Γ-convergence, Sobolev norms, and BV functions

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Abstract

We prove that the functional I_δ, defined by

$$I_\delta(g) = \iint\limits_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy, \quad \forall\, g \in L^p(\mathbb{R}^N),$$

for $p \geq 1$ and $\delta > 0$, Γ-converges in $L^p(\mathbb{R}^N)$, as δ goes to 0, for $p \geq 1$. Hereafter $|\ |$ denotes the Euclidean norm of $\mathbb{R}^N$. We also introduce a criterion to recognize BV functions which is quite close to the classic one based on the notion of essential variation on a.e. line.

1 Introduction

Recently, the following new characterization of Sobolev spaces is established in [10, Theorem 2] and [3, Theorem 1].

Theorem 1 *Let $N \geq 1$, $1 < p < +\infty$, and $g \in L^p(\mathbb{R}^N)$. Then $g \in W^{1,p}(\mathbb{R}^N)$ if and only if*

$$\varliminf_{\delta \to 0_+} I_\delta(g) < +\infty.$$

Moreover,

$$\lim_{\delta \to 0_+} I_\delta(g) = \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |Dg(x)|^p\, dx, \quad \forall\, g \in W^{1,p}(\mathbb{R}^N),$$

where $K_{N,p}$ is defined as follows

$$K_{N,p} = \int_{\mathbb{S}^{N-1}} |e \cdot \sigma|^p\, d\sigma, \tag{1.1}$$

for any $e \in \mathbb{S}^{N-1}$.

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We recall that when $p = 1$,

a) If $g \in L^1(\mathbb{R}^N)$ and $\underline{\lim}_{\delta\to 0_+} I_\delta(g) < +\infty$, then $g \in BV(\mathbb{R}^N)$ (see [3, 12]).

b) $\exists\, g \in W^{1,1}(\mathbb{R})$ such that $\lim_{\delta\to 0_+} I_\delta(g) = +\infty$ (example communicated to us by A. Ponce; see [10]).

This characterization is distinct from the one of J. Bourgain, H. Brezis, and P. Mironescu [1] (see also [5]) but it is inspired by the results of [1]. Quantities similar to I_δ appear in new estimates for the degree (see [2, 11, 6]). Further results related to Theorem 1 are presented in [12, 14] and in a recent work of D. Chiron [7].

Let $p \geq 1$ and $N \geq 1$. Define, for $g \in L^p(\mathbb{R}^N)$,

$$J(g) = \begin{cases} \dfrac{1}{p} K_{N,p} \displaystyle\int_{\mathbb{R}^N} |Dg|^p \, dx & \text{if } p > 1 \text{ and } g \in W^{1,p}(\mathbb{R}^N) \\ & (\text{resp. } p = 1 \text{ and } g \in BV(\mathbb{R}^N)), \\ +\infty & \text{otherwise.} \end{cases}$$

A natural question raised by H. Brezis (personal communication) is whether (I_δ) Γ-converges in $L^p(\mathbb{R}^N)$ to J in the sense of E. De Giorgi for $p > 1$ (see e.g. [4, 9] for an introduction of Γ-convergence). We recall that a family $(I_\delta)_{\delta\in(0,1)}$ of functionals defined on $L^p(\mathbb{R}^N)$ Γ-converges in $L^p(\mathbb{R}^N)$, as δ goes to 0, to a functional I defined on $L^p(\mathbb{R}^N)$ if and only if the following two conditions are satisfied:

(G1) For each $g \in L^p(\mathbb{R}^N)$ and for every family $(g_\delta)_{\delta\in(0,1)} \subset L^p(\mathbb{R}^N)$ such that g_δ converges to g in $L^p(\mathbb{R}^N)$ as δ goes to 0, one has

$$\underline{\lim}_{\delta\to 0} I_\delta(g_\delta) \geq I(g).$$

(G2) For each $g \in L^p(\mathbb{R}^N)$, there exists a family $(g_\delta)_{\delta\in(0,1)} \subset L^p(\mathbb{R}^N)$ such that g_δ converges to g in $L^p(\mathbb{R}^N)$ as δ goes to 0, and

$$\overline{\lim}_{\delta\to 0} I_\delta(g_\delta) \leq I(g).$$

Surprisingly, (I_δ) does not Γ-converge to J in $L^p(\mathbb{R}^N)$ for $p > 1$ but it Γ-converges to λJ for some $0 < \lambda < 1$; the same fact holds for the case $p = 1$. More precisely, we have

Theorem 2 *Let $p \geq 1$ and $N \geq 1$. Then (I_δ) Γ-converges in $L^p(\mathbb{R}^N)$ to I defined by, for all $g \in L^p(\mathbb{R}^N)$,*

$$I(g) = \begin{cases} C_{N,p} \displaystyle\int_{\mathbb{R}^N} |Dg|^p \, dx & \textit{if } p > 1 \textit{ and } g \in W^{1,p}(\mathbb{R}^N) \\ & (\textit{resp. } p = 1 \textit{ and } g \in BV(\mathbb{R}^N)), \\ +\infty & \textit{otherwise.} \end{cases}$$

Here the constant $C_{N,p}$ is defined by (1.3) below and satisfies

$$0 < C_{N,p} < \frac{1}{p} K_{N,p}. \tag{1.2}$$

For $p \geq 1$ and $N \geq 1$, the definition of the constant $C_{N,p}$ is the following

$$C_{N,p} := \inf \varliminf_{\delta \to 0} \iint_{\substack{Q^2 \\ |h_\delta(x) - h_\delta(y)| > \delta}} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy, \tag{1.3}$$

where the infimum is taken over all families of measurable functions $(h_\delta)_{\delta \in (0,1)}$ defined on the unit open cube Q of $\mathbb{R}^N$ such that h_δ converges to $h(x) \equiv \frac{x_1 + \cdots + x_N}{\sqrt{N}}$ in (Lebesgue) measure on Q as δ goes to 0. One recalls here that a family of measurable functions $(h_\delta)_{\delta \in (0,1)}$ defined on a measurable set B of $\mathbb{R}^N$ is said to converge in measure on B to a measurable function h defined on B if and only if for any $\varepsilon > 0$,

$$\lim_{\delta \to 0} |\{x \in B; |h_\delta(x) - h(x)| > \varepsilon\}| = 0.$$

Henceforth, for A a measurable subset of $\mathbb{R}^N$, $|A|$ denotes the Lebesgue measure of A.

The proof of Theorem 2 is divided into three steps:

Step 1: Proof of Property (G2).

Claim 1: Let $p \geq 1$ and $N \geq 1$. Then for each $g \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$, or $g \in BV(\mathbb{R}^N)$ if $p = 1$, there exists a family $(g_\delta)_{\delta \in (0,1)} \subset L^p(\mathbb{R}^N)$ such that g_δ converges to g in $L^p(\mathbb{R}^N)$ as δ goes to 0, and

$$\varlimsup_{\delta \to 0} I_\delta(g_\delta) \leq I(g).$$

The main steps of the proof of Claim 1 for the case $N = 1$ are:

(a) We show that (Lemma 3) there exists a family (h_δ) in $L^p(0,1)$ defined for all $\delta \in (0,1)$ (not just for a sequence $\delta_n \to 0$) such that h_δ converges to $h(x) \equiv x$ in $L^p(0,1)$ and

$$\lim_{\delta \to 0} \iint_{\substack{[0,1]^2 \\ |h_\delta(x) - h_\delta(y)| > \delta}} \frac{\delta^p}{|x-y|^{p+1}} \, dx \, dy = C_{1,p}.$$

(b) We prove Claim 1 in the case g is continuous and piecewise linear with compact support (Lemma 6). To this end, on each interval K where g is linear, using (a) we can find a family of functions $(h_{K,\delta}) \subset L^p(K)$ such that $h_{K,\delta}$ converges to g in $L^p(K)$ and

$$\lim_{\delta \to 0} \iint_{\substack{K^2 \\ |h_{K,\delta}(x) - h_{K,\delta}(y)| > \delta}} \frac{\delta^p}{|x-y|^{p+1}} \, dx \, dy = C_{1,p} |g'(x_0)|^p |K|,$$

for some $x_0 \in K$. Then we glue these functions and construct a function g_δ on $\mathbb{R}$. This is delicate since I_δ is very sensitive to jumps.

(c) We deduce Claim 1 from (b) by using the fact that if g is as in Claim 1, then there exists a sequence of continuous and piecewise linear functions with compact support (ϕ_n) such that ϕ_n converges to g in $L^p(\mathbb{R})$ and $\|D\phi_n\|_{L^p(\mathbb{R})}$ converges to $\|Dg\|_{L^p(\mathbb{R})}$ (when $p = 1$ the L^1-norm is replaced by the total mass).

The proof of Claim 1 in the general case follows from the same ideas as in the proof of the one dimensional case, but it is more complicated (see Section 3.2).

Proof of Property (G2) follows from Claim 1 and the definition of I.

Step 2: Proof of Property (G1).

Claim 2: Let $p \geq 1$ and $N \geq 1$. Then for any $g \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$ or $g \in BV(\mathbb{R}^N)$ if $p = 1$, and for any family $(g_\delta)_{\delta \in (0,1)} \subset L^p(\mathbb{R}^N)$ such that g_δ converges to g in $L^p(\mathbb{R}^N)$ as δ goes to 0, one has

$$\varliminf_{\delta \to 0} I_\delta(g_\delta) \geq I(g).$$

The proof of Claim 2 for the case $p > 1$ and $N = 1$ follows from the definition of $C_{1,p}$ and the fact that any function in $W^{1,p}(\mathbb{R})$ is locally approximately linear in the sense of measure (see e.g. [8, Theorem 4 on page 223] and the remark below it). In the case $p > 1$ and $N > 1$, one uses the same idea as in the proof of the one dimensional case. However, it is more technical (see Section 4.2). When $p = 1$, we can not directly apply the method used in the case $p > 1$. In this case, the proof relies on Lemma 28 and a new characterization of BV functions which we introduce in Section 5 (see Proposition 1). This criterion is based on and generalizes the notion of the essential variation in the one dimensional case. It may be compared with the classic one (see e.g. [8, Theorem 2 on page 220]), which is based on the notion of the essential variation on a.e. line. The idea of the proof of Claim 2 is roughly speaking the following: Let U be an open subset of $\mathbb{R}^N$ such that $\operatorname{supp} D_s g \subset U$, $D_s g$ denotes the singular part of the derivative of g, and $|U|$ is small. Then at each point $x \in \operatorname{supp} D_s g$, using the structure theorem for BV functions (see e.g. [8, Theorem 1 on page 167]) and the differentiation theorem of Radon measures (see e.g. [8, Theorem 1 on page 38]), one can find a closed cube $Q_x \subset U$ center at x such that one of its faces is orthogonal to $Dg(x)$ and

$$\|Dg\|(Q_x) \approx \|Dg \cdot \frac{Dg(x)}{|Dg(x)|}\|(Q_x).$$

Applying Lemma 28, Proposition 1, and Besicovitch's covering theorem, we can control the mass of the singular part of the derivative of g ($\|D_s g\|(U)$). The L^1-norm of the continuous part of the derivative of g outside U will be controlled as in the case $p > 1$.

Claim 3: Let $p \geq 1$, $N \geq 1$, and $g \in L^p(\mathbb{R}^N)$. Assume that there exists a family $(g_\delta)_{\delta \in (0,1)} \subset L^p(\mathbb{R}^N)$ such that g_δ converges to g in $L^p(\mathbb{R}^N)$ and

$$\varliminf_{\delta \to 0} I_\delta(g_\delta) < +\infty.$$

Then $g \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$ (resp. $g \in BV(\mathbb{R}^N)$ if $p = 1$); moreover

$$J(g) \leq C \varliminf_{\delta \to 0} I_\delta(g_\delta),$$

for some $C > 0$ depending only on N and p.

Claim 3 was proved in [12] (see [12, Theorem 3]); the proof in [12] relies heavily on the ideas of [3]. Property (G1) now follows from Claims 2 and 3.

Step 3: Proof of (1.2).

Inequality (1.2) is proved in [13]. However for the convenience of the reader, we reproduce the proof.

Let g and g_δ be defined on $\mathbb{R}^N$ by

$$g(x) = \begin{cases} |x| & \text{if } |x| \le 1, \\ 1/|x|^{2N} & \text{otherwise,} \end{cases}$$

and

$$g_\delta(x) = \begin{cases} (k+1)\delta & \text{if } k\delta \le |x| < (k+1)\delta \text{ for } 0 \le k \le [1/\delta], \\ 1/|x|^{2N} & \text{otherwise.} \end{cases}$$

Here $[1/\delta]$ denotes the largest integer less than $1/\delta$. Then g_δ converges to g in $L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ as δ goes to 0. On the other hand, it is easy to see that

$$\varliminf_{\delta \to 0} [I_\delta(g) - I_\delta(g_\delta)] \ge \varliminf_{\delta \to 0} \sum_{k=0}^{[1/\delta]} \iint\limits_{\substack{k\delta \le |x| \le (k+\frac{1}{2})\delta \\ (k+\frac{3}{2})\delta \le |y| \in (k+2)\delta}} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy,$$

$$\lim_{\delta \to 0} I_\delta(g) = \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |Dg|^p \, dx,$$

and

$$\varliminf_{\delta \to 0} \sum_{k=0}^{[1/\delta]} \iint\limits_{\substack{k\delta \le |x| \le (k+\frac{1}{2})\delta \\ (k+\frac{3}{2})\delta \le |y| \le (k+2)\delta}} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy > 0.$$

As a result, one obtains

$$\varlimsup_{\delta \to 0} I_\delta(g_\delta) < \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |Dg|^p \, dx,$$

and therefore,

$$C_{N,p} < \frac{1}{p} K_{N,p}.$$

On the other hand, as a consequence of Claims 1 and 3, one has $C_{N,p} > 0$. This completes the proof of Step 3. □

We do not know the explicit value of the constant $C_{N,p}$. But we have a guest when $N = 1$. Let h and h_n be functions defined on $(0,1)$ by $h(x) = x$

on $(0,1)$ and $h_n(x) = \frac{k-1}{n}$ if $\frac{k-1}{n} \le x < \frac{k}{n}$ for $1 \le k \le n$. An easy computation shows that

$$\lim_{n\to\infty} \iint_{\substack{[0,1]^2 \\ |h_n(x)-h_n(y)|>1/n}} \frac{1/n^p}{|x-y|^{p+1}}\, dx\, dy = c_{1,p},$$

where

$$c_{1,p} = \begin{cases} \frac{2}{p(p-1)}(1 - \frac{1}{2^{p-1}}) & \text{if } p > 1, \\ 2\ln 2 & \text{if } p = 1. \end{cases}$$

Clearly,

$$c_{1,p} \ge C_{1,p}.$$

The following open question is suggested.

Open question 1 *Is $C_{1,p}$ equal to $c_{1,p}$?*

The results of this paper were announced in [13].

The paper is organized as follows. In section 2, we prove some useful lemmas which will be used many times later. Two of them are Lemmas 1 and 2 which are quite elementary but very useful. One of interesting applications of these lemmas are Lemma 3. This lemma and the idea used in its proof plays an essential role in this paper. Section 3, which includes two subsections, will be devoted to proving Claim 1. In the first subsection (Section 3.1), we prove Claim 1 in the case $N = 1$. The proof of Claim 1 in the case $N \ge 2$ will be given in the second subsection (Section 3.2). Section 4 with two subsection will be devoted to proving Claim 2 in the case $p > 1$. The proof of Claim 2 in the case $N = 1$ and $N \ge 2$ will be presented in the first one and the second one respectively. To prove Claim 2 in the case $p = 1$, one needs a new criterion to recognize BV functions. This criterion will be introduced in Section 5. The last section, which includes four subsections, concerns the proof of Claim 2. Another definition of $C_{N,1}$ will be given in the first subsection (Section 6.1). This definition plays an important role in this case. In the second subsection (Section 6.2), we give some useful lemmas. The main goal of this part is to prove Lemma 28 which is one of the main ingredients in the proof of Claim 2. The two last subsections will be devoted to proving of Claim 2 in the one dimensional case and in the general case.

2 Preliminary

In this section, we prove some useful lemmas which will be used later.

Lemma 1 *Let $N \geq 1$, $p \geq 1$, A be a measurable subset of $\mathbb{R}^N$, and f and g be two measurable functions defined on A. Define $h = \min(f, g)$. Then*

$$\iint_{\substack{A^2 \\ |h(x)-h(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \leq \iint_{\substack{A^2 \\ |f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy + \iint_{\substack{A^2\setminus B^2 \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy, \qquad (2.1)$$

where

$$B = \{x \in A; f(x) \leq g(x)\}.$$

Moreover, if g is Lipschitz on A with a Lipschitz constant L, then

$$\iint_{\substack{A^2 \\ |h(x)-h(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \leq \iint_{\substack{A^2 \\ |f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy + CL^p|A \setminus B|. \qquad (2.2)$$

Hereafter in this paper, C denotes a positive constant depending only on N and p.

Proof: Take x and y in A. We have the following cases:

Case 1: $f(x) > g(x)$ and $f(y) > g(y)$. Then $|h(x) - h(y)| = |g(x) - g(y)|$

Case 2: $f(x) > g(x)$ and $f(y) \leq g(y)$.

Case 2.1: $f(y) > g(x)$. Then $|h(x) - h(y)| = f(y) - g(x) \leq |g(y) - g(x)|$.
Case 2.2: $f(y) \leq g(x)$. Then $|h(x) - h(y)| = g(x) - f(y) \leq |f(x) - f(y)|$.

Case 3: $f(x) \leq g(x)$ and $f(y) > g(y)$.

Case 3.1: $g(y) > f(x)$. Then $|h(x) - h(y)| = g(y) - f(x) \leq |f(y) - f(x)|$.
Case 3.2: $g(y) \leq f(x)$. Then $|h(x) - h(y)| = f(x) - g(y) \leq |g(x) - g(y)|$.

Case 4: $f(x) \leq g(x)$ and $f(y) \leq g(y)$. Then $|h(x) - h(y)| = |f(x) - f(y)|$.

Thus

$$\iint_{\substack{A^2\\ |h(x)-h(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le \iint_{\substack{A^2\\ |f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy + \iint_{\substack{A^2\setminus B^2\\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy.$$

To obtain (2.2) from (2.1), one remarks that

$$\iint_{\substack{A^2\setminus B^2\\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le 2\iint_{\substack{A\times(A\setminus B)\\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy.$$

and

$$\iint_{\substack{A\times(A\setminus B)\\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le CL^p|A\setminus B|,$$

for any Lipschitz function g with a Lipschitz constant L. $\square$

As a consequence of Lemma 1, one has

Lemma 2 *Let $N \ge 1$, $p \ge 1$, A be a measurable subset of $\mathbb{R}^N$, and f and g be two measurable functions defined on A. Define $h = \max(f, g)$. Then*

$$\iint_{\substack{A^2\\ |h(x)-h(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le \iint_{\substack{A^2\\ |f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy + \iint_{\substack{A^2\setminus B^2\\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy,$$

where

$$B = \{x \in A;\ f(x) \ge g(x)\}.$$

Moreover, if g is Lipschitz on A with a Lipschitz constant L, then

$$\iint_{\substack{A^2\\ |h(x)-h(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le \iint_{\substack{A^2\\ |f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy + CL^p|A\setminus B|. \tag{2.3}$$

Proof: Consider $\tilde{f} = -f$, $\tilde{g} = -g$, and $\tilde{h} = \min(\tilde{f}, \tilde{g})$. Then $\tilde{h} = -\max(f, g) = -h$. Applying Lemma 1 for $\tilde{f}$, $\tilde{g}$, and $\tilde{h}$, one gets

$$\iint_{\substack{A^2 \\ |\tilde{h}(x)-\tilde{h}(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \le \iint_{\substack{A^2 \\ |\tilde{f}(x)-\tilde{f}(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy + \iint_{\substack{A^2 \setminus \tilde{B}^2 \\ |\tilde{g}(x)-\tilde{g}(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy,$$

where

$$\tilde{B} = \{x \in A;\ \tilde{f}(x) \le \tilde{g}(x)\}.$$

This implies

$$\iint_{\substack{A^2 \\ |h(x)-h(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \le \iint_{\substack{A^2 \\ |f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy + \iint_{\substack{A^2 \setminus B^2 \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy,$$

since

$$B = \tilde{B}.$$

In the case that g is Lipschitz on A with a Lipschitz constant L, applying also Lemma 1 for $\tilde{f}$, $\tilde{g}$, and $\tilde{h}$, by the same manner as above, one gets (2.3). □

Here are some applications of these results.

Corollary 1 *Let $N \ge 1$, $p \ge 1$, A be a measurable subset of $\mathbb{R}^N$, f be a measurable function defined on A, and $m \in \mathbb{R}$. Define $h = \min(f, m)$. Then*

$$\iint_{\substack{A^2 \\ |h(x)-h(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \le \iint_{\substack{A^2 \\ |f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy.$$

Proof: The corollary is a consequence of Lemma 1. □

Corollary 2 *Let $N \ge 1$, $p \ge 1$, A be a measurable subset of $\mathbb{R}^N$, f be a measurable function defined on A, and $m \in \mathbb{R}$. Define $h = \max(f, m)$. Then*

$$\iint_{\substack{A^2 \\ |h(x)-h(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \le \iint_{\substack{A^2 \\ |f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy.$$

Proof: The corollary is a consequence of Lemma 2. $\square$

Corollary 3 *Let $N \geq 1$, $p \geq 1$, A be a measurable subset of $\mathbb{R}^N$, f be a measurable function defined on A, and $m_1, m_2 \in \mathbb{R}$, $m_1 < m_2$. Define $h = \min(\max(f, m_1), m_2)$. Then*

$$\iint_{\substack{A^2 \\ |h(x)-h(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \leq \iint_{\substack{A^2 \\ |f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy.$$

Proof: The corollary is a consequence of Corollaries 1 and 2. $\square$

Remark 1 Corollaries 1, 2 and 3 were observed and used in [10] and [3].

Another useful application is as follows

Corollary 4 *Let $p \geq 1$, A be a measurable subset of $\mathbb{R}^N$, f and g be two measurable function defined on A, and c be a positive number. Define $h = \min(\max(f, g - c), g + c)$. Suppose that g is Lipsichitz with a Lipschitz constant L on A. Then*

$$\iint_{\substack{A^2 \\ |h(x)-h(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \leq \iint_{\substack{A^2 \\ |f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy + CL^p|B|,$$

where

$$B := \{x \in A;\, |f(x) - g(x)| > c\}.$$

Proof: Let $h_1 = \max(f, g - c)$. Then applying Lemma 2 for f and $g - c$, one gets

$$\iint_{\substack{A^2 \\ |h_1(x)-h_1(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \leq \iint_{\substack{A^2 \\ |f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy + CL^p|B_1|,$$

where

$$B_1 := \{x \in A;\, f < g - c\}.$$

On the other hand, applying Lemma 1 for h_1 and $g + c$, one has

$$\iint_{\substack{A^2 \\ |h(x)-h(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \leq \iint_{\substack{A^2 \\ |h_1(x)-h_1(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy + CL^p|B_2|,$$

where

$$B_2 := \{x \in A;\, h_1 > g + c\}.$$

Hence it follows that

$$\iint_{\substack{A^2\\|h(x)-h(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le \iint_{\substack{A^2\\|f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy + CL^p|B_1| + CL^p|B_2|. \tag{2.4}$$

On the other hand, since $h_1 = \max(f, g-c)$, it follows from the definition of B_2 that

$$B_2 := \{x \in A;\ f > g + c\}.$$

Hence,

$$|B_1| + |B_2| = |B|.$$

Thus it follows from (2.4) that

$$\iint_{\substack{A^2\\|h(x)-h(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le \iint_{\substack{A^2\\|f(x)-f(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy + CL^p|B|.$$

□

An important application of Corollary 4 is the following

Corollary 5 *Let $p \ge 1$, A be a measurable subset of $\mathbb{R}^N$, g be a Lipschitz function defined on A, $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 0, and $(g_n)_{n\in\mathbb{N}}$ be a sequence of measurable function defined on A such that g_n converges to g in measure in A. Then there exists a sequence of measurable functions h_n defined on A such that h_n converges to g uniformly on A and*

$$\varlimsup_{n\to\infty} \iint_{\substack{A^2\\|h_n(x)-h_n(y)|>\delta_n}} \frac{\delta_n^p}{|x-y|^{N+p}}\,dx\,dy \le \varlimsup_{n\to\infty} \iint_{\substack{A^2\\|g_n(x)-g_n(y)|>\delta_n}} \frac{\delta_n^p}{|x-y|^{N+p}}\,dx\,dy.$$

Proof: Since g_n converges to g in measure in A, there exists a sequence of positive numbers $(c_n)_{n\in\mathbb{N}}$ converging to 0 such that

$$\lim_{n\to\infty} |A_n| = 0,$$

where

$$A_n := \{x \in A;\ |g_n(x) - g(x)| > c_n\}.$$

Define $h_n = \min(\max(g_n, g - c_n), g + c_n)$. Applying Corollary 4, one has

$$\iint_{\substack{A^2 \\ |h_n(x)-h_n(y)|>\delta_n}} \frac{\delta_n^p}{|x-y|^{N+p}}\, dx\, dy \leq \iint_{\substack{A^2 \\ |g_n(x)-g_n(y)|>\delta_n}} \frac{\delta_n^p}{|x-y|^{N+p}}\, dx\, dy + CL|A_n|,$$

where L is a Lipschitz constant of g. Therefore,

$$\varlimsup_{n\to\infty} \iint_{\substack{A^2 \\ |h_n(x)-h_n(y)|>\delta_n}} \frac{\delta_n^p}{|x-y|^{N+p}}\, dx\, dy$$

$$\leq \varlimsup_{n\to\infty} \iint_{\substack{A^2 \\ |g_n(x)-g_n(y)|>\delta_n}} \frac{\delta_n^p}{|x-y|^{N+p}}\, dx\, dy.$$

□

Remark 2 Corollary 5 also holds if one replaces the consequence g_n and δ_n by a family $(g_\delta)_{\delta\in(0,1)}$ and $(\delta)_{\delta\in(0,1)}$ such that g_δ converges to g in measure.

Using Corollary 5, one can prove the following

Lemma 3 *Let $N \geq 1$ and $p \geq 1$. Then there exists a family of measurable functions $(g_\delta)_{\delta\in(0,1)}$ defined on Q such that g_δ converges to $g(x) = \frac{\sum_{i=1}^N x_i}{\sqrt{N}}$ uniformly on Q and*

$$\varlimsup_{\delta\to 0} \iint_{\substack{Q\times Q \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \leq C_{N,p}.$$

Remark 3 The idea used to prove Lemma 3 is the heart matter of the method presented in this paper.

Proof: It is standard to see that there exist a sequence of positive numbers $(\delta_k)_{k\in\mathbb{N}}$ converging to 0, and a sequence of measurable functions $(g_k)_{k\in\mathbb{N}}$ converging to g in measure such that

$$\varlimsup_{k\to\infty} \iint_{\substack{Q^2 \\ |g_k(x)-g_k(y)|>\delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}}\, dx\, dy \leq C_{N,p}. \tag{2.5}$$

Using Corollary 5, one may assume that g_k converges to g uniformly as k goes to infinity. Set

$$c_k = \max(\sup_{x\in Q} |g_k(x) - g(x)|, \sqrt{\delta_k}).$$

Define

$$\begin{cases} g_{1,k} = \min\Big(\max\big(g_{0,k}(x), g(0,x_2,\dots,x_N)+2c_k\big), g(1,x_2,\dots,x_N)-2c_k\Big), \\ g_{2,k} = \min\Big(\max\big(g_{1,k}(x), g(x_1,0,\dots,x_N)+4c_k\big), g(x_1,1,\dots,x_N)-4c_k\Big), \\ \qquad\qquad \dots \\ g_{N,k} = \min\Big(\max\big(g_{N-1,k}(x), g(x_1,\dots,x_{N-1},0)+2Nc_k\big), \\ \qquad\qquad g(x_1,\dots,x_{N-1},1)-2Nc_k\Big). \end{cases}$$

with the notation $g_{0,k}=g_k$, and set

$$Q_{c_k} := \{x=(x_1,\dots,x_N)\in\mathbb{R}^N;\ c_k\le x_i\le 1-c_k \text{ for all } 1\le i\le N\}.$$

Then, since

$$\begin{cases} g(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_N)+2ic_k \le g(x)+2ic_k \\ g(x_1,\dots,x_{i-1},1,x_{i+1},\dots,x_N)-2ic_k \ge g(x)-2ic_k, \end{cases}$$

it follows that

$$g_{i,k}(x)\ge \min(g_{i-1,k}(x), g(x)-2ic_k)$$

and

$$g_{i,k}(x)\le \max(g_{i-1,k}(x), g(x)+2ic_k)$$

Then, since $g(x)-c_k\le g_{0,k}(x)\le g(x)+c_k$, one has

$$g(x)-2ic_k\le g_{i,k}(x)\le g(x)+2ic_k,$$

for $1\le i\le N$. This implies

$$g_{i,k}(x) = \begin{cases} g(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_N)+2ic_k & \text{if } 0\le x_i\le c_k, \\ g(x_1,\dots,x_{i-1},1,x_{i+1},\dots,x_N)-2ic_k, & \text{if } 1-c_k\le x_i\le 1, \end{cases}$$

since $|Dg|=1$. Thus $g_{N,k}$ is Lipschitz on Q_{c_k} with a Lipschitz constant 1 ($=|Dg|$), since the minimum and the maximum of two Lipschitz functions are Lipschitz with a Lipschitz constant equal to the maximum of the ones of these functions.

On the other hand, applying Corollary 3, one has

$$\iint_{\substack{Q^2 \\ |g_{i,k}(x)-g_{i,k}(y)|>\delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}}\,dx\,dy \le \iint_{\substack{Q^2 \\ |g_{i-1,k}(x)-g_{i-1,k}(y)|>\delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}}\,dx\,dy,$$

for all $1 \leq i \leq N$, which implies

$$\iint_{\substack{Q^2 \\ |g_{N,k}(x)-g_{N,k}(y)|>\delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}}\,dx\,dy \leq \iint_{\substack{Q^2 \\ |g_k(x)-g_k(y)|>\delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}}\,dx\,dy. \tag{2.6}$$

Let $(\tau_k)_{k\in\mathbb{N}}$ be a strictly decreasing sequence such that $\tau_0 = 1$ and $\tau_k \leq c_k\delta_k$ for $k \geq 1$. For each $\tau_{k+1} < \delta \leq \tau_k$, define $\tilde{m} = \frac{\delta_k}{\delta}$ and $m = [\tilde{m}]$, the largest integer less than $\tilde{m}$. Consider $h_\delta : Q \to \mathbb{R}$ defined as follows

$$h_\delta(x) = \frac{1}{\tilde{m}}\tilde{h}_\delta(mx),$$

where $\tilde{h}_\delta : [0,m]^N \to \mathbb{R}$ is defined by $\tilde{h}_\delta(y) = \frac{1}{\sqrt{N}}\sum_{i=1}^N (1-4ic_k)l_i + g_{N,k}(x)$ where $l_i = [y_i]$, the largest integer less than y_i, and $x = (x_1,\ldots,x_N)$ with $x_i = y_i - [l_i]$. Then

$$\iint_{\substack{Q^2 \\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = \frac{m^{p-N}}{\tilde{m}^p} \iint_{\substack{[0,m]^N\times[0,m]^N \\ |\tilde{h}_\delta(x)-\tilde{h}_\delta(y)|>\delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}}\,dx\,dy. \tag{2.7}$$

On the other hand,

$$\begin{aligned}\iint_{\substack{[0,m]^N\times[0,m]^N \\ |\tilde{h}_\delta(x)-\tilde{h}_\delta(y)|>\delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}}\,dx\,dy \leq{}& \sum_{\substack{\alpha\in\mathbb{N}^N \\ 1\leq\alpha_i\leq m}} \iint_{\substack{Q_\alpha^2 \\ |\tilde{h}_\delta(x)-\tilde{h}_\delta(y)|>\delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}}\,dx\,dy \\ &+ \sum_{\substack{\alpha\in\mathbb{N}^N \\ 1\leq\alpha_i\leq m}} \iint_{\substack{Q_\alpha\times([0,m]^N\setminus Q_\alpha) \\ |\tilde{h}_\delta(x)-\tilde{h}_\delta(y)|>\delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}}\,dx\,dy.\end{aligned}$$

Hereafter in this proof,

$$Q_\alpha := Q + (\alpha_1 - 1, \ldots, \alpha_N - 1) := \{x + (\alpha_1 - 1, \ldots, \alpha_N - 1);\ x \in Q\}.$$

for any $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{N}$, $1 \leq \alpha_i \leq m$. Thus, it follows from (2.6) that

$$\begin{aligned}\iint_{\substack{[0,m]^N \\ |\tilde{h}_\delta(x)-\tilde{h}_\delta(y)|>\delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}}\,dx\,dy \leq{}& m^N \iint_{\substack{[0,1]^N \\ |g_k(x)-g_k(y)|>\delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}}\,dx\,dy \\ &+ Cm^N(c_k + \delta_k^{\frac{p}{2}}),\end{aligned} \tag{2.8}$$

since

$$\iint_{\substack{Q_\alpha^2 \\ |\tilde h_\delta(x)-\tilde h_\delta(y)|>\delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}}\,dx\,dy = \iint_{\substack{Q^2 \\ |g_{N,k}(x)-g_{N,k}(y)|>\delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}}\,dx\,dy$$

and

$$\iint_{\substack{Q_\alpha\times([0,m]^N\setminus Q_\alpha) \\ |\tilde h_\delta(x)-\tilde h_\delta(y)|>\delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}}\,dx\,dy \le C(c_k+\delta_k^p/c_k^p) \le C(c_k+\delta_k^{\frac{p}{2}}),$$

for all $\alpha=(\alpha_1,\dots,\alpha_N)$, $\alpha_i\in\mathbb{N}$, $1\le\alpha_i\le m$. The last inequality is a consequence of the fact that $\tilde h_\delta$ is Lipschitz on the set $\{x\in[0,m]^N;\ \mathrm{dist}(x,\partial Q_\alpha)\le c_\alpha\}$ with a Lipschitz constant C depending only on N since $g_{N,k}$ is Lipschitz on Q_{c_k} with a Lipschitz constant 1. Here

$$\mathrm{dist}(x,A) := \inf_{y\in A}|x-y|,$$

for any $x\in\mathbb{R}^N$ and $A\subset\mathbb{R}^N$.

Combining (2.7) and (2.8), one gets

$$\iint_{\substack{Q^2 \\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy$$

$$\le \frac{m^p}{\tilde m^p}\left(\iint_{\substack{Q^2 \\ |g_k(x)-g_k(y)|>\delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}}\,dx\,dy + Cc_k + C\delta_k^{\frac{p}{2}}\right).$$

Therefore, it follows from (2.5) that

$$\overline{\lim_{\delta\to0}} \iint_{\substack{Q^2 \\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le C_{N,p},$$

since $\lim_{k\to\infty}\delta_k=\lim_{k\to\infty}c_k=0$.

It suffices to prove that h_δ converges to g uniformly on Q. For each $y\in[0,m]^N$, set $l_i=[y_i]$, the largest integer less than y_i, and $x=(x_1,\dots,x_N)$ with $x_i=y_i-[l_i]$. Then

$$|\tilde h_\delta(y) - \frac{\sum_{i=1}^N y_i}{\sqrt N}| = |\frac{1}{\sqrt N}\sum_{i=1}^N(1-4ic_k)l_i + g_{N,k}(x) - \frac{\sum_{i=1}^N l_i}{\sqrt N} - \frac{\sum_{i=1}^N x_i}{\sqrt N}|.$$

However,

$$|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(1-4ic_k)l_i + g_{N,k}(x) - \frac{\sum_{i=1}^N l_i}{\sqrt{N}} - \frac{\sum_{i=1}^N x_i}{\sqrt{N}}|$$

$$\leq \frac{4c_k\sum_{i=1}^N il_i}{\sqrt{N}} + |g_{N,k}(x) - \frac{\sum_{i=1}^N x_i}{\sqrt{N}}|.$$

Hence

$$|\tilde{h}_\delta(y) - \frac{\sum_{i=1}^N y_i}{\sqrt{N}}| \leq C(N)mc_k.$$

Thus it follows since $|\frac{1}{\tilde{m}}h_\delta(x) - \frac{1}{m}h_\delta(x)| \leq C(N)/\tilde{m}$ for all $x \in Q$, $1/\tilde{m} \leq c_k$, and $\lim_{k\to\infty} c_k = 0$ that h_δ uniformly converges to g in Q. □

Lemma 4 *Let $N \geq 1$ and $p \geq 1$. There exist a family of measurable functions $(g_\delta)_{\delta\in(0,1)}$ defined on Q and a family of positive numbers $(c_\delta)_{\delta\in(0,1)}$ converging to 0 such that $c_\delta \geq \sqrt{\delta}$, $|g_\delta(x) - \frac{\sum_{i=1}^N x_i}{\sqrt{N}}| \leq 2Nc_\delta$, g_δ is Lipschitz on Q_{c_δ} with a Lipschitz constant 1, and*

$$\overline{\lim_{\delta\to 0}} \iint_{\substack{Q\times Q \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \leq C_{N,p}.$$

Here Q_{c_δ} is defined by

$$Q_{c_\delta} := \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N;\ c_\delta \leq x_i \leq 1 - c_\delta \text{ for all } 1 \leq i \leq N\}.$$

Proof: Define $g(x) = \frac{\sum_{i=1}^N x_i}{\sqrt{N}}$ for $x \in Q$. By Lemma 3, there exists a family of measurable functions $(g_\delta)_{\delta\in(0,1)}$ such that g_δ converges uniformly to g in Q and

$$\overline{\lim_{\delta\to 0}} \iint_{\substack{Q\times Q \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \leq C_{N,p}.$$

Define

$$c_\delta = \max(\sup_{x\in Q}|g_\delta(x) - g(x)|, \sqrt{\delta}).$$

$$
\begin{cases}
g_{1,\delta} = \min\Big(\max\big(g_\delta(x), g(0, x_2, \dots, x_N) + 2c_\delta\big), g(1, x_2, \dots, x_N) - 2c_\delta\Big), \\
g_{2,\delta} = \min\Big(\max\big(g_{1,\delta}(x), g(x_1, 0, \dots, x_N) + 4c_\delta\big), g(x_1, 1, \dots, x_N) - 4c_\delta\Big), \\
\dots \\
g_{N,\delta} = \min\Big(\max\big(g_{N-1,\delta}(x), g(x_1, \dots, x_{N-1}, 0) + 2Nc_\delta\big), \\
\qquad\qquad g(x_1, \dots, x_{N-1}, 1) - 2Nc_\delta\Big).
\end{cases}
$$

with the notation $g_{0,\delta} = g_\delta$. Then, as in the proof of Lemma 3, $|g_\delta(x) - g(x)| \le 2Nc_\delta$ for $x \in Q$, $g_{N,\delta}$ is Lipschitz on Q_{c_δ} with a Lipschitz constant 1, and

$$
\overline{\lim_{\delta \to 0}} \iint_{\substack{Q^2 \\ |g_{N,\delta}(h)(x) - g_{N,\delta}(h)(y)| > \delta_k}} \frac{\delta_k^p}{|x-y|^{N+p}} \, dx \, dy \le C_{N,p}.
$$

□

3 Proof of Claim 1

3.1 Proof of Claim 1 in the case $N = 1$

The following lemma is a consequence of Lemma 4. This lemma will be used in the proof of Lemma 6 which is the main point in the proof of Claim 1 in the one dimensional case.

Lemma 5 *Let g be an affine function defined on a nonempty bounded interval $(a, b) \subset \mathbb{R}$. Then there exists a family $(g_\delta)_{\delta \in (0,1)}$ and a family $(c_\delta)_{\delta \in (0,1)}$ converging to 0 such that $c_\delta \ge \sqrt{\delta}$, $|g_\delta(x) - g(x)| \le 2l|g'|c_\delta$, $l := b - a$, for all $x \in (a, b)$, g_δ is constant on the interval $(a, a + lc_\delta)$ and on the interval $(b - lc_\delta, b)$, and*

$$
\overline{\lim_{\delta \to 0}} \iint_{\substack{[a,b]^2 \\ |g_\delta(x) - g_\delta(y)| > \delta}} \frac{\delta^p}{|x-y|^{p+1}} \, dx \, dy \le C_{1,p}|g'|^p(b-a).
$$

Proof: If g is constant on (a, b), take $g_\delta(x) = g(x)$ for all $x \in (a, b)$ and $c_\delta = \sqrt{\delta}$. Otherwise, consider the function $h : [0, 1] \mapsto \mathbb{R}$ defined by

$$
h(t) = \frac{g(lt + a) - g(a)}{lg'(lt + a)}, \quad \forall\, t \in [0, 1].
$$

Then $h(t) = t$ on $[0,1]$. Hence, by Lemma 4, there exists a family of measurable functions $(h_\delta)_{\delta\in(0,1)}$ and a family of positive numbers $(c_\delta)_{\delta\in(0,1)}$ converging to 0 such that $c_\delta \geq \sqrt{\delta}$, $|h_\delta(t) - h(t)| \leq 2c_\delta$, h_δ is Lipschitz on $(0, c_\delta)$ and $(1 - c_\delta, 1)$ with a Lipschitz constant 1, and

$$\overline{\lim_{\delta\to 0}} \iint_{\substack{[0,1]^2 \\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{p+1}}\, dx\, dy \leq C_{1,p}.$$

Define $g_\delta : [a,b] \mapsto \mathbb{R}$ by

$$g_\delta(x) = l g'(x) h_\delta\left(\frac{x-a}{l}\right) + g(a), \quad \forall\, x \in [a,b].$$

Then, since

$$|g_\delta(x) - g(x)| = l|g'(x)| \cdot |h_\delta(t) - h(t)|,$$

with $t = \frac{x-a}{l}$, it follows that $|g_\delta(x) - g(x)| \leq 2l|g'|c_\delta$ for all $x \in (a,b)$, g_δ is constant on the interval $(a, a + lc_\delta)$ and on the interval $(b - lc_\delta, b)$, and

$$\overline{\lim_{\delta\to 0}} \iint_{\substack{[a,b]^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{p+1}}\, dx\, dy \leq C_{1,p}|g'|^p(b-a).$$

□

Lemma 6 *Let g be a continuous piecewise linear function on $\mathbb{R}$ with compact support. Then there exists a family of measurable functions $(g_\delta)_{\delta\in(0,1)}$ such that g_δ converges to g in $L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and*

$$\overline{\lim_{\delta\to 0}} I_\delta(g_\delta) \leq I(g).$$

Proof: Since g is a continuous piecewise linear function with compact support, there exist $a_1 < a_2 < \cdots < a_m$ such that g is affine on $[a_i, a_{i+1}]$, $1 \leq i < m-1$, and $g(x) = 0$ if $x < a_1$ or $x > a_m$.

By Lemma 5, there exist a family of positive numbers $(c_{i,\delta})_{\delta\in(0,1)}$, $1 \leq i \leq m-1$, converging to 0 and a family of measurable functions $(h_{i,\delta})_{\delta\in(0,1)}$ such that $c_{i,\delta} \geq \sqrt{\delta}$, $|h_{i,\delta}(x) - g(x)| \leq 2l_i\|g'\|_{L^\infty(\mathbb{R})}c_{i,\delta}$, $l_i = a_{i+1} - a_i$, $h_{i,\delta}$ is constant on the interval $(a_i, a_i + c_{i,\delta}l_i)$ and on the interval $(a_{i+1} - c_{i,\delta}l_i, a_{i+1})$, and

$$\overline{\lim_{\delta\to 0}} \iint_{\substack{[a_i,a_{i+1}]^2 \\ |h_{i,\delta}(x)-h_{i,\delta}(y)|>\delta}} \frac{\delta^p}{|x-y|^{p+1}}\, dx\, dy \leq C_{1,p} \int_{a_i}^{a_{i+1}} |g'|^p\, dx.$$

Consider the continuous function g_δ defined by

$$g_\delta(x) = \begin{cases} h_{i,\delta}(x) & \text{if } x \in [a_i + c_{i,\delta} l_i/2, a_{i+1} - c_{i,\delta} l_i/2], \\ 0 & \text{if } x \le a_1 - c_{1,\delta}/2 \text{ or } x \ge a_m + c_{m-1,\delta}/2, \\ \text{affine} & \text{otherwise.} \end{cases}$$

Define

$$c_\delta = \sup_{1 \le i \le m-1} c_{i,\delta}.$$

Then, by the manner used to obtain estimate (2.8), one has

$$\iint_{\substack{\mathbb{R}^2 \\ |g_\delta(x) - g_\delta(y)| > \delta}} \frac{\delta^p}{|x-y|^{p+1}} \, dx \, dy \le \sum_{i=1}^{m-1} \iint_{\substack{[a_i, a_{i+1}]^2 \\ |h_{i,\delta}(x) - h_{i,\delta}(y)| > \delta}} \frac{\delta^p}{|x-y|^{p+1}} \, dx \, dy$$
$$+ C(g)(c_\delta + \delta^{\frac{p}{2}}).$$

Therefore, since $\lim_{\delta \to 0} c_\delta = 0$,

$$\overline{\lim_{\delta \to 0}} \iint_{\substack{\mathbb{R}^2 \\ |g_\delta(x) - g_\delta(y)| > \delta}} \frac{\delta^p}{|x-y|^{p+1}} \, dx \, dy \le C_{1,p} \int_{\mathbb{R}} |g'|^p \, dx.$$

Finally, it follows from the definition of g_δ that g_δ converges to g in $L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ since $h_{i,\delta}$ converges to g uniformly on each $[a_i, a_{i+1}]$, $1 \le i \le m-1$. $\square$

Using Lemma 6, one can prove

Proof of Claim 1: Consider a sequence of functions $(h_k)_{k \in \mathbb{N}} \subset W^{1,p}(\mathbb{R})$ such that h_k is continuous piecewise linear with compact support, h_k converges to g in $L^p(\mathbb{R})$, as k goes to infinity, and $\lim_{k \to \infty} I(h_k) = I(g)$. For each k, by Lemma 6, there exists a family of functions $(h_{k,\delta})_{\delta \in (0,1)} \subset L^p(\mathbb{R})$ such that $h_{k,\delta}$ converges to h_k in $L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and

$$\overline{\lim_{\delta \to 0}} I_\delta(h_{k,\delta}) \le I(h_k).$$

Then there exists a strictly decreasing sequence of positive numbers $(\tau_k)_{k \in \mathbb{N}}$ converging to 0 such that $\tau_0 = 1$ and

$$\begin{cases} \|h_{k,\delta} - h_k\|_{L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})} \le \dfrac{1}{k}, \\ I_\delta(h_{k,\delta}) - I(h_k) \le \dfrac{1}{k}, \end{cases} \quad \forall\, 0 < \delta < \tau_k.$$

Define $g_\delta = h_{k,\delta}$ for $\tau_{k+1} < \delta \le \tau_k$. Then g_δ converges to g in $L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and

$$\overline{\lim_{\delta\to 0}}\, I_\delta(g_\delta) \le I(g).$$

□

3.2 Proof of Claim 1 in the case $N \ge 2$

The following lemma follows from the definition of $C_{N,p}$.

Lemma 7 *Let $\tilde{Q}$ be the image of the cube Q by a dilatation and a translation. Then*

$$\inf \underline{\lim_{\delta\to 0}} \iint_{\substack{\tilde{Q}^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = C_{N,p}|\tilde{Q}|,$$

where the infimum is taken over all families of measurable functions $(g_\delta)_{\delta\in(0,1)}$ defined on $\tilde{Q}$ such that g_δ converges to $g(x) \equiv \frac{x_1+\cdots+x_N}{\sqrt{N}}$ in measure as δ goes to 0.

Proof: Without loss of generality, one may assume that $\tilde{Q} = aQ := \{ax \in \mathbb{R}^N; x \in Q\}$, for some positive constant a. Then, it follows from the change of variables formula,

$$\underline{\lim_{\delta\to 0}} \iint_{\substack{\tilde{Q}^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = a^N \underline{\lim_{\delta\to 0}} \iint_{\substack{Q^2 \\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy,$$

where $h_\delta : Q \mapsto \mathbb{R}$ is defined by

$$h_\delta(x) = g_\delta(ax)/a, \quad \forall\, x \in Q.$$

The rest of the proof is clear. Its detail is left for the reader. □

The following lemma deals with a covering result. The result is quite classic for experts but the author can not find the reference for it. For the convenience of the reader, the proof is presented.

Lemma 8 *Let Ω be a nonempty open subset of $\mathbb{R}^N$ and B be a nonempty bounded open subset of $\mathbb{R}^N$ with $|\partial B| = 0$. Then there exists a collection of open subsets of $(B_i)_{i\in\mathbb{N}}$ such that $B_i \subset \Omega$, B_i is the image of B by a dilatation and a translation, $B_i \cap B_j = \emptyset$ for $i \neq j$, and $\sum_{i\in\mathbb{N}} |B_i| = |\Omega|$.*

Proof: Let $\tilde{B}$ be an image of B by a dilatation and a translation such that the closure of $\tilde{B}$ is included in Q. Since B is open, $|\tilde{B}| = c > 0$. Set $\Omega_0 = \Omega$. Consider a collection $(Q_{1,i})_{i\in\mathbb{N}}$ such that $Q_{1,i} \subset \Omega_0$, $Q_{1,i}$ is an image of Q by a dilatation and a translation, $Q_{1,i} \cap Q_{1,j} = \emptyset$ for $i \neq j$, and $|\Omega| = \sum_{i\in\mathbb{N}} |Q_{1,i}|$. The existence of this collection follows from [15, the assertion d in page 50]. Then there exists a collection of disjoint subsets $(B_{1,i})_{i\in\mathbb{N}}$ such that $B_{1,i}$ is an image of B by a translation and a dilatation, $\bar{B}_{1,i} \subset Q_{1,i}$ and $|B_{1,i}| = c|Q_{1,i}|$. This implies that $\sum_{i\in\mathbb{N}} |B_{1,i}| = c|\Omega_0|$. Set $\Omega_1 = \bigcup_{i\in\mathbb{N}}(Q_{1,i} \setminus \bar{B}_{1,i})$. Then Ω_1 is open and $|\Omega_1| = (1-c)|\Omega_0|$ (since $|\partial B| = 0$). Continuing this process, one finds collections of sets $(Q_{k,i})_{i\in\mathbb{N}}$ and $(B_{k,i})_{i\in\mathbb{N}}$, and an open subset Ω_k of $\mathbb{R}^N$ such that $Q_{k,i}$ and $B_{k,i}$ are images of Q and B respectively by a dilatation and a translation, $Q_{k,i} \subset \Omega_{k-1}$, $Q_{k,i} \cap Q_{k,j} = \emptyset$ for $i \neq j$, $\sum_{i\in\mathbb{N}} |Q_{k,i}| = |\Omega_{k-1}|$, $\bar{B}_{k,i} \subset Q_{k,i}$, $|B_{k,i}| = c|Q_{k,i}|$, and $\Omega_k = \bigcup_{i\in\mathbb{N}}(Q_{k,i} \setminus \bar{B}_{k,i})$. Set

$$a_m = \sum_{k=1}^{m} \sum_{i\in\mathbb{N}} |B_{k,i}|.$$

Then since

$$\sum_{k=1}^{m} \sum_{i\in\mathbb{N}} |B_{k,i}| = c(|\Omega| - \sum_{k=1}^{m-1} \sum_{i\in\mathbb{N}} |B_{k,i}|) + \sum_{k=1}^{m-1} \sum_{i\in\mathbb{N}} |B_{k,i}|,$$

one has

$$a_m = c(|\Omega| - a_{m-1}) + a_{m-1}. \tag{3.1}$$

It is easy to see that a_m is increasing and bounded from above. Hence a_m converges to a. Thus from (3.1), $a = |\Omega|$. The conclusion follows by taking the collection $(B_{i,k})_{(i,k)\in\mathbb{N}^2}$. □

Let $A_1, A_2, \dots, A_m$ be disjoint open $(N+1)$-simplices in $\mathbb{R}^N$ such that every coordinate component of any vertex of A_i is equal to 0 or 1, $A_i \cap A_j = \emptyset$ for $i \neq j$,

$$\bar{Q} = \bigcup_{i=1}^{m} \bar{A}_i,$$

and

$$A_1 = \left\{ x = (x_1, \dots, x_N) \in \mathbb{R}^N;\ x_i > 0 \text{ for all } 1 \leq i \leq N, \text{ and } \sum_{i=1}^{N} x_i < 1 \right\}.$$

Lemma 9 *We have*

$$\inf \varliminf_{\delta\to 0} \iint\limits_{\substack{A_1\times A_1 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = C_{N,p}|A_1|,$$

where the infimum is taken over all families of measurable functions $(g_\delta)_{\delta\in(0,1)}$ *defined on* A_1 *such that* g_δ *converges to* $g(x) \equiv \frac{\sum_{i=1}^N x_i}{\sqrt{N}}$ *in measure as* δ *goes to 0. Moreover, there exists a family of measurable functions* $(h_\delta)_{\delta\in(0,1)}$ *such that* h_δ *converges to* $\frac{\sum_{i=1}^N x_i}{\sqrt{N}}$ *uniformly in* A_1 *and*

$$\varlimsup_{\delta\to 0} \iint\limits_{\substack{A_1\times A_1 \\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = C_{N,p}|A_1|.$$

Proof: Set

$$\tilde{C}_{N,p} = \inf \varliminf_{\delta\to 0} \iint_{\substack{A_1^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy,$$

where the infimum is taken over all families of measurable functions $(g_\delta)_{\delta\in(0,1)}$ defined on A_1 such that g_δ converges to $g(x) \equiv \frac{\sum_{i=1}^N x_i}{\sqrt{N}}$ in measure as δ goes to 0.

First we claim that $\tilde{C}_{N,p} \geq C_{N,p}|A_1|$. In fact, let $(g_\delta)_{\delta\in(0,1)}$ be a family of measurable functions which converges to $\frac{\sum_{i=1}^N x_i}{\sqrt{N}}$ in measure on A_1. By Lemma 8, there exists a sequence of sets $(Q_i)_{i\in\mathbb{N}}$ such that Q_i is the image of Q by a dilation and a translation, $Q_i \cap Q_j = \emptyset$ for $i \neq j$, $Q_i \subset A_1$, and

$$|A_1| = \sum_{i\in\mathbb{N}} |Q_i|.$$

Applying Lemma 7, one has

$$\varliminf_{\delta\to 0} \iint_{\substack{Q_i^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \geq C_{N,p}|Q_i|.$$

This implies

$$\varliminf_{\delta\to 0} \iint_{\substack{A_1^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \geq C_{N,p}\sum_{i\in\mathbb{N}}|Q_i| = C_{N,p}|A_1|.$$

Thus

$$\tilde{C}_{N,p} \geq C_{N,p}|A_1|.$$

We claim that $\tilde{C}_{N,p} \leq C_{N,p}|A_1|$ and there exists a family of measurable functions $(h_\delta)_{\delta\in(0,1)}$ defined on A_1 such that h_δ converges to $\frac{\sum_{i=1}^N x_i}{\sqrt{N}}$ uniformly on A_1, and

$$\overline{\lim_{\delta\to 0}} \iint_{\substack{A_1^2 \\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = C_{N,p}|A_1|.$$

We prove this by contradiction. Suppose that this is not true. Then there exists $\varepsilon_0 > 0$ such that

$$\overline{\lim_{\delta\to 0}} \iint_{\substack{A_1^2 \\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \geq (C_{N,p}+\varepsilon_0)|A_1|, \tag{3.2}$$

for any family of measurable functions $(h_\delta)_{\delta\in(0,1)}$ such that h_δ uniformly converges to g on A_1. Let $(g_\delta)_{\delta\in(0,1)}$ be a family of measurable functions defined on Q such that g_δ converges to $\frac{\sum_{i=1}^N x_i}{\sqrt{N}}$ uniformly on Q and

$$\overline{\lim_{\delta\to 0}} \iint_{\substack{Q^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = C_{N,p}.$$

The existence of $(g_\delta)_{\delta\in(0,1)}$ is affirmed by Lemma 3. From Lemma 8, there exists a collection of sets $(A_{1,i})_{i\in\mathbb{N}}$ such that $A_{1,i}$ is the image of A_1 by a dilatation and a translation for every $i \in \mathbb{N}$, $A_{1,i} \cap A_{1,j} = \emptyset$ for $i \neq j$, $A_{1,i} \subset Q$, and

$$|Q| = \sum_{i\in\mathbb{N}} |A_{1,i}|.$$

Then, as in the proof of Lemma 7, it follows from (3.2) that

$$\overline{\lim_{\delta\to 0}} \iint_{\substack{A_{1,i}^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \geq (C_{N,p}+\varepsilon_0)|A_{1,i}|.$$

This implies

$$\overline{\lim_{\delta\to 0}} \iint_{\substack{Q^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \geq (C_{N,p}+\varepsilon_0)\sum_{i\in\mathbb{N}}|A_{1,i}| = (C_{N,p}+\varepsilon_0)|Q|.$$

This contradicts the choice of (g_δ). $\square$

A generalized version of Lemma 9, which will be useful, is the following

Lemma 10 *Let g be an affine function defined on A_1. Then*

$$\inf \varliminf_{\delta\to 0} \iint\limits_{\substack{A_1\times A_1\\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = C_{N,p}|Dg|^p|A_1|,$$

where the infimum is taken over all families of measurable functions $(g_\delta)_{\delta\in(0,1)}$ defined on A_1 such that g_δ converges to g in measure. Moreover, there exists a family of measurable functions $(h_\delta)_{\delta\in(0,1)}$ defined on A_1 such that h_δ converges to g uniformly on A_1 and

$$\varlimsup_{\delta\to 0} \iint\limits_{\substack{A_1\times A_1\\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy = C_{N,p}|Dg|^p|A_1|.$$

Proof: Suppose that $g(x) = \langle a, x\rangle + c$, for some $a\in\mathbb{R}^N$ and $c\in\mathbb{R}$. Hereafter $\langle .,.\rangle$ denotes the usual scalar product in $\mathbb{R}^N$. Without loss of generality, suppose that $|a| = 1$ and $c = 0$, since there is nothing to prove in the case $|a| = 0$. Let R be a rotation such that $g(Re) = \sqrt{N}$ for $e = (1,\dots,1)\in\mathbb{R}^N$. Let $(g_\delta)_{\delta\in(0,1)}$ be a family of measurable functions defined on $R(Q)$ such that g_δ converges to g in measure on $R(Q)$. Define $h_\delta : Q\mapsto\mathbb{R}$ and $h : Q\mapsto\mathbb{R}$ as follows

$$h_\delta(x) = g_\delta(Rx) \quad\text{and}\quad h(x) = g(Rx), \quad \forall\, x\in Q.$$

Then h_δ converges to h in measure on Q, $|Dh| = 1$, and $h(e) = g(Re) = \sqrt{N}$. Hence since $b_1 = \cdots = b_N = \frac{1}{\sqrt{N}}$ if $\sum_{i=1}^N b_i = \sqrt{N}$ and $\sum_{i=1}^N b_i^2 = 1$, it follows that $h(x) = \frac{\sum_{i=1}^N x_i}{\sqrt{N}}$. Thus, from the change of variables formula and the definition of $C_{N,p}$, one has

$$\varliminf_{\delta\to 0} \iint\limits_{\substack{R(Q)\times R(Q)\\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \ge C_{N,p}. \tag{3.3}$$

Consider a collection of sets $(B_i)_{i\in\mathbb{N}}$ such that B_i is an image of $R(Q)$ by a dilatation and a translation, $B_i\cap B_j = \emptyset$ for $i\ne j$, $B_i\subset A_1$, and

$$|A_1| = \sum_{i\in\mathbb{N}} |B_i|.$$

(see Lemma 8). Let (g_δ) be a family of measurable functions such that g_δ converges to g in measure on A_1. Then, as in the proof of Lemma 7, it follows from (3.3) that

$$\underline{\lim}_{\delta\to 0} \iint_{\substack{B_i^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \geq C_{N,p}|B_i|.$$

This implies

$$\underline{\lim}_{\delta\to 0} \iint_{\substack{A_1^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \geq \sum_{i\in\mathbb{N}} C_{N,p}|B_i| = C_{N,p}|A_1|. \tag{3.4}$$

It now suffices to prove that there exists a family of measurable functions $(g_\delta)_{\delta\in(0,1)}$ defined on A_1 such that g_δ converges to g uniformly in A_1, and

$$\overline{\lim}_{\delta\to 0} \iint_{\substack{A_1^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \leq C_{N,p}|A_1|. \tag{3.5}$$

Let $(h_\delta)_{\delta\in(0,1)}$ be a family of measurable function defined on Q such that h_δ converges to h uniformly on Q and

$$\overline{\lim}_{\delta\to 0} \iint_{\substack{Q^2 \\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \leq C_{N,p}.$$

Define $g_\delta : R(Q) \mapsto \mathbb{R}$ by $g_\delta(x) = h_\delta(R^{-1}x)$ for $x \in R(Q)$. Then g_δ converges to g uniformly in $R(Q)$ and

$$\overline{\lim}_{\delta\to 0} \iint_{\substack{R(Q)\times R(Q) \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \leq C_{N,p}. \tag{3.6}$$

Using (3.6) and (3.4), as in the proof of Lemma 9, one has g_δ uniformly converges to g in A_1 and satisfies (3.5). The detail is left to the reader. □

The following lemma plays the same role as Lemma 5 for the one dimensional case.

Lemma 11 *Let g be an affine function defined on A_1 such that the normal derivative $\frac{\partial g}{\partial n}$ does not vanish along the boundary of A_1. Then there exists a family of measurable functions $(g_\delta)_{\delta\in(0,1)}$ defined on A_1 and a family of* *positive numbers* $(c_\delta)_{\delta\in(0,1)}$ *converging to 0 such that* $c_\delta \geq \sqrt{\delta}$, $|g_\delta(x) - g(x)| \leq 8N(1+|Dg|)c_\delta$ *for all* $x \in A_1$, g_δ *is Lipschitz on* A_{1,c_δ}, *where*

$$A_{1,c_\delta} := \{x \in A_1; \operatorname{dist}(x, A_1^c) \leq c_\delta\},$$

with a Lipschitz constant $|Dg|$, *and*

$$\overline{\lim_{\delta\to 0}} \iint\limits_{\substack{A_1\times A_1\\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \leq C_{N,p}|Dg|^p|A_1|.$$

Proof: We adapt here the proof of Lemma 4. By Lemma 10, there exists a family of measurable functions $(g_\delta)_{\delta\in(0,1)}$ such that g_δ converges to g uniformly in A_1 as δ goes to 0, and

$$\overline{\lim_{\delta\to 0}} \iint\limits_{\substack{A_1\times A_1\\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy = C_{N,p}|Dg|^p|A_1|.$$

Set

$$c_\delta = \max(\|g_\delta - g\|_{L^\infty(A_1)}, \sqrt{\delta}), \quad l_\delta = 2(|Dg|c_\delta + c_\delta),$$

and

$$g_{0,\delta} = g_\delta,$$

for $\delta \in (0,1)$. For $i = 1, 2, \dots, N$, define

$$g_{i,\delta}(x) = \begin{cases} \max\big(g_{i-1,\delta}(x), g(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_N) + il_\delta\big) \text{ if } \frac{\partial g}{\partial x_i} > 0, \\ \min\big(g_{i-1,\delta}(x), g(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_N) - il_\delta\big) \text{ if } \frac{\partial g}{\partial x_i} < 0. \end{cases}$$

Set $e = (\frac{1}{\sqrt{N}}, \dots, \frac{1}{\sqrt{N}})$ and define

$$g_{N+1,\delta}(x) = \begin{cases} \max\big(g_{N,\delta}(x), g(z(x)) + (N+1)l_\delta\big) & \text{if } \frac{\partial g}{\partial e} < 0 \\ \min\big(g_{N,\delta}(x), g(z(x)) - (N+1)l_\delta\big) & \text{if } \frac{\partial g}{\partial e} > 0. \end{cases}$$

Here for each $x \in \mathbb{R}^N$, $z(x) = x - \langle x, e\rangle e + e$, i.e. $z(x)$ denotes the projection of x on the hyperplane P which is orthogonal to e and contains e. Then, as in the proof of Lemma 3, $|g_{i,\delta}(x) - g(x)| \leq il_\delta$ for $x \in A_1$. Thus

$$g_{i,\delta}(x) = \begin{cases} g(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_N) + il_\delta & \text{if } \frac{\partial g}{\partial x_i} > 0 \\ g(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_N) - il_\delta & \text{if } \frac{\partial g}{\partial x_i} < 0, \end{cases}$$

for $1 \le i \le N$, for any $x \in A_1$ such that $0 \le x_i \le c_\delta$, and

$$g_{N+1,\delta}(x) = \begin{cases} g(z(x)) + (N+1)l_\delta & \text{if } \frac{\partial g}{\partial e} < 0 \\ g(z(x)) - (N+1)l_\delta & \text{if } \frac{\partial g}{\partial e} > 0, \end{cases}$$

where $z(x) = x - \langle x, e\rangle e + e$, for any $x \in A_1$ such that $0 \le |x - z(x)| \le c_\delta$. Then $g_{N+1,\delta}$ is Lipschitz on A_{1,c_δ} with a Lipschitz constant $|Dg|$ (see the proof of Lemma 3 for these arguments).

It now suffices to prove

$$\overline{\lim_{\delta\to 0}} \iint\limits_{\substack{A_1\times A_1 \\ |g_{N+1,\delta}(x)-g_{N+1,\delta}(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le \overline{\lim_{\delta\to 0}} \iint\limits_{\substack{A_1\times A_1 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy. \tag{3.7}$$

For $1 \le i \le N$, since if $\frac{\partial g}{\partial x_i} > 0$,

$$\lim_{\delta\to 0} |\{x \in A_1;\, g_{i-1,\delta}(x) \le g(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_N) + il_\delta\}| = 0,$$

and

$$\lim_{\delta\to 0} |\{x \in A_1;\, g_{i-1,\delta}(x) \ge g(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_N) - il_\delta\}| = 0,$$

if $\frac{\partial g}{\partial x_i} < 0$, it follows from Lemmas 1 and 2 that, for $1 \le i \le N$,

$$\overline{\lim_{\delta\to 0}} \iint\limits_{\substack{A_1\times A_1 \\ |g_{i,\delta}(x)-g_{i,\delta}(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le \overline{\lim_{\delta\to 0}} \iint\limits_{\substack{A_1\times A_1 \\ |g_{i-1,\delta}(x)-g_{i-1,\delta}(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy. \tag{3.8}$$

Similarly,

$$\varlimsup_{\delta\to 0} \iint\limits_{\substack{A_1\times A_1\\ |g_{N+1,\delta}(x)-g_{N+1,\delta}(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy$$

$$\leq \varlimsup_{\delta\to 0} \iint\limits_{\substack{A_1\times A_1\\ |g_{N,\delta}(x)-g_{N,\delta}(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy. \tag{3.9}$$

Thus (3.7) follows from (3.8) and (3.9). □

Using the same ideas as in the proof of Lemma 11, one gets

Lemma 12 *Let $i \in \{1,\dots,m\}$ and g be an affine function defined on A_i such that $\frac{\partial g}{\partial n} \neq 0$ along the boundary of A_i. Then there exists a family of measurable functions $(g_\delta)_{\delta\in(0,1)}$ defined on A_i and a family of positive numbers $(c_\delta)_{\delta\in(0,1)}$ converging to 0 such that $c_\delta \geq \sqrt{\delta}$, $|g_\delta(x)-g(x)| \leq 8N(|Dg|+1)c_\delta$ for all $x \in A_i$, g_δ is Lipschitz on A_{i,c_δ}, where*

$$A_{i,c_\delta} := \{x \in A_i;\, \mathrm{dist}(x, A_i^c) \leq c_\delta\},$$

with a Lipschitz constant $|Dg|$, and

$$\varlimsup_{\delta\to 0} \iint\limits_{\substack{A_i\times A_i\\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \leq C_{N,p}|Dg|^p|A_i|.$$

To approximate a smooth function by a continuous piecewise linear function and to be able to apply Lemma 12, which plays a role as Lemma 3 in the case $N=1$, one introduces the following

Definition 1 For each $k \in \mathbb{N}$, K is called a k-net of $\mathbb{R}^N$ if and only if there exist $l \in \mathbb{N}^N$ and $i \in \{1,2,\dots,m\}$ such that $K = \frac{1}{2^k}A_i + \frac{l}{2^k}$.

Hereafter, for any two subsets A and B of $\mathbb{R}^N$ and a real number c, one defines

$$cA = \{ca \in \mathbb{R}^N;\, a \in A\}$$

and

$$A + B := \{a+b \in \mathbb{R}^N;\, a \in A \text{ and } b \in B\}.$$

When B is a set containing only a vector v, one writes $A+v$ instead of $A+\{v\}$.

Definition 2 A continuous function g on $\mathbb{R}^N$ is said to be a continuous piecewise linear function defined on k-nets if and only if g is affine on each k-net of $\mathbb{R}^N$.

A variant of Lemma 5 for the general case is the following

Lemma 13 *Let K be a k-net of $\mathbb{R}^N$ and g be an affine function defined on K such that $\frac{\partial g}{\partial n} \neq 0$ along the boundary of K. Then there exists a family of measurable functions $(g_\delta)_{\delta\in(0,1)}$ defined on K and a family of positive numbers $(c_\delta)_{\delta\in(0,1)}$ converging to 0 such that $c_\delta \geq \sqrt{\delta}$, $|g_\delta(x) - g(x)| \leq 2^{-k+3}N(|Dg|+1)c_\delta$ for all $x \in K$, g_δ is Lipschitz function on $K_{2^{-k}c_\delta}$, where*

$$K_{2^{-k}c_\delta} := \{x \in K;\, \mathrm{dist}(x, K^c) \leq 2^{-k}c_\delta\},$$

with a Lipschitz constant $|Dg|$, and

$$\varlimsup_{\delta\to 0} \iint\limits_{\substack{K\times K \\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \leq C_{N,p}|Dg|^p|K|.$$

Proof: Without loss of generality, one may assume that $K = 2^{-k}A_i$ for some $1 \leq i \leq m$. Set $g_k(x) = 2^k g(2^{-k}x)$ for $x \in A_i$. Then by Lemma 12, there exist a family of measurable functions $(g_{k,\delta})_{\delta\in(0,1)}$ and a family $(c_\delta)_{\delta\in(0,1)}$ converging to 0 such that $c_\delta \geq \sqrt{\delta}$, $|g_{k,\delta}(x) - g_k(x)| \leq 8N(|Dg|+1)c_\delta$ for all $x \in A_i$, $g_{k,\delta}$ is Lipschitz on A_{i,c_δ} with a Lipschitz constant $|Dg|$ (since $|Dg_k| = |Dg|$), and

$$\varlimsup_{\delta\to 0} \iint\limits_{\substack{A_i\times A_i \\ |g_{k,\delta}(x)-g_{k,\delta}(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \leq C_{N,p}|Dg|^p|A_i|.$$

Set $g_\delta(x) = 2^{-k}g_{k,\delta}(2^k x)$ for $x \in K$. Then $|g_\delta - g| \leq 2^{-k+3}N(|Dg|+1)c_\delta$ on K, g_δ is Lipschitz on $K_{2^{-k}c_\delta}$ with a Lipschitz constant $|Dg|$ (since $|Dg_\delta| = |Dg_{k,\delta}|$), and

$$\varlimsup_{\delta\to 0} \iint_{\substack{K\times K \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \leq C_{N,p}|Dg|^p|K|.$$

□

A variant of Lemma 6 for the general case is the following

Lemma 14 *Let g be a continuous piecewise linear function on k-nets with compact support such that on each k-net $\frac{\partial g}{\partial n} \neq 0$ along the boundary of that k-net unless g is constant on this one. Then there exists a family of measurable functions $(g_\delta)_{\delta\in(0,1)} \subset L^p(\mathbb{R}^N)$ such that g_δ converges to g in $L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and*

$$\overline{\lim_{\delta\to 0}} \iint\limits_{\substack{\mathbb{R}^N\times\mathbb{R}^N \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le C_{N,p}\int_{\mathbb{R}^N} |Dg|^p\,dx.$$

Proof: For each k-net K_i, if g is not constant on K_i, by Lemma 13, there exist a family of measurable functions $(h_{i,\delta})_{\delta\in(0,1)}$ defined on K_i and a family of positive numbers $(c_{i,\delta})_{\delta\in(0,1)}$ converging to 0 such that $c_{i,\delta} \ge \sqrt{\delta}$, $|h_{i,\delta}(x) - g(x)| \le 2^{-k+3}N(\|Dg\|_\infty + 1)c_{i,\delta}$ for $x \in K_i$, $h_{i,\delta}$ is Lipschitz on $K_{i,2^{-k}c_{i,\delta}}$ with a Lipschitz constant $\|Dg\|_{L^\infty(\mathbb{R}^N)}$, and

$$\overline{\lim_{\delta\to 0}} \iint_{\substack{K_i^2 \\ |h_{i,\delta}(x)-h_{i,\delta}(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le C_{N,p}\int_{K_i} |Dg|^p\,dx.$$

Hereafter in this proof, for $\tau > 0$,

$$K_{i,\tau} := \{x \in K_i;\, \mathrm{dist}(x, K_i^c) \ge \tau\}.$$

If g is constant on K_i, define $h_{i,\delta} = g$ on K_i, $c_{i,\delta} = \sqrt{\delta}$. Then there exists a family of measurable functions $(g_\delta)_{\delta\in(0,1)}$ such that $g_\delta = h_{i,\delta}$ on each $K_{i,2^{-k-1}c_{i,\delta}}$, g_δ is Lipschitz on $\mathbb{R}^N \setminus \bigcup \bar{K}_{i,2^{-k}c_{i,\delta}}$ with a Lipschitz constant $C(\|Dg\|_\infty + 1)$ for some $C = C(N) > 0$, and $\operatorname{supp} g_\delta \subset \operatorname{supp} g + B_{2^{-k}c_\delta}$, $c_\delta := \sup_i c_{i,\delta}$. Here for any $r > 0$, we denote

$$B_r = \{x \in \mathbb{R}^N;\, |x| \le r\}.$$

Hence

$$\iint\limits_{\substack{\mathbb{R}^N\times\mathbb{R}^N \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le \sum_i \iint_{\substack{K_i^2 \\ |h_{i,\delta}(x)-h_{i,\delta}(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy + C(k,g)(c_\delta + \delta^{\frac{p}{2}}),$$

since

$$\iint\limits_{\substack{K_i\times(\mathbb{R}^N\setminus K_i) \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le C\left(c_\delta(\|Dg\|_\infty + 1)^p + \frac{\delta^p}{2^{-k}\sqrt{\delta^p}}\right).$$

This implies

$$\overline{\lim_{\delta\to 0}} \iint\limits_{\substack{\mathbb{R}^N\times\mathbb{R}^N \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le C_{N,p}\int_{\mathbb{R}^N}|Dg|^p\,dx.$$

□

We are now ready to prove

Proof of Claim 1: Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of smooth function with compact support in $\mathbb{R}^N$ such that g_n converges to g in $L^p(\mathbb{R}^N)$ and $\|Dg_n\|_{L^p(\mathbb{R}^N)}$ converges to $\|Dg\|_{L^p(\mathbb{R}^N)}$ (when $p=1$, the L^1-norm is replaced by the total mass). For each $n\in\mathbb{N}$, let $(g_{k,n})_{k\in\mathbb{N}}$ be a sequence of functions defined on $\mathbb{R}^N$ such that $g_{k,n}$ is a continuous piecewise linear function with compact support defined on k-nets, $g_{k,n}$ converges to g_n in $W^{1,p}(\mathbb{R}^N)$. Without loss of generality, one may assume that $\frac{\partial g_{k,n}}{\partial n}\neq 0$ along the boundary of on each k-net unless $g_{k,n}$ is constant on this one. Applying Lemma 14, one finds a family $(g_{\delta,k,n})_{\delta\in(0,1)}\subset L^p(\mathbb{R}^N)$ such that $g_{\delta,k,n}$ converges to $g_{k,n}$ in $L^p(\mathbb{R}^N)$, as δ goes to 0, and

$$\overline{\lim_{\delta\to 0}}\, I_\delta(g_{\delta,k,n}) \le C_{N,p}\int_{\mathbb{R}^N}|Dg_{k,n}|^p\,dx.$$

The rest of the proof, which is quite standard, is left to the reader (see the argument in the proof for the case $N=1$). □

Remark 4 Using the same method as in the proof of Claim 1, we also prove that for any U open subset of $\mathbb{R}^N$ with Lipschitz boundary, for each $g\in W^{1,p}(U)$ ($p\ge 1$), there exists a family of measurable functions $(g_\delta)_{\delta\in(0,1)}$ defined on U such that g_δ converges to g in $L^p(U)$ and

$$\overline{\lim_{\delta\to 0}} \iint\limits_{\substack{U^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le C_{N,p}\int_U|Dg|^p\,dx.$$

4 Proof of Claim 2 in the case $p>1$

4.1 Proof of Claim 2 in the case $p>1$ and $N=1$

The proof of Claim 2 in the case $p>1$ and $N=1$ is heavily based on the following lemma, which is a consequence of the definition of $C_{1,p}$.

Lemma 15 *any $\varepsilon > 0$, there exist three positive numbers $\tilde{\delta}_1(\varepsilon)$, $\tilde{\delta}_2(\varepsilon)$, and $\tilde{\delta}_3(\varepsilon)$ such that if g is a measurable function defined on $[0,1]$,*

$$|\{x \in [0,1];\, |g(x) - x| > \tilde{\delta}_1(\varepsilon)\}| < \tilde{\delta}_2(\varepsilon),$$

and $\delta < \tilde{\delta}_3(\varepsilon)$, then

$$\iint_{\substack{[0,1]^2 \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{p+1}}\, dx\, dy \geq C_{1,p} - \varepsilon.$$

Proof: We prove this lemma by contradiction. Suppose it is not true. Then there exist a positive number ε_0, a sequence of measurable functions $(g_k)_{k\in\mathbb{N}}$ defined on $[0,1]$, and a sequence $(\delta_k)_{k\in\mathbb{N}}$ converging to 0 such that

$$|\{x \in [0,1];\, |g_k(x) - x| > \frac{1}{k}\}| < \frac{1}{k},$$

and

$$I_{\delta_k}(g_k) < C_{1,p} - \varepsilon_0, \quad \forall\, k \in \mathbb{N}.$$

Thus g_k converges to x in measure and

$$\varliminf_{k\to\infty} I_{\delta_k}(g_k) < C_{1,p} - \varepsilon_0.$$

This contradicts the definition of $C_{1,p}$. □

As a consequence of Lemma 15, one has

Lemma 16 *For any $\varepsilon > 0$, there exist three positive numbers δ_1, δ_2, and δ_3 depending only on ε such that if g is a measurable function on $[a,b]$,*

$$|\{x \in [a,b];\, |g(x) - (c(x-x_0)+d)| > |c|(b-a)\delta_1\}| < (b-a)\delta_2, \tag{4.1}$$

for some $x_0 \in \mathbb{R}$, and $\delta < |c|(b-a)\delta_3$ then

$$\iint_{\substack{[a,b]^2 \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{p+1}}\, dx\, dy \geq (C_{1,p} - \varepsilon)|c|^p(b-a).$$

Proof: Let $\delta_1 = \tilde{\delta}_1(\varepsilon)$, $\delta_2 = \tilde{\delta}_2(\varepsilon)$, and $\delta_3 = \tilde{\delta}_3(\varepsilon)$, where $\tilde{\delta}_i(\varepsilon)$ is a positive constant corresponding to ε in Lemma 15 for $i = 1,2,3$. Without loss of generality, one may assume that $c > 0$ and $d = 0$. Then

$$|\{x \in [a,b];\, |h(x) - (x-a)| > \tilde{\delta}_1(\varepsilon)(b-a)\}| < \tilde{\delta}_2(\varepsilon)(b-a),$$

where $h : [a,b] \to \mathbb{R}$ is defined by

$$h(x) = \frac{g(x) + c(x_0 - a)}{c}, \quad \forall\, x \in [a,b].$$

Consider $h_1 : [0,1] \to \mathbb{R}$ defined as follows

$$h_1(t) = \frac{h((b-a)t + a)}{b-a}, \quad \forall\, t \in [0,1].$$

Then

$$|\{t \in [0,1];\ |h_1(t) - t| > \tilde{\delta}_1(\varepsilon)\}| < \tilde{\delta}_2(\varepsilon).$$

Hence applying Lemma 15, one has

$$\iint_{\substack{[0,1]^2 \\ |h_1(t)-h_1(s)|>\delta}} \frac{\delta^p}{|t-s|^{p+1}}\, dt\, ds \geq (C_{1,p} - \varepsilon),$$

for all $\delta < \tilde{\delta}_3$. This implies

$$\iint_{\substack{[a,b]^2 \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{p+1}}\, dx\, dy \geq (C_{1,p} - \varepsilon)|c|^p(b-a),$$

for all $\delta < c(b-a)\delta_3$. □

Using Lemma 16, one can prove

Proof of Claim 2: Fix $\varepsilon > 0$ (arbitrary) and let δ_i be a positive constant corresponding to ε in Lemma 16 for $i = 1, 2, 3$.

For $m \in \mathbb{N}_+$, define

$$A_m = \Big\{x \in [-m,m];\ \frac{1}{h}\int_x^{x+h} |g'(s) - g'(x)|^p\, ds \to 0 \text{ as } h \to 0$$
$$\text{and } \frac{1}{m} < |g'(x)| < m\Big\}.$$

Then

$$\lim_{m\to\infty} \int_{\mathbb{R}\setminus A_m} |g'|^p\, dx = 0.$$

Hence there exist $m \in \mathbb{N}_+$ and an open set $U \subset \mathbb{R}$ such that $\mathbb{R} \setminus A_m \subset U$ and

$$\int_U |g'|^p\, dx \leq \varepsilon. \tag{4.2}$$

Set

$$B_m = \mathbb{R} \setminus U.$$

Then B_m is a compact subset of $\mathbb{R}$. Fix $x \in B_m$. Then $x \in A_m$. Thus $\exists\, h_x \in (0,1)$ such that

$$\left| \frac{1}{h} \int_x^{x+h} |g'(s) - g'(x)|^p \, ds \right| \leq \frac{\min\{\delta_1, \varepsilon\}}{2m}, \quad \forall\, h \in [-h_x, h_x],$$

Since $B_m \subset \bigcup_{x \in B_m} (x - h_x, x + h_x)$ and B_m is a compact subset of $\mathbb{R}$, there exist $c_i \in B_m$ and $[a_i, b_i]$ for all $1 \leq i \leq k$ for some $k \in \mathbb{N}$ such that $[a_i, b_i] \subset [c_i - h_{c_i}, c_i + h_{c_i}]$, $c_i \in (a_i, b_i)$, $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ for $i \neq j$, and $B_m \subset \bigcup_{1 \leq i \leq k} [a_i, b_i]$. In fact, consider a family $\{(x_i - h_{x_i}, x_i + h_{x_i})\}_{i=1}^k$ which satifies

$$\begin{cases} B \subset \bigcup_{i=1}^k (x_i - h_{x_i}, x_i + h_{x_i}), \\ x_1 < x_2 < \cdots < x_k, \\ (x_i - h_{x_i}, x_i + h_{x_i}) \not\subset \bigcup_{j \neq i} (x_j - h_{x_j}, x_j + h_{x_j}). \end{cases}$$

Set

$$\begin{cases} c_1 = x_1, \\ a_1 = x_1 - h_{x_1}, \\ b_1 = x_1 + h_{x_1} \text{ if } x_1 + h_{x_1} < x_2, \dfrac{x_1 + x_2}{2} \text{ otherwise}, \end{cases}$$

$$\begin{cases} c_2 = x_2, \\ a_2 = \max(b_1, x_2 - h_{x_2}), \\ b_2 = x_2 + h_{x_2} \text{ if } x_2 + h_{x_2} < x_3, \dfrac{x_2 + x_3}{2} \text{ otherwise}, \end{cases}$$

$\ldots$

$$\begin{cases} c_k = x_k, \\ a_k = \max(b_{k-1}, x_k - h_{x_k}), \\ b_k = x_k + h_{x_k}. \end{cases}$$

Then the family $\{c_i, [a_i, b_i]\}_{i=1}^k$ verifies the properties.

We have

$$g(x) = g'(c_i)(x - c_i) + g(c_i) + o(x - c_i),$$

where $|o(x-c_i)| \le \frac{\delta_1}{2m}|x-c_i|$. Thus

$$|g(x) - [g'(c_i)(x-c_i) + g(c_i)]| \le \frac{\delta_1}{2m}|b_i - a_i|, \quad \forall\, x \in (a_i, b_i).$$

On the other hand, since g_δ converges to g in measure,

$$|\{x \in [a_i, b_i]; |g_\delta(x) - g(x)| > \frac{\delta_1}{2m}|b_i - a_i|\}| < \delta_2 |b_i - a_i|,$$

when δ is sufficiently small. Then

$$|\{x \in [a_i, b_i]; |g_\delta(x) - [g'(c_i)(x-c_i) + g(c_i)]| > \frac{\delta_1}{m}|b_i - a_i|\}| < \delta_2 |b_i - a_i|.$$

Applying Lemma 16, one has

$$\varliminf_{\delta \to 0} \iint_{\substack{[a_i,b_i]^2 \\ |g_\delta(x) - g_\delta(y)| > \delta}} \frac{\delta^p}{|x-y|^{p+1}}\, dx\, dy \ge [C_{1,p} - \varepsilon](b_i - a_i)|g'(c_i)|^p,$$

which implies

$$\varliminf_{\delta \to 0} \iint_{\substack{\mathbb{R}^2 \\ |g_\delta(x) - g_\delta(y)| > \delta}} \frac{\delta^p}{|x-y|^{p+1}}\, dx\, dy \ge \sum_{i=1}^{k} [C_{1,p} - \varepsilon](b_i - a_i)|g'(c_i)|^p.$$

However,

$$\begin{aligned}\int_{a_i}^{b_i} |g'(x)|^p\, dx \le& (b_i - a_i)|g'(c_i)|^p + C_p \int_{a_i}^{b_i} |g'(x) - g'(c_i)|^p\, dx \\ \le& (b_i - a_i)|g'(c_i)|^p + C_p (b_i - a_i)\varepsilon/m.\end{aligned}$$

Hence

$$\varliminf_{\delta \to 0} \iint_{\substack{\mathbb{R}^2 \\ |g_\delta(x) - g_\delta(y)| > \delta}} \frac{\delta^p}{|x-y|^{p+1}}\, dx\, dy \ge [C_{1,p} - \varepsilon] \int_{B_m} |g'(x)|^p\, dx - C_p \varepsilon. \quad (4.3)$$

Therefore, since $\varepsilon > 0$ is arbitrary, it follows from (4.2) and (4.3) that

$$\varliminf_{\delta \to 0} \iint_{\substack{\mathbb{R}^2 \\ |g_\delta(x) - g_\delta(y)| > \delta}} \frac{\delta^p}{|x-y|^{p+1}}\, dx\, dy \ge C_{1,p} \int_{\mathbb{R}} |g'(x)|^p\, dx.$$

□

4.2 Proof of Claim 2 in the case $p > 1$ and $N \geq 2$

The proof of Claim 2 in the case $p > 1$ and $N \geq 2$ follows the ideas used in the proof for the case $p > 1$ and $N = 1$. However, the proof is more complicated.

Lemma 17 *For any $\varepsilon > 0$, there exist three positive numbers $\tilde{\delta}_1(\varepsilon)$, $\tilde{\delta}_2(\varepsilon)$, and $\tilde{\delta}_3(\varepsilon)$ such that if g is a measurable function defined on Q,*

$$|\{x \in Q;\, |g(x) - \frac{\sum_{i=1}^N x_i}{\sqrt{N}}| > \tilde{\delta}_1(\varepsilon)\}| < \tilde{\delta}_2(\varepsilon),$$

and $\delta < \tilde{\delta}_3(\varepsilon)$, then

$$\iint_{\substack{Q^2 \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy \geq C_{N,p} - \varepsilon.$$

Proof: The proof is the same as the one of Lemma 15. □

To obtain a generalized version of Lemma 17, one needs the following

Lemma 18 *Let g be an affine function defined on Q. Then*

$$\inf \varliminf_{\delta \to 0} \iint_{\substack{Q^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy = C_{N,p}|Dg|^p,$$

where the infimum is taken over all families of measurable functions $(g_\delta)_{\delta \in (0,1)}$ defined on Q such that g_δ converges to g in measure. Moreover, there exists a family of measurable functions $(h_\delta)_{\delta \in (0,1)}$ defined on Q such that h_δ converges to g uniformly in Q and

$$\varlimsup_{\delta \to 0} \iint_{\substack{Q^2 \\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy = C_{N,p}|Dg|^p.$$

Proof: The proof is similar to the one of Lemma 10. □

A generalized version of Lemma 17 is as follows.

Lemma 19 *For any $\varepsilon > 0$, there exist three positive numbers $\hat{\delta}_1(\varepsilon)$, $\hat{\delta}_2(\varepsilon)$, and $\hat{\delta}_3(\varepsilon)$ such that if g is a measurable function defined on Q,*

$$|\{x \in Q;\, |g(x) - (\langle a, x\rangle + b)| > \hat{\delta}_1(\varepsilon)\}| < \hat{\delta}_2(\varepsilon),$$

for some $a \in \mathbb{R}^N$ and $b \in \mathbb{R}$ such that $|a| = 1$, and $\delta < \hat{\delta}_3(\varepsilon)$, then

$$\iint_{\substack{Q^2 \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \geq C_{N,p} - \varepsilon.$$

We recall here that $\langle .,. \rangle$ denotes the usual scalar product in $\mathbb{R}^N$.

Proof: We prove this by contradiction. Suppose this is not true. Then there exist $\varepsilon_0 > 0$, a sequence of measurable functions $(g_n)_{n\in\mathbb{N}}$, a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$, a sequence $(b_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, and a sequence $(\delta_n)_{n\in\mathbb{N}}$ converging to 0 such that $|a_n| = 1$,

$$|\{x \in Q;\, |g_n(x) - (\langle a_n, x\rangle + b_n)| > \frac{1}{n}\} < \frac{1}{n},$$

and

$$\iint_{\substack{Q^2 \\ |g_n(x)-g_n(y)|>\delta_n}} \frac{\delta_n^p}{|x-y|^{N+p}}\,dx\,dy < C_{N,p} - \varepsilon_0.$$

Without loss of generality, one may assume that $b_n = 0$ for $n \in \mathbb{N}$. Since $|a_n| = 1$, there exist $a \in \mathbb{R}^N$ and a subsequence (a_{n_k}) of (a_n) such that a_{n_k} converges to a and $|a| = 1$. Then g_{n_k} converges to $\langle a, . \rangle$ in measure on Q and

$$\iint_{\substack{Q^2 \\ |g_{n_k}(x)-g_{n_k}(y)|>\delta_{n_k}}} \frac{\delta_{n_k}^p}{|x-y|^{N+p}}\,dx\,dy < C_{N,p} - \varepsilon_0.$$

This contradicts Lemma 18. $\square$

The following lemma, which is a variant of Lemma 16 in high dimensional cases, is a consequence of Lemma 19.

Lemma 20 *Let $\varepsilon > 0$ and $\tilde{Q}$ be the image of Q by a translation and a dilatation. Let l be the length of any edge of $\tilde{Q}$. Then there exist three positive numbers δ_1, δ_2, and δ_3 depending only on ε such that if g is a measurable function defined on $\tilde{Q}$,*

$$|\{x \in \tilde{Q};\, |g(x) - (\langle a, x - x_0\rangle + b)| > l|a|\delta_1\}| < \delta_2|\tilde{Q}|,$$

and $\delta < l|a|\delta_3$, for some $a \in \mathbb{R}^N$, $x_0 \in \mathbb{R}^N$, and $b \in \mathbb{R}$, then

$$\iint_{\substack{\tilde{Q}^2 \\ |g(x)-g(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \geq (C_{N,p} - \varepsilon)|a|^p|\tilde{Q}|.$$

Proof: The proof is similar to the one of Lemma 16. For the convenience of the reader, we present the proof in detail. Let $\delta_1 = \hat{\delta}_1(\varepsilon)$, $\delta_2 = \hat{\delta}_2(\varepsilon)$, and $\delta_3 = \hat{\delta}_3(\varepsilon)$, where $\hat{\delta}_i(\varepsilon)$ is a positive constant corresponding to ε in Lemma 19 for $i = 1, 2, 3$. Without loss of generality, one may assume that $b = 0$ and $|a| > 0$. Let $\tau \in \mathbb{R}^N$ be such that

$$\tilde{Q} = lQ + \tau.$$

Then

$$|\{x \in \tilde{Q};\, |h(x) - \langle \hat{a}, x - \tau\rangle| > \hat{\delta}_1(\varepsilon) l\}| < \hat{\delta}_2(\varepsilon)|\tilde{Q}|,$$

where $\hat{a} = a/|a|$ and $h : \tilde{Q} \to \mathbb{R}$ defined by

$$h(x) = \frac{g(x) + \langle a, x_0 - \tau\rangle}{|a|}.$$

Define $h_1 : Q \to \mathbb{R}$ as follows

$$h_1(y) = \frac{h(ly + \tau)}{l},$$

Then

$$|\{y \in Q;\, |h_1(y) - \langle \hat{a}, y\rangle| > \tilde{\delta}_1(\varepsilon)\}| < \tilde{\delta}_2(\varepsilon).$$

Applying Lemma 19, one has

$$\iint_{\substack{Q^2 \\ |h_1(x) - h_1(y)| > \delta}} \frac{\delta^p}{|x - y|^{N+p}}\, dx\, dy \geq (C_{N,p} - \varepsilon),$$

for all $\delta < \delta_3$. This implies

$$\iint_{\substack{\tilde{Q}^2 \\ |g(x) - g(y)| > \delta}} \frac{\delta^p}{|x - y|^{N+p}}\, dx\, dy \geq (C_{N,p} - \varepsilon)|a|^p|\tilde{Q}|,$$

for all $\delta < l|a|\delta_3$. □

Using Lemma 20, one can prove

Proof of Claim 2: For $x \in \mathbb{R}^N$ and $r > 0$, let $Q(x, r)$ denote the open cube center at x with the length of each its edge equal to $2r$, i.e.

$$Q(x, r) := \{y = (y_1, \ldots, y_N) \in \mathbb{R}^N;\, |y_i - x_i| < r \text{ for all } 1 \leq i \leq N\}.$$

For $n = 1, 2, 3, \ldots$, we let P_n be the set of all $x \in \mathbb{R}^N$ whose coordinates are integral multiples of 2^{-n}, and we let $\mathbf{\Omega}_n$ be the collection of all 2^{-n} open cubes with corners at points of P_n. For $x \in \mathbb{R}^N$, set

$$\tau_n(x) := \sup_{\substack{Q' \in \mathbf{\Omega}_k; k \geq n; \\ x \in Q'}} \fint_{Q'} |Dg(y) - Dg(x)|^p \, dy,$$

and

$$\rho_n(x) := \sup_{0<r<1/2^n} \frac{1}{r} \fint_{Q(x,r)} |g(y) - g(x) - Dg(x)(y-x)| \, dy.$$

For each $m \in \mathbb{N}$, consider the set

$$A_m = \Big\{ x \in [-m, m]^N ; 1/m \leq |Dg(x)| \leq m, \lim_{n \to \infty} \rho_n(x) = 0, \\ \text{and } \lim_{n\to\infty} \tau_n(x) = 0 \Big\}.$$

Then it follows from [8, Theorem 1 page 228], the theory of maximal functions, and Egorov's theorem that there exist $m \in \mathbb{N}_+$ and a compact set $B_m \subset A_m$ such that ρ_n and τ_n converge to 0 uniformly on B_m, and

$$\int_{\mathbb{R}^N \setminus B_m} |Dg|^p \, dx \leq \varepsilon \int_{\mathbb{R}^N} |Dg|^p \, dx. \tag{4.4}$$

Fix $\varepsilon > 0$ (arbitrary), let δ_1, δ_2, and δ_3 be three positive constants corresponding to ε in Lemma 20 . Take k such that $\rho_k(x)$ and $\tau_k(x)$ are less than ε_1 for all $x \in B_m$, where ε_1 is a small constant defined later. Let $\mathbf{J}$ be the collection of $Q' \in \mathbf{\Omega}_k$ such that $Q' \cap B_m \neq \emptyset$. Choose ε_1 sufficiently small ($\varepsilon_1 = \varepsilon_1(\varepsilon, m, \delta_1, \delta_2)$). Then for $Q' \in \mathbf{J}$ and $x \in Q' \cap B_m$, one has

$$\{y \in Q'; |g(y) - g(x) - Dg(x)(y-x)| > \frac{\delta_1}{2m} |Q'|^{\frac{1}{N}}\} < \frac{\delta_2}{2} |Q'|,$$

since $\rho_k(x) < \varepsilon_1$, and

$$|Dg(x)|^p |Q'| \geq (1-\varepsilon) \int_{Q'} |Dg|^p \, dy, \tag{4.5}$$

since $\tau_k(x) < \varepsilon_1$ and $|Dg(x)| > \frac{1}{m}$. Since g_δ converges to g in measure, we have

$$\{y \in Q'; |g_\delta(y) - g(y)| > \frac{\delta_1}{2m} |Q'|^{\frac{1}{N}}\} < \frac{\delta_2}{2} |Q'|,$$

when δ small. This implies

$$\{y \in Q'; |g_\delta(y) - g(x) - Dg(x)(y-x)| > \frac{\delta_1}{m}|Q'|^{\frac{1}{N}}\} < \delta_2 |Q'|,$$

when δ small. Hence, applying Lemma 20, one gets

$$\varliminf_{\delta\to 0} \iint_{\substack{Q'^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \geq (C_{N,p} - \varepsilon)|Dg(x)|^p |Q'|,$$

which implies, by (4.5),

$$\varliminf_{\delta\to 0} \iint_{\substack{Q'^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \geq (C_{N,p} - \varepsilon)(1-\varepsilon)\int_{Q'} |Dg|^p\,dy. \quad (4.6)$$

Thus since

$$\varliminf_{\delta\to 0} \iint_{\substack{\mathbb{R}^N\times\mathbb{R}^N \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \geq \sum_{Q'\in\mathbf{J}} \varliminf_{\delta\to 0} \iint_{\substack{Q'^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy,$$

it follows from (4.4) and (4.6) that

$$\varliminf_{\delta\to 0} \iint_{\substack{\mathbb{R}^N\times\mathbb{R}^N \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \geq (C_{N,p} - \varepsilon)(1-\varepsilon)^2\int_{\mathbb{R}^N} |Dg|^p\,dx.$$

Therefore, since $\varepsilon > 0$ is arbitrary,

$$\varliminf_{\delta\to 0} \iint_{\substack{\mathbb{R}^N\times\mathbb{R}^N \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \geq C_{N,p}\int_{\mathbb{R}^N} |Dg|^p\,dx.$$

□

Remark 5 In fact, we proved that

$$\varliminf_{\delta\to 0} \iint_{\substack{U\times U \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \geq C_{N,p}\int_{U} |Dg|^p\,dx,$$

for any $p \geq 1$, for any $g \in W^{1,p}(U)$ and for any family of measurable functions $(g_\delta)_{\delta\in(0,1)}$ such that g_δ converges to g in measure. The proof also shows that

$$\varliminf_{\delta\to 0} \iint_{\substack{U\times U \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta}{|x-y|^{N+p}}\,dx\,dy \geq C_{N,p}\int_{U} |D_a g|\,dx,$$

for any $g \in BV(U)$ and for any family of measurable functions $(g_\delta)_{\delta\in(0,1)}$ such that g_δ converges to g in measure.

5 A criterion to recognize BV functions

In this section, we introduce a criterion to recognize BV functions which will be useful in the proof of Claim 2 in the case $p = 1$. We first present the following notation, which is motivated from the one of Lebesgue points.

Definition 3 Let $g \in L^1(\prod_{i=1}^N (a_i, b_i))$ $(a_i < b_i)$ and $t \in (a_1, b_1)$. Then the surface $x_1 = t$ is said to be a Lebesgue surface of g if and only if for almost every $z' \in \prod_{i=2}^N (a_i, b_i)$, (t, z') is a Lebesgue point of g, the restriction of g on the surface $x_1 = t$ is integrable with respect to $(N-1)$-Hausdorff measure, and

$$\lim_{\varepsilon \to 0_+} \fint_{t-\varepsilon}^{t+\varepsilon} \int_{\prod_{i=2}^N (a_i,b_i)} |g(s, z') - g(t, z')| \, dz' \, ds = 0.$$

For $i = 2, \dots, N$, we also define the notion of the Lesbegue surface for surfaces $x_i = t$ with $t \in (a_i, b_i)$ by the similar manner.

Using the theory of maximal functions (see e.g. [16]) and Fubini's theorem, one can prove

Lemma 21 *Let $g \in L^1(\prod_{i=1}^N (a_i, b_i))$ and $i \in \{1, \dots, N\}$. Then for almost every $t \in (a_i, b_i)$, the surface $x_i = t$ is a Lebesgue surface of g.*

Proof: Without loss of generality, one may suppose that $i = 1$. It suffices to prove, for any $\tau > 0$,

$$|\{t \in (a_1, b_1);\ \lim_{\varepsilon \to 0_+} \fint_{t-\varepsilon}^{t+\varepsilon} \int_{\prod_{i=2}^N (a_i,b_i)} |g(s, z') - g(t, z')| \, dz' \, ds > \tau\}| \le \tau. \quad (5.1)$$

We follow the idea used in the theory of maximal functions (see e.g. [16, page 8]). Define

$$Mg(t, 1) = \sup_{\substack{\varepsilon > 0; \\ (t-\varepsilon, t+\varepsilon) \subset (a_1, b_1)}} \fint_{t-\varepsilon}^{t+\varepsilon} \int_{\prod_{i=2}^N (a_i,b_i)} |g(s, z')| \, dz' \, ds,$$

for each $t \in (a_1, b_1)$. As in the theory of maximal functions, one has

$$|\{t \in (a_1, b_1); Mg(t, 1) > \tau\}| \le \frac{C}{\tau} \int_{\prod_{i=1}^N (a_i,b_i)} |g| \, dx. \quad (5.2)$$

Let f be a continuous function on $\prod_{i=1}^N [a_i, b_i]$ such that

$$\int_{\prod_{i=1N(a_i,b_i)}} |f - g|\, dx \le \frac{1}{C}\tau^2, \tag{5.3}$$

where C is the constant in (5.2). Then

$$\lim_{\varepsilon\to 0_+} \fint_{t-\varepsilon}^{t+\varepsilon} \int_{\prod_{i=2}^N (a_i,b_i)} |f(s,z') - f(t,z')|\, dz'\, ds = 0,$$

Hence it follows that

$$\{t \in (a_1, b_1);\ \lim_{\varepsilon\to 0_+} \fint_{t-\varepsilon}^{t+\varepsilon} \int_{\prod_{i=2}^N (a_i,b_i)} |g(s,z') - g(t,z')|\, dz'\, ds > \tau\}$$

$$\subset \{t \in (a_1, b_1);\ \lim_{\varepsilon\to 0_+} \fint_{t-\varepsilon}^{t+\varepsilon} \int_{\prod_{i=2}^N (a_i,b_i)} |(g-f)(s,z') - (g-f)(t,z')|\, dz'\, ds > \tau\}.$$

Thus from (5.2) and (5.3), one has

$$|\{t \in (a_1, b_1);\ \lim_{\varepsilon\to 0_+} \fint_{t-\varepsilon}^{t+\varepsilon} \int_{\prod_{i=2}^N (a_i,b_i)} |g(s,z') - g(t,z')|\, dz'\, ds > \tau\}| \le \tau.$$

This completes the proof of (5.1). □

The following definition will be used in Proposition 1 which deals with a criterion to recognize BV functions.

Definition 4 Let $g \in L^1(\prod_{i=1}^N (a_i, b_i))$. The essential variation of g in the first direction is defined as follows

$$\operatorname{ess} V(g, 1) = \sup \left\{ \sum_{i=1}^m \int_{\prod_{i=2}^N [a_i,b_i]} |g(t_{i+1}, x') - g(t_i, x')|\, dx' \right\},$$

where the supremum is taken over all finite partitions $\{a_1 < t_1 < \cdots < t_{m+1} < b_1\}$ such that the surface $x_1 = t_k$ is a Lebesgue surface of g for $1 \le k \le m+1$. For $2 \le j \le N$, we also define $\operatorname{ess} V(g, j)$ the essential variation of g in the j^{th} direction by the similar manner.

The following proposition gives a way to recognize BV functions.

Proposition 1 *Let $g \in L^1(\prod_{i=1}^N (a_i, b_i))$. Then $g \in BV(\prod_{i=1}^N (a_i, b_i))$ if and only if*

$$\operatorname{ess} V(g, j) < +\infty, \quad \forall\, 1 \le j \le N.$$

Moreover, for $g \in BV(\prod_{i=1}^N (a_i, b_i))$,

$$\operatorname{ess} V(g, j) = \|Dg \cdot e_j\|(\prod_{i=1}^N (a_i, b_i)), \quad \forall\, 1 \le j \le N.$$

Here $Dg \cdot e_j$ denotes the derivative of g with respect to the variable x_j and $\|Dg \cdot e_j\|(\prod_{i=1}^N (a_i, b_i))$ denotes its total mass.

Proof: Suppose that $\operatorname{ess} V(g, j) < +\infty$ for all $1 \le j \le N$. One claims that $g \in BV(\prod_{i=1}^N (a_i, b_i))$ and

$$\|Dg \cdot e_j\|(\prod_{i=1}^N (a_i, b_i)) \le \operatorname{ess} V(g, j), \quad \forall\, 1 \le j \le N. \tag{5.4}$$

Let (ρ_ε) be the standard sequence of smooth mollifiers in $\mathbb{R}$. Fix $\varepsilon > 0$ and set $g_\varepsilon(x) = \int_{\mathbb{R}} g(x_1 - s, x')\rho_\varepsilon(s)\, ds$. Choose any $a_1 + \varepsilon < t_1 < \cdots < t_{m+1} < b_1 - \varepsilon$. One has

$$\begin{aligned}
\sum_{k=1}^{m} \int_{\prod_{i=2}^N [a_i, b_i]} & |g_\varepsilon(t_{k+1}, x') - g_\varepsilon(t_k, x')|\, dx' \\
&= \sum_{k=1}^{m} \int_{\prod_{i=2}^N [a_i, b_i]} \left| \int_{-\varepsilon}^{\varepsilon} \rho_\varepsilon(s)[g(t_{k+1} - s, x') - g(t_k - s, x')]\, ds \right| dx' \\
&\le \sum_{k=1}^{m} \int_{\prod_{i=2}^N [a_i, b_i]} \int_{-\varepsilon}^{\varepsilon} \rho_\varepsilon(s)|g(t_{k+1} - s, x') - g(t_k - s, x')|\, ds\, dx'.
\end{aligned} \tag{5.5}$$

On the other hand, for almost every $t \in (a_1, b_1)$, the surface $x_1 = t$ is a Lebesgue surface of g (see Proposition 21). Hence

$$\sum_{k=1}^{m} \int_{\prod_{i=2}^N [a_i, b_i]} \int_{-\varepsilon}^{\varepsilon} \rho_\varepsilon(s)|g(t_{k+1} - s, x') - g(t_k - s, x')|\, ds\, dx' \le \operatorname{ess} V(g, 1). \tag{5.6}$$

Combining (5.5) and (5.6) yields

$$\int_{a_1+\varepsilon}^{b_1-\varepsilon} \int_{\prod_{i=2}^N [a_i,b_i]} |Dg_\varepsilon \cdot e_1| \, dx' \, ds \le \operatorname{ess} V(g,1).$$

This implies

$$\|Dg \cdot e_1\|(\prod_{i=1}^N (a_i,b_i)) \le \operatorname{ess} V(g,1).$$

Similarly,

$$\|Dg \cdot e_j\|(\prod_{i=1}^N (a_i,b_i)) \le \operatorname{ess} V(g,j), \quad \forall\, 2 \le j \le N.$$

Therefore, $g \in BV(\prod_{i=1}^N (a_i,b_i))$ and (5.4).

We now suppose that $g \in BV(\prod_{i=1}^N (a_i,b_i))$. One claims that

$$\operatorname{ess} V(g,j) \le \|Dg \cdot e_j\|(\prod_{i=1}^N (a_i,b_i)), \quad \forall\, 1 \le j \le N. \tag{5.7}$$

In fact, consider a partition of $\{a_1 < t_1 < \cdots < t_{m+1} < b_1\}$ with each surface $x_1 = t_k$ is a Lebesgue surface of g. Then

$$\begin{aligned}\sum_{k=1}^m \int_{\prod_{i=2}^N [a_i,b_i]} & |g(t_{k+1},x') - g(t_k,x')| \, dx' \\ &= \lim_{\varepsilon \to 0} \sum_{k=1}^m \int_{\prod_{i=2}^N [a_i,b_i]} |g_\varepsilon(t_{k+1},x') - g_\varepsilon(t_k,x')| \, dx'.\end{aligned}$$

However,

$$\sum_{k=1}^m \int_{\prod_{i=2}^N [a_i,b_i]} |g_\varepsilon(t_{k+1},x') - g_\varepsilon(t_k,x')| \, dx' \le \int_{t_1-\varepsilon}^{t_{m+1}+\varepsilon} \int_{\prod_{i=2}^N [a_i,b_i]} |Dg_\varepsilon \cdot e_1| \, dx.$$

and

$$\int_{t_1-\varepsilon}^{t_{m+1}+\varepsilon} \int_{\prod_{i=2}^N [a_i,b_i]} |Dg_\varepsilon \cdot e_1| \, dx \le \|Dg \cdot e_1\|(\prod_{i=1}^N (a_i,b_i)).$$

Thus it follows that

$$\sum_{k=1}^{m} \int_{\prod_{i=2}^{N}[a_i,b_i]} |g(t_{k+1}, x') - g(t_k, x')|\, dx' \leq \|Dg \cdot e_1\|(\prod_{i=1}^{N}(a_i, b_i)),$$

which implies

$$\operatorname{ess} V(g, 1) \leq \|Dg \cdot e_1\|(\prod_{i=1}^{N}(a_i, b_i)).$$

Similarly,

$$\operatorname{ess} V(g, j) \leq \|Dg \cdot e_j\|(\prod_{i=1}^{N}(a_i, b_i)), \quad \forall\, 2 \leq j \leq N.$$

Thus (5.7) is proved.

The conclusion of Proposition 1 now follows from (5.4) and (5.7). □

6 The proof of Claim 2 in the case $p = 1$

6.1 Another definition of $C_{N,1}$

Define

$$\mathbf{b}_{N,1} := \inf \varliminf_{\delta \to 0} \iint_{\substack{Q^2 \\ |g_\delta(x) - g_\delta(y)| > \delta}} \frac{\delta}{|x-y|^{N+1}}\, dx\, dy.$$

where the infimum is taken over all family of measurable functions $(g_\delta)_{\delta \in (0,1)}$ such that g_δ converges to $H_{\frac{1}{2}}$ in measure as δ goes to 0. Here and afterwards $H_c(x) := H(x_1 - c, x')$ for any $c \in \mathbb{R}$, where H is the function defined by

$$H(x) = \begin{cases} 0 & \text{if } x_1 < 0, \\ 1 & \text{otherwise.} \end{cases}$$

We have

Proposition 2

$$\mathbf{b}_{N,1} = C_{N,1}.$$

To see this, we first prove

Lemma 22 *Let $(g_\delta)_{\delta\in(0,1)}$ be a family of measurable functions defined on Q such that g_δ converges to $g \equiv x_1$ in measure on Q as δ goes to 0. Then*

$$\varliminf_{\delta\to 0} \iint_{\substack{Q^2\\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy \geq C_{N,1}.$$

Proof: See Remark 5. □

On the other hand, one has

Lemma 23 *There exist a sequence of measurable functions (ψ_k) and a sequence of positive numbers (τ_k) converging to 0 such that ψ_k converges to $g \equiv x_1$ in measure on Q, and*

$$\varliminf_{k\to\infty} \iint_{\substack{Q^2\\ |\psi_k(x)-\psi_k(y)|>\tau_k}} \frac{\tau_k}{|x-y|^{N+1}}\,dx\,dy = \mathbf{b}_{N,1}.$$

Proof: From the definition of $\mathbf{b}_{N,1}$, there exist a sequence (δ_k) converging to 0 and a sequence of measurable functions (g_k) converging in measure to $H_{\frac{1}{2}}$ as k goes to infinity such that

$$\lim_{k\to\infty} \iint_{\substack{Q^2\\ |g_k(x)-g_k(y)|>\delta_k}} \frac{\delta_k}{|x-y|^{N+1}}\,dx\,dy = \mathbf{b}_{N,1}. \tag{6.1}$$

Since g_k converges to $H_{\frac{1}{2}}$ in measure, there exists a sequence of positive numbers $(c_k)_{k\in\mathbb{N}}$ converging to 0 such that

$$\lim_{k\to\infty} \frac{|\{x\in Q; |g_k(x) - H_{\frac{1}{2}}(x)| \geq c_k\}|}{c_k} = 0. \tag{6.2}$$

Set

$$h_{1,k}(x) = \begin{cases} c_k & \text{if } x_1 < \frac{1}{2} - c_k,\\ 1 + c_k & \text{if } x_1 > \frac{1}{2},\\ \dfrac{1}{c_k}(x_1 - \dfrac{1}{2} + c_k) + c_k & \text{otherwise,} \end{cases}$$

and

$$h_{2,k}(x) = \begin{cases} -c_k & \text{if } x_1 < \frac{1}{2},\\ 1 - c_k & \text{if } x_1 > \frac{1}{2} + c_k,\\ \dfrac{1}{c_k}(x_1 - \dfrac{1}{2}) - c_k & \text{otherwise.} \end{cases}$$

Define

$$g_{1,k} = \min\big(\max(g_k, h_{2,k}), h_{1,k}\big),$$

and

$$g_{2,k} = \min\big(\max(g_{1,k}, c_k), 1 - c_k\big).$$

Then $g_{2,k}(x) = c_k$ for all x such that $x_1 < \frac{1}{2} - c_k$, $g_{2,k}(x) = 1 - c_k$ for all x with $x_1 > \frac{1}{2} + c_k$, and

$$\lim_{k\to\infty} \iint_{\substack{Q^2 \\ |g_{2,k}(x)-g_{2,k}(y)|>\delta_k}} \frac{\delta_k}{|x-y|^{N+1}}\,dx\,dy = \mathbf{b}_{N,1}. \tag{6.3}$$

In fact, it suffices to prove (6.3). By Corollary 3,

$$\iint_{\substack{Q^2 \\ |g_{2,k}(x)-g_{2,k}(y)|>\delta_k}} \frac{\delta_k}{|x-y|^{N+1}}\,dx\,dy \le \iint_{\substack{Q^2 \\ |g_{1,k}(x)-g_{1,k}(y)|>\delta_k}} \frac{\delta_k}{|x-y|^{N+1}}\,dx\,dy. \tag{6.4}$$

Since $|Dh_{1,k}(x)| \le \frac{1}{c_k}$ and $|Dh_{2,k}(x)| \le \frac{1}{c_k}$ for all $x \in Q$, it follows from (6.2), and Lemmas 1 and 2 that

$$\varlimsup_{k\to\infty} \iint_{\substack{Q^2 \\ |g_{1,k}(x)-g_{1,k}(y)|>\delta_k}} \frac{\delta_k}{|x-y|^{N+1}}\,dx\,dy \le \varlimsup_{k\to\infty} \iint_{\substack{Q^2 \\ |g_k(x)-g_k(y)|>\delta_k}} \frac{\delta_k}{|x-y|^{N+1}}\,dx\,dy. \tag{6.5}$$

Combining (6.1), (6.4), and (6.5) yields

$$\lim_{k\to\infty} \iint_{\substack{Q^2 \\ |g_{2,k}(x)-g_{2,k}(y)|>\delta_k}} \frac{\delta_k}{|x-y|^{N+1}}\,dx\,dy \le \mathbf{b}_{N,1}.$$

From the definition of $\mathbf{b}_{N,1}$, this implies (6.3).

Define $h_k : Q \mapsto \mathbb{R}$ by

$$h_k(x) = \frac{g_{2,k}(x) - c_k}{1 - c_k},$$

and set $\varepsilon_k = \delta_k/(1-c_k)$. Then ε_k converges to 0 as k goes to infinity, and

$$\overline{\lim_{k\to\infty}} \iint_{\substack{Q^2 \\ |h_k(x)-h_k(y)|>\varepsilon_k}} \frac{\varepsilon_k}{|x-y|^{N+1}}\,dx\,dy$$

$$= \overline{\lim_{k\to\infty}} \iint_{\substack{Q^2 \\ |g_k(x)-g_k(y)|>\delta_k}} \frac{\delta_k}{|x-y|^{N+1}}\,dx\,dy,$$

which implies, from (6.3),

$$\overline{\lim_{k\to\infty}} \iint_{\substack{Q^2 \\ |h_k(x)-h_k(y)|>\varepsilon_k}} \frac{\varepsilon_k}{|x-y|^{N+1}}\,dx\,dy = \mathbf{b}_{N,1}. \tag{6.6}$$

For each $n \in \mathbb{N}$ (arbitrary), consider the sequence $f_k : Q \mapsto \mathbb{R}$ defined as follows

$$f_k(x) = \frac{1}{n} h_k(x_1 - \frac{i}{n} + \frac{1}{2} - \frac{1}{2n}, x') + \frac{i}{n} \text{ if } x_1 \in [\frac{i}{n}, \frac{i+1}{n}], \quad \forall\, 0 \le i \le n-1.$$

We claim that

$$\lim_{k\to\infty} \iint_{\substack{Q^2 \\ |f_k(x)-f_k(y)|>\varepsilon_k/n}} \frac{\varepsilon_k/n}{|x-y|^{N+1}}\,dx\,dy = \mathbf{b}_{N,1}, \tag{6.7}$$

and

$$\int_Q |f_k(x) - x_1|\,dx \le \frac{1}{n}. \tag{6.8}$$

Indeed, (6.8) is clear from the definition of f_k and the fact that $0 \le h_k(x) \le 1$ for all $x \in Q$. It suffices to prove (6.7). One has

$$\iint_{\substack{Q^2 \\ |f_k(x)-f_k(y)|>\varepsilon_k/n}} \frac{\varepsilon_k/n}{|x-y|^{N+1}}\,dx\,dy = \sum_{i=0}^{n-1} \iint_{\substack{x_1 \in [\frac{i}{n}, \frac{i+1}{n}] \\ y_1 \in [\frac{i}{n}, \frac{i+1}{n}] \\ |f_k(x)-f_k(y)|>\varepsilon_k/n}} \frac{\varepsilon_k/n}{|x-y|^{N+1}}\,dx\,dy$$

$$+ \sum_{i=0}^{n-1} \iint_{\substack{x_1 \in [\frac{i}{n}, \frac{i+1}{n}] \\ y_1 \notin [\frac{i}{n}, \frac{i+1}{n}] \\ |f_k(x)-f_k(y)|>\varepsilon_k/n}} \frac{\varepsilon_k/n}{|x-y|^{N+1}}\,dx\,dy.$$

On the other hand, since $h_k(x) = 0$ if $x_1 < \frac{1}{2} - c_k$, $h_k(x) = 1$ if $x_1 > \frac{1}{2} + c_k$, and c_k converges to 0 as k goes to infinity, it follows from the definition of f_k and (6.6) that

$$\iint\limits_{\substack{x_1 \in [\frac{i}{n}, \frac{i+1}{n}] \\ y_1 \in [\frac{i}{n}, \frac{i+1}{n}] \\ |f_k(x) - f_k(y)| > \varepsilon_k/n}} \frac{\varepsilon_k/n}{|x-y|^{N+1}}\,dx\,dy = \frac{1}{n} \iint\limits_{\substack{x_1 \in [\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}] \\ x_1 \in [\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}] \\ |h_k(x) - h_k(y)| > \varepsilon_k}} \frac{\varepsilon_k}{|x-y|^{N+1}}\,dx\,dy,$$

$$\lim_{k \to \infty} \iint\limits_{\substack{x_1 \in [\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}] \\ x_1 \in [\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}] \\ |h_k(x) - h_k(y)| > \varepsilon_k}} \frac{\varepsilon_k}{|x-y|^{N+1}}\,dx\,dy = \mathbf{b}_{N,1},$$

and

$$\lim_{k \to \infty} \iint\limits_{\substack{x_1 \in [\frac{i}{n}, \frac{i+1}{n}] \\ y_1 \notin [\frac{i}{n}, \frac{i+1}{n}] \\ |f_k(x) - f_k(y)| > \varepsilon_k/n}} \frac{\varepsilon_k/n}{|x-y|^{N+1}}\,dx\,dy = 0.$$

Hence

$$\iint\limits_{\substack{Q^2 \\ |f_k(x) - f_k(y)| > \varepsilon_k/n}} \frac{\varepsilon_k/n}{|x-y|^{N+1}}\,dx\,dy = \mathbf{b}_{N,1},$$

which is (6.7).

From (6.7) and (6.8), there exist a sequence $(\psi_k)_{k \in \mathbb{N}}$ converging to $g \equiv x_1$ in measure and a sequence $(\tau_k)_{k \in \mathbb{N}}$ converging to 0 such that

$$\lim_{k \to \infty} \iint\limits_{\substack{Q^2 \\ |\psi_k(x) - \psi_k(y)| > \tau_k}} \frac{\tau_k}{|x-y|^{N+1}}\,dx\,dy = \mathbf{b}_{N,1}.$$

□

As a consequence of Lemmas 22 and 23, one has

Corollary 6

$$C_{N,1} \leq \mathbf{b}_{N,1}.$$

Similarly,

Lemma 24 *There exist a sequence of measurable functions (ψ_k) and a sequence of positive numbers (τ_k) converging to 0 such that ψ_k converges to $H_{\frac{1}{2}}$ in measure on Q and*

$$\varliminf_{k\to\infty} \iint_{\substack{Q^2 \\ |\psi_k(x)-\psi_k(y)|>\tau_k}} \frac{\tau_k}{|x-y|^{N+1}}\,dx\,dy = C_{N,1}.$$

Proof: Let (g_n) be a sequence of functions defined on Q as follows

$$g_n(x) = \begin{cases} 0 & \text{if } x_1 \le \frac{1}{2} - \frac{1}{n}, \\ n[x_1 - \frac{1}{2} + \frac{1}{n}] & \text{if } \frac{1}{2} - \frac{1}{n} < x_1 \le \frac{1}{2}, \\ 1 & \text{otherwise,} \end{cases} \qquad \forall\, n \in \mathbb{N}.$$

For each n, there exists a family of measurable functions $(g_{n,\delta})_{\delta\in(0,1)}$ defined on Q such that $g_{n,\delta}$ converges to g_n in measure and

$$\lim_{\delta\to 0} \iint_{\substack{Q^2 \\ |g_{n,\delta}(x)-g_{n,\delta}(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy = C_{N,1}.$$

(see Remark 4).

Therefore, there exist a sequence of measurable functions (ψ_k) and a sequence of positive numbers (δ_k) converging to 0 such that ψ_k converges to $H_{\frac{1}{2}}$ in measure on Q and

$$\varliminf_{k\to\infty} \iint_{\substack{Q^2 \\ |\psi_k(x)-\psi_k(y)|>\tau_k}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy = C_{N,1}.$$

□

As a consequence of Lemma 24 and the definition of $\mathbf{b}_{N,1}$, one gets

Corollary 7

$$\mathbf{b}_{N,1} \le C_{N,1}.$$

Remark 6 This corollary can also be deduced from Lemma 27.

We are ready to give

Proof of Proposition 2: Proposition is a direct consequence of Corollaries 6 and 7. □

6.2 Some useful lemmas

In this section, we prove some useful lemmas which are needed in the proof of Claim 2 in the case $p = 1$. Our main goal is to prove Lemma 28. From the definition of $\mathbf{b}_{N,1}$, one has

Lemma 25 *For any $\varepsilon > 0$, there exist three positives numbers δ_1, δ_2, and δ_3 such that if $g \in L^1(Q)$,*

$$|\{x \in Q;\, |g(x) - H_{\frac{1}{2}}(x)| > \delta_1\}| < \delta_2,$$

and $\delta < \delta_3$, then

$$\iint_{\substack{Q^2 \\ |g_\delta(x) - g_\delta(y)| > \delta}} \frac{\delta}{|x-y|^{N+1}}\, dx\, dy \geq \mathbf{b}_{N,1} - \varepsilon.$$

Proof: This proof is similar to the one of Lemma 15. The detail is left to the reader. □

As a consequence of Lemma 25, one has

Lemma 26 *For any $\varepsilon > 0$, there exist three positives numbers δ_1, δ_2, and δ_3 such that if $g \in L^1(\prod_{i=1}^N [a_i, b_i])$ $(a_i < b_i)$,*

$$|\{x \in \prod_{i=1}^N [a_i, b_i];\, |g(x) - (cH_{a_1 + \frac{b_1 - a_1}{2}}(x) + d)| > |c|\delta_1\}| < \delta_2 \prod_{i=1}^N (b_i - a_i),$$

and $\delta < |c|\delta_3$, for some c and d in $\mathbb{R}$, then

$$\iint_{\substack{\prod_{i=1}^N [a_i, b_i] \times \prod_{i=1}^N [a_i, b_i] \\ |g(x) - g(y)| > \delta}} \frac{\delta}{|x-y|^{N+1}}\, dx\, dy \geq |c|(\mathbf{b}_{N,1} - \varepsilon).$$

Proof: Let δ_1, δ_2, and δ_3 be three positive constants corresponding to ε in Lemma 25. Without loss of generality, one may assume that $c > 0$ and $d = 0$. Define $h : \prod_{i=1}^N [a_i, b_i] \to \mathbb{R}$ as follows

$$h(x) = \frac{g(x)}{c}, \quad \forall\, x \in \prod_{i=1}^N [a_i, b_i].$$

Then

$$|\{x \in \prod_{i=1}^N [a_i, b_i];\, |h(x) - H_{a_1 + \frac{b_1 - a_1}{2}}(x)| > \delta_1\}| < \delta_2 \prod_{i=1}^N (b_i - a_i).$$

Define $h_1 : Q \to \mathbb{R}$ by

$$h_1(t) = h((b_1 - a_1)t_1 + a_1, \ldots, (b_N - a_N)t_N + a_N), \quad \forall\, t \in Q.$$

Then

$$|\{t \in Q;\, |h_1(t) - H_{\frac{1}{2}}(t)| > \delta_1\}| < \delta_2.$$

Applying Lemma 25, one has

$$\iint_{\substack{Q^2 \\ |h_1(t)-h_1(s)|>\delta}} \frac{\delta}{|t-s|^{N+1}}\, dt\, ds \geq (\mathbf{b}_{N,1} - \varepsilon), \quad \forall\, \delta < \delta_3,$$

which implies

$$\iint_{\substack{\prod_{i=1}^N [a_i,b_i] \times \prod_{i=1}^N [a_i,b_i] \\ |g(x)-g(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\, dx\, dy \geq c(\mathbf{b}_{N,1} - \varepsilon),$$

for all $\delta < c\delta_3$. $\square$

The following lemma plays an important role in the proof of Lemma 28.

Lemma 27 *Let $(g_\delta)_{\delta \in (0,1)} \subset L^1(\prod_{i=1}^N [a_i, b_i])$. Assume that $g_\delta(x) \leq 0$ for x with $x_1 \leq a_1 + \delta$ and $g_\delta(x) \geq c$ for x such that $x_1 \geq b_1 - \delta$. Then*

$$\varliminf_{\delta \to 0} \iint_{\substack{\prod_{i=1}^N [a_i,b_i] \times \prod_{i=1}^N [a_i,b_i] \\ |g(x)-g(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\, dx\, dy \geq c\mathbf{b}_{N,1}.$$

Proof: Without loss of generality, by Corollary 3, one may assume that $g_\delta(x) = 0$ for all x such that $x_1 < a_1 + \delta$ and $g(x) = c$ for all x such that $x_1 \geq b_1 - \delta$. For $\varepsilon > 0$, let δ_2 be a positive constant corresponding to ε in Lemma 26. Set $Q_\delta := [a_1 - \frac{b_1 - a_1}{2\delta_2}, b_1 + \frac{b_1 - a_1}{2\delta_2}] \times \prod_{i=2}^N [a_i, b_i]$. Define $h_\delta : Q_\delta \mapsto \mathbb{R}$ by

$$h_\delta(x) = \begin{cases} 0 & \text{if } x_1 \in (a_1 - \frac{b_1-a_1}{2\delta_2}, a_1), \\ g_\delta(x) & \text{if } x_1 \in (a_1, b_1), \\ c & \text{if } x_1 \in (b_1, b_1 + \frac{b_1-a_1}{2\delta_2}). \end{cases}$$

Then, applying Lemma 25 for the function h_δ, one has

$$\iint_{\substack{Q_\delta^2 \\ |h_\delta(x)-h_\delta(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\, dx\, dy \geq c(\mathbf{b}_{N,1} - \varepsilon),$$

which implies, since $g_\delta(x) = 0$ if $x_1 < a_1 + \delta$ and $g_\delta(x) = c$ if $x_1 > b_1 - \delta$,

$$\iint\limits_{\substack{\prod_{i=1}^N [a_i,b_i] \times \prod_{i=1}^N [a_i,b_i] \\ |g_\delta(x) - g_\delta(y)| > \delta}} \frac{\delta}{|x-y|^{N+1}}\, dx\, dy \geq c(\mathbf{b}_{N,1} - \varepsilon) - \frac{C\delta \prod_{i=1}^N (b_i - a_i)}{\delta_2} \ln(1 + \frac{1}{\delta}),$$

when δ is small. Hence

$$\varliminf_{\delta \to 0} \iint\limits_{\substack{\prod_{i=1}^N [a_i,b_i] \times \prod_{i=1}^N [a_i,b_i] \\ |g_\delta(x) - g_\delta(y)| > \delta}} \frac{\delta}{|x-y|^{N+1}}\, dx\, dy \geq c(\mathbf{b}_{N,1} - \varepsilon).$$

Therefore,

$$\varliminf_{\delta \to 0} \iint\limits_{\substack{\prod_{i=1}^N [a_i,b_i] \times \prod_{i=1}^N [a_i,b_i] \\ |g_\delta(x) - g_\delta(y)| > \delta}} \frac{\delta}{|x-y|^{N+1}}\, dx\, dy \geq c\mathbf{b}_{N,1},$$

since $\varepsilon > 0$ is arbitrary. □

The following lemma plays a crucial role in the proof of Claim 2 in the case $p = 1$.

Lemma 28 *Let $g \in L^1(\prod_{i=1}^N (a_i, b_i))$ and $(g_\delta)_{\delta \in (0,1)} \subset L^1(\prod_{i=1}^N [a_i, b_i])$ ($a_i < b_i$) such that g_δ converges to g in measure on $\prod_{i=1}^N [a_i, b_i]$. Then for any t_1 and t_2 in (a_1, b_1) such that the surface $x_1 = t_i$ ($i = 1, 2$) is a Lebesgue surface of g, one has*

$$\varliminf_{\delta \to 0} \iint\limits_{\substack{[(t_1,t_2) \times \Pi_{i=2}^N (a_i,b_i)]^2 \\ |g_\delta(x) - g_\delta(y)| > \delta}} \frac{\delta}{|x-y|^{N+1}}\, dx\, dy \geq \mathbf{b}_{N,1} \int\limits_{\prod_{i=2}^N (a_i,b_i)} |g(t_2, x') - g(t_1, x')|\, dx'.$$

Proof: Fix $\tau > 0$ (arbitrary). Let A be the set of all elements $z' \in \Pi_{i=2}^N [a_i, b_i]$ such that (t_1, z') is a Lebesgue point of $g_{|x_1 = t_1}$, (t_2, z') is a Lebesgue point of $g_{|x_1 = t_2}$, and, (t_1, z') and (t_2, z') are Lebesgue points of g. For each $z' \in A$, let $Q'(z') \subset \mathbb{R}^{N-1}$ be an open cube center at z' such that

$$\begin{cases} (i) \quad |\{y' \in Q'(z');\ |g(t_1, y') - g(t_1, z')| \geq \tau/2\}| \leq \tau |Q'(z')|, \\ (ii) \quad \fint_{Q'(z')} |g(t_1, y') - g(t_1, z')|\, dy' \leq \tau, \end{cases} \tag{6.9}$$

$$\begin{cases} (i) & |\{y' \in Q'(z');\ |g(t_2,y') - g(t_2,z')| \geq \tau/2\}| \leq \tau |Q'(z')|, \\ (ii) & \displaystyle ⨍_{Q'(z')} |g(t_2,y') - g(t_2,z')|\, dy' \leq \tau, \end{cases} \tag{6.10}$$

and

$$\begin{cases} |\{(x_1,y') \in (t_1,t_1+2l) \times Q'(z');\ |g(x_1,y') - g(t_1,z')| \geq \tau/2\}| \\ \qquad \leq \tau l |Q'(z')|, \\ |\{(x_1,y') \in (t_2-2l,t_2) \times Q'(z');\ |g(x_1,y') - g(t_2,z')| \geq \tau/2\}| \\ \qquad \leq \tau l |Q'(z')|, \end{cases} \tag{6.11}$$

where $l = \frac{1}{2}|Q'(z')|^{\frac{1}{N-1}}$.

Since g_δ converges to g in measure, it follows from (6.11) that, when δ is small,

$$\begin{cases} (i) & |\{(x_1,y') \in (t_1,t_1+2l) \times Q'(z');\ |g_\delta(x_1,y') - g(t_1,z')| \geq \tau\}| \\ & \qquad \leq 2\tau l |Q'(z')|, \\ (ii) & |\{(x_1,y') \in (t_2-2l,t_2) \times Q'(z');\ |g_\delta(x_1,y') - g(t_2,z')| \geq \tau\}| \\ & \qquad \leq 2\tau l |Q'(z')|. \end{cases} \tag{6.12}$$

One claims that

$$\varliminf_{\delta\to 0} \iint\limits_{\substack{[(t_1,t_2)\times Q'(z')]^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\, dx\, dy \geq \mathbf{b}_{N,1}[1 - C\tau]|g(t_2,z') - g(t_1,z')||Q'(z')| - C\tau|Q'(z')|. \tag{6.13}$$

In fact, without loss of generality, one may assume that $g(t_1,z') < g(t_2,z') + 3\tau$ and $g(t_1,z') \leq g_\delta(x) \leq g(t_2,z')$ (by Corollary 3). Define $f_1 : (t_1,t_2) \times Q'(z') \mapsto \mathbb{R}$ by

$$f_1(y) = \begin{cases} g(t_1,z') + \tau & \text{if } y_1 \leq t_1 + l, \\ g(t_2,z') & \text{if } y_1 \geq t_1 + 2l, \\ \frac{g(t_2,z') - g(t_1,z') - \tau}{l}(y_1 - l) + g(t_1,z') + \tau & \text{otherwise.} \end{cases}$$

and set

$$h_{1,\delta} = \min(g_\delta, f_1).$$

Then, from (6.12-i), after applying Lemma 1, one gets

$$\iint\limits_{\substack{[(t_1,t_2)\times Q'(z')]^2\\ |h_{1,\delta}(x)-h_{1,\delta}(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy \le \iint\limits_{\substack{[(t_1,t_2)\times Q'(z')]^2\\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy + C\tau[g(t_2,z')-g(t_1,z')]|Q'(z')|, \tag{6.14}$$

since $f_{1,\delta}$ is a Lipschitz function with a Lipschitz constant $\frac{g(t_2,z')-g(t_1,z')}{l}$.

Define

$$g_{1,\delta} = \max(g(t_1,z')+\tau, h_{1,\delta}).$$

Applying Corollary 2, one has

$$\iint\limits_{\substack{[(t_1,t_2)\times Q'(z')]^2\\ |g_{1,\delta}(x)-g_{1,\delta}(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy \le \iint\limits_{\substack{[(t_1,t_2)\times Q'(z')]^2\\ |h_{1,\delta}(x)-h_{1,\delta}(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy. \tag{6.15}$$

Combining (6.14) and (6.15) yields

$$\iint\limits_{\substack{[(t_1,t_2)\times Q'(z')]^2\\ |g_{1,\delta}(x)-g_{1,\delta}(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy \le \iint\limits_{\substack{[(t_1,t_2)\times Q'(z')]^2\\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy + C\tau[g(t_2,z')-g(t_1,z')]|Q'(z')|. \tag{6.16}$$

Similarly, define $g_2 : (t_1,t_2)\times Q'(z') \mapsto \mathbb{R}$ by

$$f_2(y) = \begin{cases} g(t_1,z') & \text{if } y_1 \le t_2 - 2l, \\ g(t_2,z') - \tau & \text{if } y_1 \ge t_2 - l, \\ \frac{g(t_2,z')-g(t_1,z')-\tau}{l}(y_1 - \tau_2 + 2l) + g(t_1,z') & \text{otherwise}, \end{cases}$$

and set $h_{2,\delta} = \max(g_{1,\delta}, f_2)$ and $g_{2,\delta} = \min(h_{2,\delta}, g(t_2,z')-\tau)$. Then

$$\iint\limits_{\substack{[(t_1,t_2)\times Q'(z')]^2\\ |g_{2,\delta}(x)-g_{2,\delta}(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy \le \iint\limits_{\substack{[(t_1,t_2)\times Q'(z')]^2\\ |g_{1,\delta}(x)-g_{1,\delta}(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy + C\tau[g(t_2,z')-g(t_1,z')]|Q'(z')|. \tag{6.17}$$

Combining (6.16) and (6.17) yields

$$\iint\limits_{\substack{[(t_1,t_2)\times Q'(z')]^2\\ |g_{2,\delta}(x)-g_{2,\delta}(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy \le \iint\limits_{\substack{[(t_1,t_2)\times Q'(z')]^2\\ |g_{\delta}(x)-g_{\delta}(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy + C\tau[g(t_2,z')-g(t_1,z')]|Q'(z')|. \tag{6.18}$$

On the other hand, by Lemma 27,

$$\varliminf_{\delta\to 0} \iint\limits_{\substack{[(t_1,t_2)\times Q'(z')]^2\\ |g_{2,\delta}(x)-g_{2,\delta}(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy \ge \mathbf{b}_{N,1}[g(t_2,z')-g(t_1,z')-2\tau]|Q'(z')|. \tag{6.19}$$

Hence combining (6.18) and (6.19) yields (6.13).

On the other hand, from (6.9-ii), (6.10-ii), and (6.13), one has

$$\varliminf_{\delta\to 0} \iint\limits_{\substack{[(t_1,t_2)\times Q'(x')]^2\\ |g_{\delta}(x)-g_{\delta}(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy$$

$$\ge \mathbf{b}_{N,1}[1-C\tau]\int_{Q'(x')} |g(t_2,x')-g(t_1,x')|\,dx' - C\tau|Q'(x')|.$$

Therefore, applying Besicovitch's covering theorem, one has

$$\varliminf_{\delta\to 0} \iint\limits_{\substack{[(t_1,t_2)\times \Pi_{i=2}^N(a_i,b_i)]^2\\ |g_{\delta}(x)-g_{\delta}(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy$$

$$\ge \mathbf{b}_{N,1}[1-C\tau]\int\limits_{\Pi_{i=2}^N[a_i,b_i]} |g(t_2,x')-g(t_1,x')|\,dx' - C\tau\Pi_{i=2}^N(b_i-a_i).$$

Since $\tau > 0$ is arbitrary, one has

$$\varliminf_{\delta\to 0} \iint\limits_{\substack{[(t_1,t_2)\times \Pi_{i=2}^N(a_i,b_i)]^2\\ |g_{\delta}(x)-g_{\delta}(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy$$

$$\ge \mathbf{b}_{N,1}\int\limits_{\Pi_{i=2}^N[a_i,b_i]} |g(t_2,x')-g(t_1,x')|\,dx'.$$

□

Remark 7 It is surprising that the inequality in Lemma 28 involves the constant $C_{N,1}$ (since $\mathbf{b}_{N,1} = C_{N,1}$), although $C_{N,1}$ is defined by a process depending on a smooth function.

6.3 Proof of Claim 2 in the case $p = 1$ and $N = 1$

For each $\varepsilon > 0$, consider a set of intervals $\{(a_i, b_i)\}_{i=1}^k$ such that $b_i < a_{i+1}$ ($a_0 = b_0 = -\infty$ and $a_{k+1} = b_{k+1} = +\infty$), a_i and b_i are Lebesgue points of g for $1 \le i \le k$,

$$\sum_{i=1}^{k} |g(b_i) - g(a_i)| \ge (1 - \varepsilon)\|D_s g\|, \tag{6.20}$$

and

$$\int_{\mathbb{R}\setminus\bigcup_{i=1}^k (a_i,b_i)} |D_a g|\, dx \ge (1-\varepsilon) \int_{\mathbb{R}} |D_a g|\, dx. \tag{6.21}$$

We recall here that $D_a g$ and $D_s g$ are respectively the absolutely continuous part and the singular part of the derivative Dg of g.

One has (see Remark 5)

$$\varliminf_{\delta\to 0} \iint_{\substack{(b_i,a_{i+1})^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta}{|x-y|^2}\, dx\, dy \ge C_{1,1} \int_{(b_i,a_{i+1})} |D_a g|\, dx. \tag{6.22}$$

On the other hand, by Lemma 6.23,

$$\varliminf_{\delta\to 0} \iint_{\substack{(a_i,b_i)^2 \\ |g(x)-g(y)|>\delta}} \frac{\delta}{|x-y|^2}\, dx\, dy \ge \mathbf{b}_{1,1} |g(b_i) - g(a_i)|. \tag{6.23}$$

Hence combining (6.20), (6.21), (6.22), and (6.23), one obtains

$$\varliminf_{\delta\to 0} \iint_{\substack{\mathbb{R}^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta}{|x-y|^2}\, dx\, dy \ge C_{1,1}\|Dg\|,$$

since $\varepsilon > 0$ is arbitrary and $\mathbf{b}_{1,1} = C_{1,1}$ (see Proposition 2). □

6.4 Proof of Claim 2 in the case $p = 1$ and $N \geq 2$

We recall that for each $g \in BV(\mathbb{R}^N)$, $\|Dg\|$ is a Radon measure on $\mathbb{R}^N$ and there exist a $\|Dg\|$-measurable function $\sigma : \mathbb{R}^N \mapsto \mathbb{R}^N$ such that

$$\begin{cases} |\sigma(x)| = 1 \, \|Dg\| \, a.e., \\ \displaystyle\int_{\mathbb{R}^N} f \mathrm{div}\, \psi = -\int_{\mathbb{R}^N} \psi \cdot \sigma \, d\|Dg\|, \quad \forall \psi \in C^1_{\mathrm{c}}(\mathbb{R}^N; \mathbb{R}^N) \end{cases}$$

(see e.g. [8, Theorem 1 on page 167]). Set

$$A = \mathrm{supp}\, D_s g.$$

Then $|A| = 0$ and for $\|Dg\|$ a.e. $x \in A$, one has, by [8, Theorem 1 on page 39],

$$\lim_{r \to 0} \frac{\|Dg \cdot \sigma(x)\|(Q(x, \sigma(x), r))}{\|Dg\|(Q(x, \sigma(x), r))} = 1. \tag{6.24}$$

Hereafter for any $(x, \sigma, r) \in \mathbb{R}^N \times \mathbb{S}^{N-1} \times (0, +\infty)$, $Q(x, \sigma, r)$ denotes the closed cube center at x with the length of its sides equal to $2r$ such that one of its faces is orthogonal to σ. Let B be the set of all $x \in A$ such that (6.24) is satisfied. Then

$$\|Dg\|(B) = \|Dg\|(A).$$

Fix $\varepsilon > 0$ (arbitrary). Let U be the open subset of $\mathbb{R}^N$ such that $A \subset U$ and

$$\|D_a g\|(U) \leq \varepsilon \|D_a g\|(\mathbb{R}^N). \tag{6.25}$$

Thus by Besicovitch's covering theorem, there exists a family of cubes $(Q(x_i, \sigma(x_i), r_i))_{i \in \mathbb{N}}$ such that $x_i \in B$,

$$Q(x_i, \sigma(x_i), r_i) \cap Q(x_j, \sigma(x_j), r_j) = \emptyset, \quad \text{for } i \neq j, \tag{6.26}$$

$$\frac{\|Dg \cdot \sigma(x_i)\|(Q(x_i, \sigma(x_i), r_i))}{\|Dg\|(Q(x_i, \sigma(x_i), r_i))} \geq 1 - \varepsilon, \tag{6.27}$$

$$\bigcup_{i \in \mathbb{N}} Q(x_i, \sigma(x_i), r_i) \subset U, \tag{6.28}$$

and

$$\|Dg\|(B) \leq \|Dg\|(\bigcup_{i \in \mathbb{N}} Q(x_i, \sigma(x_i), r_i)).$$

Thus

$$\|Dg\|(A) \leq \frac{1}{1 - \varepsilon} \sum_{i \in \mathbb{N}} \|Dg \cdot \sigma(x_i)\|(Q(x_i, \sigma(x_i), r_i)).$$

Take k sufficiently big such that

$$\sum_{i\in\mathbb{N}} \|Dg\cdot\sigma(x_i)\|Q(x_i,\sigma(x_i),r_i) \le (1+\varepsilon)\sum_{i=1}^{k} \|Dg\cdot\sigma(x_i)\|(Q(x_i,\sigma(x_i),r_i)).$$

Then

$$\|Dg\|(A) \le \frac{1+\varepsilon}{1-\varepsilon}\sum_{i=1}^{k} \|Dg\cdot\sigma(x_i)\|(Q(x_i,\sigma(x_i),r_i)). \tag{6.29}$$

From (6.26) and (6.28), there exists $\tau > 0$ sufficiently small such that

$$Q(x_i,\sigma(x_i),(1+\tau)r_i) \cap Q(x_j,\sigma(x_j),(1+\tau)r_j) = \emptyset, \quad \text{for } i \ne j,$$

and

$$\bigcup_{i=1}^{k} Q(x_i,\sigma(x_i),(1+\tau)r_i) \subset U.$$

Thus it follows from (6.25) that

$$\|D_a g\|(\bigcup_{i=1}^{k} Q(x_i,\sigma(x_i),(1+\tau)r_i)) \le \varepsilon\|D_a g\|(\mathbb{R}^N). \tag{6.30}$$

Applying Lemma 28 and Proposition 1, one has

$$\mathbf{b}_{N,1}\|Dg\cdot\sigma(x_i)\|(Q(x_i,\sigma(x_i),r_i)) \le \varliminf_{\delta\to 0} \iint\limits_{\substack{[Q(x_i,\sigma(x_i),(1+\tau)r_i)]^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy.$$

Hence it follows from (6.29) that

$$\mathbf{b}_{N,1}\|Dg\|(A) \le \frac{1+\varepsilon}{1-\varepsilon}\varliminf_{\delta\to 0} \iint\limits_{\substack{[\bigcup_{i=1}^{k} Q(x_i,\sigma(x_i),(1+\tau)r_i)]^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy. \tag{6.31}$$

On the other hand, one gets (see Remark 5)

$$C_{N,1}\|D_a g\|(\mathbb{R}^N \setminus [\bigcup_{i=1}^{k} Q(x_i,\sigma(x_i),(1+\tau)r_i)])$$
$$\le \varliminf_{\delta\to 0} \iint\limits_{\substack{\left[\mathbb{R}^N\setminus[\bigcup_{i=1}^{k} Q(x_i,\sigma(x_i),(1+\tau)r_i)]\right]^2 \\ |g_\delta(x)-g_\delta(y)|>\delta}} \frac{\delta}{|x-y|^{N+1}}\,dx\,dy. \tag{6.32}$$

Combining (6.30), (6.31), and (6.32) yields

$$\varliminf_{\delta \to 0} \iint\limits_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ |g_\delta(x) - g_\delta(y)| > \delta}} \frac{\delta}{|x-y|^{N+1}} \, dx \, dy \geq \frac{1-\varepsilon}{1+\varepsilon} \left[C_{N,1} \|D_a g\|(\mathbb{R}^N) + \mathbf{b}_{N,1} \|D_s g\|(\mathbb{R}^N) \right].$$

Therefore, since $\varepsilon > 0$ is arbitrary and $\mathbf{b}_{N,1} = C_{N,1}$ (see Proposition 2),

$$\varliminf_{\delta \to 0} I_\delta(g_\delta) \geq I(g).$$

□

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