explanation. What must then be true about the Laurent coefficients of $g(z)$?

Answer g has a pole at 0. This is because if $\{z_n\}$ is any non-zero sequence of non-zero complex numbers with $\lim_{n\to\infty} z_n = 0$, we know that $\lim_{n\to\infty} \frac{1}{z_n} = \infty$, so by the hypothesis assumed about f, we know $\lim_{n\to\infty} f\left(\frac{1}{z_n}\right) = \infty$ so $\lim_{n\to\infty} g(z_n) = \infty$. This is exactly the defining behavior of a pole. The Laurent series of g must have only finitely many non-zero coefficients associated to negative powers of z, so there is some N with $a_n = 0$ when $n > N$. The inequality is turned around because of the way the series of f and g are related.

c) Suppose that an entire function $f(z)$ has the property that if $\{z_n\}$ is *any* sequence in $\mathbb{C}$ with $\lim_{n\to\infty} z_n = \infty$, then $\lim_{n\to\infty} f(z_n) = \infty$. Show that $f(z)$ must be a non-constant polynomial.

Answer The previous answer shows there is N so that $f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{N} a_n z^n$. The finite sum is a polynomial. A constant polynomial doesn't have the behavior given, so f is a non-constant polynomial.

(18) 4. Suppose $\{f_n(z)\}$ is a sequence of analytic functions in the unit disc (z's with $|z| < 1$). Also suppose there is a function $f(z)$ defined on the unit disc so that if n is a positive integer and if $|z| < 1$ then $|f_n(z) - f(z)| < \frac{1}{n}$.

a) Prove that $f(z)$ is analytic in the unit disc.

Answer Use Morera's Theorem. Suppose C is a simple closed curve in the unit disc. Then $\left|\int_C (f(z) - f_n(z))\, dz\right| \leq \int_C |f(z) - f_n(z)|\, dz \leq \frac{\text{Length of } C}{n}$. But (Cauchy's Theorem) we know $\int_C f_n(z)\, dz = 0$. Therefore $\left|\int_C f(z)\, dz\right| \leq \frac{\text{Length of } C}{n}$ for all positive integers, n. This can only happen if $\int_C f(z)\, dz = 0$. Morera's Theorem then implies $f(z)$ is analytic.

b) Prove that the sequence $\{f_n'(0)\}$ converges and that its limit is $f'(0)$.

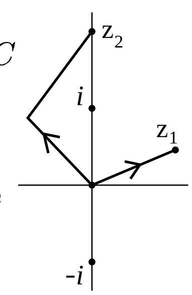
Answer Suppose C is the circle of radius $\frac{1}{2}$ centered at 0. We know from a) that $f(z)$ is analytic. Then the Cauchy Integral Formula for Derivatives tells us that $f'(0) - f_n(0) = \frac{1}{2\pi i} \int_C \frac{f(s) - f_n(s)}{(s-z)^2} ds$. The ML estimate gives us $|f(0) - f_n(0)| \leq \frac{\pi}{2\pi} \frac{\frac{1}{n}}{\frac{1}{4}} = \frac{1}{2n}$. But $\frac{1}{2n} \to 0$ as $n \to \infty$ so $\lim_{n\to\infty} f_n'(0) = f'(0)$.

c) Give an example of a sequence of differentiable functions $\{g_n(x)\}$ in *real calculus* defined for $-1 < x < 1$ with $\lim_{n\to\infty} g_n(x) = 0$ for every x in $(-1, 1)$ but the sequence $\{g_n'(0)\}$ does *not* converge.

Answer Suppose $g_n(x) = \frac{(-1)^n}{n} \sin(nx)$. Then certainly since the max of $|g_n(x)| = \frac{1}{n}$ (x is *real* here!) we know that $\lim_{n\to\infty} g_n(x) = 0$ for every x. But since $g_n'(0) = (-1)^n \cos(n \cdot 0) = (-1)^n$, the sequence $\{g_n'(0)\}$ does *not* converge.

(18) 5. In "real calculus", $\arctan'(x) = \frac{1}{1+x^2}$ and $\arctan(0) = 0$. Define $A(z)$ for complex z as follows:

If z is not $\pm i$, then $A(z) = \int_C \frac{1}{1+w^2}\, dw$ where C is a curve (*any curve*: C need not be simple!) from 0 to z not passing through $\pm i$.



a) If z is not equal to $\pm i$, explain briefly why there must be such a curve.

Comment Pictures accompanied by some written discussion (with complete sentences!) will be sufficient.

Answer When $z \neq \pm i$, consider z's reachable from 0 with one line segment. The situation for such z_1 is shown. Other z's (with $\text{Re}(z) = 0$ and and $|\text{Im}(z)| > 1$) need a piecewise linear path with two pieces as is shown with z_2.
More briefly, the question is answered by noticing that U is a connected open set.

b) Will the result of the integration always be the same, no matter which eligible curve C is chosen? Explain why or why not.

Comment First discuss any values of $A(1)$ as completely as possible. This then implies results about $A(z)$ for all $z \neq \pm i$ and those results should be presented.

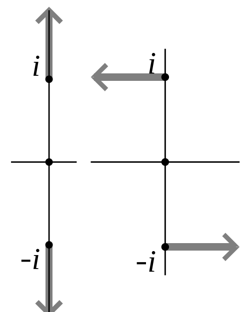
Answer The result will not always be the same. Since $\frac{1}{1+z^2} = \frac{-\frac{i}{2}}{z-i} + \frac{\frac{i}{2}}{z+i}$, the Residue Theorem implies when a curve goes from 0 to z, every time it winds counterclockwise around i it picks up a value (residue!) of $2\pi i \cdot \left(-\frac{i}{2}\right) = \pi$ and, around $-i$, the value picked up is $-\pi$. Therefore the values of $A(1)$ could be the "standard" value from calculus (that's $\frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2}\, dx$, the usual calculus integral) with any integer multiple of π added.

Generally, there will be infinitely many values of $A(z)$ as described: a specific number obtained by using any one path, C, and then any integer multiple of π added to that number because the path C can be modified as desired by loops around the singularities.

c) The answer to b) will show that $A(z)$ is "multivalued". Explain how the domain of $A(z)$ can be restricted so that this defect is repaired: find a largest connected open set, U, containing $\mathbb{R}$ so $A(z)$ is a (single-valued!) function whose derivative is $\frac{1}{1+z^2}$ and also $A(z)$ is the usual $\arctan(x)$ when $z = x$ is a real number.

Comment Many different U's are valid solutions to this problem. Explain why the suggested U verifies the requirements.

Answer Two candidates for U are shown to the right. The immediate right picture shows two closed halflines on the negative imaginary axis excluded and what remains is U. Then Cauchy's Theorem implies that the integral along any two curves from 0 to z in the part of the complex plane remaining will be the same: there will be no way to go "around" either i or $-i$, and therefore no "pickup" of $\pm\pi$ can occur. Since the real axis is inside the set, the integral along any C will be the same as the usual real integral, giving the standard value of arctan.



Another candidate for U is shown to the far right. Two infinite closed horizontal halflines are deleted from $\mathbb{C}$ to create U. The same logic applies, and the $A(z)$ defined on this set will satisfy all the requirements given.

(16) 6. Suppose $u(x, y)$ is a *harmonic function* defined for all of $\mathbb{R}^2$, and that $u(x, y)$ is bounded above: there is a real number M so that $u(x, y) \leq M$ for all (x, y) in $\mathbb{R}^2$. Prove that $u(x, y)$ must be constant.

Hint Math 403 mostly investigates *analytic functions*. We discussed creating an analytic function function $f(z)$ with $\mathrm{Re}\,(f(z)) = u(x, y)$ under certain circumstances. Show that there is such an *entire* function in this case. Then explain why this $f(z)$ must be constant.

Answer In Lecture #10 on February 10 we (painfully) created a harmonic conjugate for every function harmonic in a disc. $\mathbb{C}$ is a disc of radius ∞, so $u(x, y)$ has a harmonic conjugate $v(x, y)$ in $\mathbb{C}$, and the function $f(z) = u(x, y) + iv(x, y)$ is entire. Since $u(x, y) \leq M$, the range of f (this result was previously assigned as problem 20, section 2.4, or see the discussion after Liouville's Theorem in Lecture #17 for a different approach) is contained in a halfplane, and therefore $f(z)$ is constant and so is $u(x, y)$.

Comment What does this say about using such u's as models for steady-state heat flow?