

# $\Gamma$ -convergence and Sobolev norms

## $\Gamma$ -convergence et normes de Sobolev

Hoai-Minh NGUYEN <sup>a</sup>

<sup>a</sup>*Rutgers University, Dept. of Math., Hill Center, Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA*

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### Abstract

We study a  $\Gamma$ -convergence problem related to a new characterization of Sobolev spaces  $W^{1,p}(\mathbb{R}^N)$  ( $p > 1$ ) established in H-M. Nguyen [Some new characterizations of Sobolev spaces, J. Funct. Anal. 237 (2006) 689–720] and J. Bourgain and H-M. Nguyen [A new characterization of Sobolev spaces, C. R. Acad. Sci. Paris, Ser. I 343 (2006), 75–80]. We can also handle the case  $p = 1$  which was out of reach previously.

### Résumé

On étudie un problème de  $\Gamma$ -convergence qui apparaît naturellement en liaison avec les travaux de H-M. Nguyen [Some new characterizations of Sobolev spaces, J. Funct. Anal. 237 (2006) 689–720], et J. Bourgain et H-M. Nguyen [A new characterization of Sobolev spaces, C. R. Acad. Sci. Paris, Ser. I 343 (2006), 75–80] concernant des nouvelles caractérisations des espaces de Sobolev  $W^{1,p}(\mathbb{R}^N)$  ( $p > 1$ ). On peut aussi traiter le cas  $p = 1$  qui était inaccessible précédemment.

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### Version française abrégée

Soient  $p \geq 1$  et  $\delta > 0$ . Posons

$$I_\delta(g) = \iint_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ |g(x) - g(y)| > \delta}} \frac{\delta^p}{|x - y|^{N+p}} dx dy, \quad \forall g \in L^p(\mathbb{R}^N).$$

Ci-après  $|\cdot|$  désigne la norme euclidienne de  $\mathbb{R}^N$ . Récemment la caractérisation suivante des espaces de Sobolev a été établie dans [10, Théorème 2] et [3, Théorème 1].

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*Email address:* hoaiminh@math.rutgers.edu (Hoai-Minh NGUYEN).

**Théorème 0.1** Soient  $N \geq 1$ ,  $1 < p < +\infty$ , et  $g \in L^p(\mathbb{R}^N)$ . Alors  $g \in W^{1,p}(\mathbb{R}^N)$  si et seulement si

$$\liminf_{\delta \rightarrow 0_+} I_\delta(g) < +\infty.$$

De plus

$$\lim_{\delta \rightarrow 0_+} I_\delta(g) = \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |Dg|^p dx, \quad \forall g \in W^{1,p}(\mathbb{R}^N),$$

où  $K_{N,p}$  est définie par

$$K_{N,p} = \int_{\mathbb{S}^{N-1}} |e \cdot \sigma|^p d\sigma,$$

pour tout  $e \in \mathbb{S}^{N-1}$ .

Rappelons aussi que lorsque  $p = 1$ , on a

a) Si  $g \in L^1(\mathbb{R}^N)$  et  $\liminf_{\delta \rightarrow 0_+} I_\delta(g) < +\infty$ , alors  $g \in BV(\mathbb{R}^N)$  (voir [3,12]).

b)  $\exists g \in W^{1,1}(\mathbb{R})$  telle que  $\lim_{\delta \rightarrow 0_+} I_\delta(g) = +\infty$  (exemple communiqué par A. Ponce; voir [10]).

Le résultat principal de cette Note est

**Théorème 0.2** Soient  $p \geq 1$  et  $N \geq 1$ . Alors  $(I_\delta)$   $\Gamma$ -converge dans  $L^p(\mathbb{R}^N)$  quand  $\delta$  tend vers 0 vers la fonctionnelle  $I$  définie sur  $L^p(\mathbb{R}^N)$  par

$$I(g) = \begin{cases} C_{N,p} \int_{\mathbb{R}^N} |Dg|^p dx & \text{si } p > 1 \text{ et } g \in W^{1,p}(\mathbb{R}^N) \text{ (resp. } p = 1 \text{ et } g \in BV(\mathbb{R}^N)), \\ +\infty & \text{sinon.} \end{cases}$$

Ici, la constante  $C_{N,p}$  est définie par (2) et vérifie  $0 < C_{N,p} < \frac{1}{p} K_{N,p}$ .

## 1. Introduction and the main result

For  $p \geq 1$  and  $\delta > 0$ , define

$$I_\delta(g) = \iint_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ |g(x) - g(y)| > \delta}} \frac{\delta^p}{|x - y|^{N+p}} dx dy, \quad \forall g \in L^p(\mathbb{R}^N).$$

Hereafter  $|\cdot|$  denotes the Euclidean norm of  $\mathbb{R}^N$ . Recently, the following new characterization of Sobolev spaces was established in [10, Theorem 2] and [3, Theorem 1].

**Theorem 1** Let  $N \geq 1$ ,  $1 < p < +\infty$ , and  $g \in L^p(\mathbb{R}^N)$ . Then  $g \in W^{1,p}(\mathbb{R}^N)$  if and only if

$$\liminf_{\delta \rightarrow 0_+} I_\delta(g) < +\infty.$$

Moreover,

$$\lim_{\delta \rightarrow 0_+} I_\delta(g) = \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |Dg|^p dx, \quad \forall g \in W^{1,p}(\mathbb{R}^N),$$

where  $K_{N,p}$  is given by

$$K_{N,p} = \int_{\mathbb{S}^{N-1}} |e \cdot \sigma|^p d\sigma,$$

for any  $e \in \mathbb{S}^{N-1}$ .

We recall that when  $p = 1$ ,

- a) If  $g \in L^1(\mathbb{R}^N)$  and  $\liminf_{\delta \rightarrow 0^+} I_\delta(g) < +\infty$ , then  $g \in BV(\mathbb{R}^N)$  (see [3,12]).  
b)  $\exists g \in W^{1,1}(\mathbb{R})$  such that  $\lim_{\delta \rightarrow 0^+} I_\delta(g) = +\infty$  (example communicated to us by A. Ponce ; see [10]).

This characterization is distinct from the one of J. Bourgain, H. Brezis, and P. Mironescu [1] (see also [5]) but it is inspired by the results of [1]. Quantities similar to  $I_\delta$  appear in new estimates for the degree (see [2,11,6]). Further results related to Theorem 1 are presented in [12,14] and in a recent work of D. Chiron [7].

Let  $p \geq 1$  and  $N \geq 1$ . Define, for  $g \in L^p(\mathbb{R}^N)$ ,

$$J(g) = \begin{cases} \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |Dg|^p dx & \text{if } p > 1 \text{ and } g \in W^{1,p}(\mathbb{R}^N) \text{ (resp. } p = 1 \text{ and } g \in BV(\mathbb{R}^N)), \\ +\infty & \text{otherwise.} \end{cases}$$

A natural question raised by H. Brezis (personal communication) is whether  $(I_\delta)$   $\Gamma$ -converges in  $L^p(\mathbb{R}^N)$  to  $J$  in the sense of E. De Giorgi for  $p > 1$  (see e.g. [4,9] for an introduction of  $\Gamma$ -convergence). We recall that a family  $(I_\delta)_{\delta \in (0,1)}$  of functionals defined on  $L^p(\mathbb{R}^N)$   $\Gamma$ -converges in  $L^p(\mathbb{R}^N)$ , as  $\delta$  goes to 0, to a functional  $I$  defined on  $L^p(\mathbb{R}^N)$  if and only if the following two conditions are satisfied :

- (G1) For each  $g \in L^p(\mathbb{R}^N)$  and for every family  $(g_\delta)_{\delta \in (0,1)} \subset L^p(\mathbb{R}^N)$  such that  $g_\delta$  converges to  $g$  in  $L^p(\mathbb{R}^N)$  as  $\delta$  goes to 0, one has

$$\liminf_{\delta \rightarrow 0} I_\delta(g_\delta) \geq I(g).$$

- (G2) For each  $g \in L^p(\mathbb{R}^N)$ , there exists a family  $(g_\delta)_{\delta \in (0,1)} \subset L^p(\mathbb{R}^N)$  such that  $g_\delta$  converges to  $g$  in  $L^p(\mathbb{R}^N)$  as  $\delta$  goes to 0, and

$$\limsup_{\delta \rightarrow 0} I_\delta(g_\delta) \leq I(g).$$

Surprisingly,  $(I_\delta)$  does not  $\Gamma$ -converge to  $J$  in  $L^p(\mathbb{R}^N)$  for  $p > 1$  but it  $\Gamma$ -converges to  $\lambda J$  for some  $0 < \lambda < 1$ ; the same fact holds for the case  $p = 1$ . More precisely, we have

**Theorem 2** *Let  $p \geq 1$  and  $N \geq 1$ . Then  $(I_\delta)$   $\Gamma$ -converges in  $L^p(\mathbb{R}^N)$  to  $I$  defined by, for all  $g \in L^p(\mathbb{R}^N)$ ,*

$$I(g) = \begin{cases} C_{N,p} \int_{\mathbb{R}^N} |Dg|^p dx & \text{if } p > 1 \text{ and } g \in W^{1,p}(\mathbb{R}^N) \text{ (resp. } p = 1 \text{ and } g \in BV(\mathbb{R}^N)), \\ +\infty & \text{otherwise.} \end{cases}$$

Here the constant  $C_{N,p}$  is defined by (2) below and satisfies

$$0 < C_{N,p} < \frac{1}{p} K_{N,p}. \quad (1)$$

For  $p \geq 1$  and  $N \geq 1$ , the definition of the constant  $C_{N,p}$  is the following

$$C_{N,p} := \inf \liminf_{\delta \rightarrow 0} \iint_{Q^2} \frac{\delta^p}{|x-y|^{N+p}} dx dy, \quad (2)$$

$|h_\delta(x) - h_\delta(y)| > \delta$

where the infimum is taken over all families of measurable functions  $(h_\delta)_{\delta \in (0,1)}$  defined on the unit open cube  $Q$  of  $\mathbb{R}^N$  such that  $h_\delta$  converges to  $h(x) \equiv \frac{x_1 + \dots + x_N}{\sqrt{N}}$  in (Lebesgue) measure on  $Q$  as  $\delta$  goes to 0.

## 2. Sketch of the proof

The proof is quite long (about 40 pages) and it is divided into three steps :

**Step 1** : Proof of Property (G2).

Claim 1 : Let  $p \geq 1$  and  $N \geq 1$ . Then for each  $g \in W^{1,p}(\mathbb{R}^N)$  if  $p > 1$ , or  $g \in BV(\mathbb{R}^N)$  if  $p = 1$ , there exists a family  $(g_\delta)_{\delta \in (0,1)} \subset L^p(\mathbb{R}^N)$  such that  $g_\delta$  converges to  $g$  in  $L^p(\mathbb{R}^N)$  as  $\delta$  goes to 0, and

$$\limsup_{\delta \rightarrow 0} I_\delta(g_\delta) \leq I(g).$$

The proof of Claim 1 is quite involved. We mention here main steps of the proof for the case  $N = 1$  :

- (a) We show that there exists a family  $(h_\delta)$  in  $L^p(0,1)$  defined for all  $\delta \in (0,1)$  (not just for a sequence  $\delta_n \rightarrow 0$ ) such that  $h_\delta$  converges to  $h(x) \equiv x$  in  $L^p(0,1)$  and

$$\lim_{\delta \rightarrow 0} \iint_{\substack{[0,1]^2 \\ |h_\delta(x) - h_\delta(y)| > \delta}} \frac{\delta^p}{|x - y|^{p+1}} dx dy = C_{1,p}.$$

- (b) We prove Claim 1 in the case  $g$  is continuous and piecewise linear with compact support. To this end, on each interval  $K$  where  $g$  is linear, using (a) we can find a family of functions  $(h_{K,\delta}) \subset L^p(K)$  such that  $h_{K,\delta}$  converges to  $g$  in  $L^p(K)$  and

$$\lim_{\delta \rightarrow 0} \iint_{\substack{K^2 \\ |h_{K,\delta}(x) - h_{K,\delta}(y)| > \delta}} \frac{\delta^p}{|x - y|^{p+1}} dx dy = C_{1,p} |g'(x_0)|^p |K|,$$

for some  $x_0 \in K$ . Then we glue these functions and construct a function  $g_\delta$  on  $\mathbb{R}$ . This is delicate since  $I_\delta$  is very sensitive to jumps.

- (c) We deduce Claim 1 from (b) by using the fact that if  $g$  is as in Claim 1, then there exists a sequence of continuous and piecewise linear functions with compact support  $(\phi_n)$  such that  $\phi_n$  converges to  $g$  in  $L^p(\mathbb{R})$  and  $\|D\phi_n\|_{L^p(\mathbb{R})}$  converges to  $\|Dg\|_{L^p(\mathbb{R})}$  (when  $p = 1$  the  $L^1$ -norm is replaced by the total mass).

Proof of Property (G2) follows from Claim 1 and the definition of  $I$ .

**Step 2** : Proof of Property (G1).

Claim 2 : Let  $p \geq 1$  and  $N \geq 1$ . Then for any  $g \in W^{1,p}(\mathbb{R}^N)$  if  $p > 1$  or  $g \in BV(\mathbb{R}^N)$  if  $p = 1$ , and for any family  $(g_\delta)_{\delta \in (0,1)} \subset L^p(\mathbb{R}^N)$  such that  $g_\delta$  converges to  $g$  in  $L^p(\mathbb{R}^N)$  as  $\delta$  goes to 0, one has

$$\liminf_{\delta \rightarrow 0} I_\delta(g_\delta) \geq I(g).$$

The proof of Claim 2 for the case  $p > 1$  and  $N = 1$  follows from the definition of  $C_{1,p}$  and the fact that any function in  $W^{1,p}(\mathbb{R})$  is locally approximately linear in the sense of measure (see e.g. [8, Theorem 4 on page 223] and the remark below it). In the case  $p > 1$  and  $N > 1$ , one uses the same idea as in the one dimensional case. However, it is more technical. When  $p = 1$ , we can not directly apply the method used in the case  $p > 1$ . In this case, the proof relies on some new ingredients and a new characterization of BV functions which we introduce in [13]. In the proof, we also use the structure theorem for BV functions (see e.g. [8, Theorem 1 on page 167]), the differentiation theorem of Radon measures (see e.g. [8, Theorem 1 on page 38]) and Besicovitch's covering theorem.

Claim 3 : Let  $p \geq 1$ ,  $N \geq 1$ , and  $g \in L^p(\mathbb{R}^N)$ . Assume that there exists a family  $(g_\delta)_{\delta \in (0,1)} \subset L^p(\mathbb{R}^N)$  such that  $g_\delta$  converges to  $g$  in  $L^p(\mathbb{R}^N)$  and

$$\liminf_{\delta \rightarrow 0} I_\delta(g_\delta) < +\infty.$$

Then  $g \in W^{1,p}(\mathbb{R}^N)$  if  $p > 1$  (resp.  $g \in BV(\mathbb{R}^N)$  if  $p = 1$ ); moreover

$$J(g) \leq C \liminf_{\delta \rightarrow 0} I_\delta(g_\delta),$$

for some  $C > 0$  depending only on  $N$  and  $p$ .

Claim 3 was proved in [12] (see [12, Theorem 3]); the proof in [12] relies heavily on the ideas of [3]. Property (G1) now follows from Claims 2 and 3.

**Step 3** : Proof of (1).

Let  $g$  and  $g_\delta$  be defined on  $\mathbb{R}^N$  by

$$g(x) = \begin{cases} |x| & \text{if } |x| \leq 1, \\ 1/|x|^{2N} & \text{otherwise,} \end{cases}$$

and

$$g_\delta(x) = \begin{cases} (k+1)\delta & \text{if } k\delta \leq |x| < (k+1)\delta \text{ for } 0 \leq k \leq [1/\delta], \\ 1/|x|^{2N} & \text{otherwise.} \end{cases}$$

Here  $[1/\delta]$  denotes the largest integer less than  $1/\delta$ . Then  $g_\delta$  converges to  $g$  in  $L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  as  $\delta$  goes to 0. On the other hand, it is easy to see that

$$\liminf_{\delta \rightarrow 0} [I_\delta(g) - I_\delta(g_\delta)] \geq \liminf_{\delta \rightarrow 0} \sum_{k=0}^{[1/\delta]} \iint_{\substack{k\delta \leq |x| \leq (k+\frac{1}{2})\delta \\ (k+\frac{3}{2})\delta \leq |y| \leq (k+2)\delta}} \frac{\delta^p}{|x-y|^{N+p}} dx dy,$$

$$\lim_{\delta \rightarrow 0} I_\delta(g) = \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |Dg|^p dx,$$

and

$$\liminf_{\delta \rightarrow 0} \sum_{k=0}^{[1/\delta]} \iint_{\substack{k\delta \leq |x| \leq (k+\frac{1}{2})\delta \\ (k+\frac{3}{2})\delta \leq |y| \leq (k+2)\delta}} \frac{\delta^p}{|x-y|^{N+p}} dx dy > 0.$$

As a result, one obtains

$$\limsup_{\delta \rightarrow 0} I_\delta(g_\delta) < \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |Dg|^p dx,$$

and therefore,

$$C_{N,p} < \frac{1}{p} K_{N,p}.$$

On the other hand, as a consequence of Claims 1 and 3, one has  $C_{N,p} > 0$ . This completes the proof of Step 3.  $\square$

We do not know the explicit value of the constant  $C_{N,p}$ . But we have a guest when  $N = 1$ . Let  $h$  and  $h_n$  be functions defined on  $(0, 1)$  by  $h(x) = x$  on  $(0, 1)$  and  $h_n(x) = \frac{k-1}{n}$  if  $\frac{k-1}{n} \leq x < \frac{k}{n}$  for  $1 \leq k \leq n$ . An easy computation shows that

$$\lim_{n \rightarrow \infty} \iint_{\substack{[0,1]^2 \\ |h_n(x) - h_n(y)| > 1/n}} \frac{1/n^p}{|x-y|^{p+1}} dx dy = c_{1,p},$$

where

$$c_{1,p} = \begin{cases} \frac{2}{p(p-1)} \left(1 - \frac{1}{2^{p-1}}\right) & \text{if } p > 1, \\ 2 \ln 2 & \text{if } p = 1. \end{cases}$$

Clearly,

$$c_{1,p} \geq C_{1,p}.$$

The following open question is suggested.

**Open question 1** *Is  $C_{1,p}$  equal to  $c_{1,p}$  ?*

The detailed proofs of the results discussed in this Note will be presented in [13].

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