Some inequalities related to Sobolev norms

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Abstract

In this paper, we study some properties of Sobolev spaces in the context of the new characterizations given in [11, 5, 13]. We establish some results in the spirit of the Poincaré inequality, the Sobolev inequality, and the Rellich-Kondrachov compactness theorem, where $\int_{\mathbb{R}^N} |\nabla g|^p \, dx$ is replaced by some quantity of the type

$$I_\delta(g) = \underset{|g(x)-g(y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy.$$

1 Introduction

We first recall some new characterizations of Sobolev spaces. The first one is as follows

Proposition 1 *[11, Theorem 2] Let $1 < p < +\infty$. Then*

a. There exists a constant $C_{N,p}$ depending only on N and p such that

$$I_\delta(g) \le C_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx, \quad \forall\, \delta > 0, \forall\, g \in W^{1,p}(\mathbb{R}^N).$$

b. If $g \in L^p(\mathbb{R}^N)$ satisfies

$$\limsup_{\delta \to 0_+} I_\delta(g) < +\infty,$$

then $g \in W^{1,p}(\mathbb{R}^N)$.

c. Moreover, for any $g \in W^{1,p}(\mathbb{R}^N)$,

$$\lim_{\delta \to 0_+} I_\delta(g) = \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx,$$

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where $K_{N,p}$ *is defined by*

$$K_{N,p} = \int_{\mathbb{S}^{N-1}} |e \cdot \sigma|^p \, d\sigma, \tag{1.1}$$

for any $e \in \mathbb{S}^{N-1}$.

Remark 1 The conclusion in Assertion (b) of Proposition 1 also holds under the weaker hypothesis as follows

$$\liminf_{\delta \to 0_+} I_\delta(g) < +\infty.$$

This assertion was proved by J. Bourgain and H-M. Nguyen in [5]. Our proof is totally different and more delicate than the one of Assertion (b) in Proposition 1 given in [11].

The second one is a generalized version of Proposition 1.

Proposition 2 *[13, Theorem 1] Let* $1 < p < +\infty$ *and* $(F_n)_{n\in\mathbb{N}}$ *be a sequence of functions from* $[0, +\infty)$ *into* $[0, +\infty)$ *such that*

i. $F_n(t)$ *is a non-decreasing function with respect to* t *on* $[0, +\infty)$*, for all* $n \in \mathbb{N}$.

ii. $\displaystyle\int_0^1 F_n(t)t^{-(p+1)} \, dt = 1$*, for all* $n \in \mathbb{N}$.

iii. $F_n(t)$ *converges uniformly to* 0 *on every compact subset of* $(0, +\infty)$ *as* n *goes to infinity.*

Then

a. If $g \in W^{1,p}(\mathbb{R}^N)$*, then for every* $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x-y|^{N+p}} \, dx \, dy \le C_{N,p} \int_0^\infty F_n(t)t^{-(p+1)} \, dt \int_{\mathbb{R}^N} |\nabla g|^p \, dx,$$

where $C_{N,p}$ *is a positive constant depending only on* N *and* p.

b. If $g \in L^p(\mathbb{R}^N)$ *satisfies*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x-y|^{N+p}} \, dx \, dy < +\infty,$$

then $g \in W^{1,p}(\mathbb{R}^N)$ *and*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x-y|^{N+p}} \, dx \, dy \ge K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx.$$

c. Moreover, if

$$\limsup_{n\to\infty} \int_0^\infty F_n(t) t^{-(p+1)}\, dt < +\infty,$$

then, for any $g \in W^{1,p}(\mathbb{R}^N)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x-y|^{N+p}}\, dx\, dy = K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p\, dx.$$

Here $K_{N,p}$ *is defined by* (1.1).

Remark 2 Proposition 1 follows from Proposition 2 by choosing

$$F_n(t) = \begin{cases} 0 & \text{if } 0 \le t \le \delta_n, \\ \dfrac{p\delta_n^p}{1-\delta_n^p} & \text{otherwise.} \end{cases} \tag{1.2}$$

Remark 3 Hypotheses (i)-(iii) of the sequence (F_n) are necessary. These were discussed in [13, Remark 4].

Thus it is natural to demand whether or not variants of the Poincaré inequality, the Sobolev inequality, and the Rellich-Kondrachov theorem hold in these contexts. The answer is positive. The first result motivated by Proposition 1 and the Sobolev inequality is the following

Theorem 1 *Let* $v \in L^p(\mathbb{R}^N)$, $1 \le p < N$, *and* $\delta > 0$ *such that*

$$I_\delta(v) = \iint\limits_{|v(x)-v(y)|>\delta} \frac{1}{|x-y|^{N+p}}\, dx\, dy < +\infty.$$

Then $v \in L^q(\mathbb{R}^N)$ *with* $q = \frac{Np}{N-p}$. *Moreover, there exist three positive constants* C, C_1 *and* C_2, *depending only on* N *and* p, *such that*

$$\int\limits_{|v|>\delta} |v|^q\, dx \le C\left([I_\delta(v)]^{\frac{N}{N-p}} + \delta^q |\{C_1\delta < |v| < C_2\delta\}| \right). \tag{1.3}$$

Remark 4 Fix $v \in C_c^\infty(\mathbb{R}^N)$. Letting δ go to 0 in (1.3), one recovers the Sobolev inequality in the case $1 \le p < N$. When $1 < p < N$, Theorem 1 is strictly sharper than Sobolev's imbedding, since $I_\delta(v) \le C_{N,p} \int_{\mathbb{R}^N} |\nabla v|^p\, dx$ for all $p > 1$ and for all $\delta > 0$ (see Proposition 1).

Remark 5 When $p = N$, one obtains that $v \in BMO(\mathbb{R}^N)$, the mean bounded oscillation spaces, if $I_\delta(v) < +\infty$ for some $\delta > 0$. Moreover, there exists a positive constant C such that

$$|v|_{BMO} := \sup_Q \fint_Q \fint_Q |v(x) - v(y)|\, dx\, dy \le C\left(I_\delta^{\frac{1}{N}}(v) + \delta \right),$$

where supremun is taken over all the cubes of $\mathbb{R}^N$ (see Theorem 3).

A variant of Theorem 1 in the context of the fractional Sobolev spaces is as follows.

Proposition 3 *Let* $v \in L^p(\mathbb{R}^N)$, $p \geq 1$, *and* $s \geq 0$ *such that* $sp < N$ *and*

$$\iint\limits_{|v(x)-v(y)|>\delta} \frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}}\, dx\, dy < +\infty. \tag{1.4}$$

Then $v \in L^q(\mathbb{R}^N)$ *with* $q = \frac{Np}{N-sp}$. *Moreover, there exist two positive constants* C *and* λ, *depending only on* N *and* p, *such that*

$$\int\limits_{|v|>\delta} |v|^q\, dx \leq C\Big(\Big[\iint\limits_{|v(x)-v(y)|>\delta} \frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}}\, dx\, dy\Big]^{\frac{N}{N-sp}} + \delta^q |\{\delta < |v| < \lambda\delta\}|\Big). \tag{1.5}$$

Remark 6 Letting δ go to 0 in (1.5), one obtains the Sobolev inequality for the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ with $0 < s < 1$ and $p > 1$.

The second result inspired by Proposition 1 and the Rellich-Kondrachov theorem (see also [2, Theorem 4]) is the following

Theorem 2 *Let* $f_n : \mathbb{R}^N \to \mathbb{R}$ *be a bounded sequence of functions in* $L^p(\mathbb{R}^N)$ *and* δ_n *be a sequence of positive numbers converging to* 0 *such that*

$$\liminf_{n\to\infty} I_{\delta_n}(f_n) = \liminf_{n\to\infty} \underset{|f_n(x)-f_n(y)|>\delta_n}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta_n^p}{|x-y|^{N+p}}\, dx\, dy < +\infty.$$

Then there exist a subsequence (f_{n_k}) *of* (f_n) *and* $f \in L^p(\mathbb{R}^N)$ *such that* f_{n_k} *converges to* f *in* $L^p_{\mathrm{loc}}(\mathbb{R}^N)$. *Moreover,* $f \in W^{1,p}(\mathbb{R}^N)$ *for* $p > 1$ *resp.* $f \in BV(\mathbb{R}^N)$ *for* $p = 1$ *and there exists a positive constant* C *such that*

$$\int_{\mathbb{R}^N} |\nabla f|^p\, dx \leq C \liminf_{n\to\infty} I_{\delta_n}(f_n). \tag{1.6}$$

One of the main ingredient in the proof of Theorems 1 and 2 is the following

Theorem 3 *Let* f *be a measurable function defined on a bounded cube* $Q \subset \mathbb{R}^N$ *into* $\mathbb{R}$. *Then for each* $p \geq 1$, *there exists* $C_{p,N} > 0$ *such that*

$$\iint_{Q^2} |f(x)-f(y)|^p\, dx\, dy \leq C_{p,N}\Big(|Q|^{\frac{N+p}{N}} \underset{|f(x)-f(y)|>\delta}{\iint_{Q^2}} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy + \delta^p |Q|^2\Big),\ \forall\, \delta > 0. \tag{1.7}$$

Remark 7 A variant of estimate (1.7) which was shown by J. Bourgain, H. Brezis, and P. Mironesu when δ is small, $p = 1$, and $f \in C(I, \mathbb{R})$ in the one dimensional case (see [3]) is as follows

$$\int_I \int_I |f(x) - f(y)| \, dx \, dy \leq C\Big(|I|^2 \underset{|e^{if(x)} - e^{if(y)}| > \delta}{\int_I \int_I} \frac{1}{|x-y|^2} \, dx \, dy + \delta |I|^2\Big),$$

where C is a positive constant. The continuity of f is necessary in this case. In the above inequality, the LHS can be bounded by the RHS for all $\delta < \sqrt{3}$ up to a constant depending only on δ, and the constant $\sqrt{3}$ is optimal. These are established in [7].

Remark 8 Theorem 3 is the heart of the matter in this paper. Estimate (1.7) can be seen as a variant of the Poincaré inequality. It implies directly the assertion in Remark 5.

In order to prove Theorem 1, one needs also some result (Proposition 4) in the theory of sharp functions (see [16]) which will be recalled in Section 3.

For what is motivated by Proposition 2, one has

Theorem 4 *Let* $1 \leq p < N$, $F : [0, +\infty) \to [0, +\infty)$ *be a non-decreasing function such that*

$$F(1) + \int_0^1 F(t) t^{-(p+1)} \, dt = 1,$$

and $v \in L^p(\mathbb{R}^N)$*. Assume that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(|v(x) - v(y)|)}{|x-y|^{N+p}} \, dx \, dy < +\infty.$$

Then $v \in L^q(\mathbb{R}^N)$ *with* $q = \frac{Np}{N-p}$*. Moreover, there exist three positive constants* C, C_1, *and* C_2, *depending only on* N *and* p, *such that*

$$C \underset{|v| > F(1)}{\int} |v|^q \, dx \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(|v(x) - v(y)|)}{|x-y|^{N+p}} \, dx \, dy$$
$$+ F(1)^q |\{C_1 F(1) < |v| < C_2 F(1)\}|.$$

Theorem 5 *Let* $1 \leq p < N$, $F_n : [0, +\infty) \to [0, +\infty)$ *be a sequence of non-decreasing functions such that* $\lim_{n \to \infty} F_n(1) = 0$,

$$F_n(1) + \int_0^1 F_n(t) t^{-(p+1)} \, dt = 1,$$

and $f_n : \mathbb{R}^N \to \mathbb{R}$ *be a bounded sequence of functions in* $L^p(\mathbb{R}^N)$*. Assume that*

$$\liminf_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|f_n(x) - f_n(y)|)}{|x-y|^{N+p}} \, dx \, dy < +\infty.$$

Then there exist a subsequence (f_{n_k}) *of* (f_n) *and* $f \in L^p(\mathbb{R}^N)$ *such that* f_{n_k} *converges to* f *in* $L^p_{\rm loc}(\mathbb{R}^N)$.

Theorems 4 and 5 are prove by the same manner used as in the proof of Theorems 1 and 2. However in place of Theorem 3, one uses the following

Theorem 6 *Let* $F : [0, +\infty) \to [0, +\infty)$ *be a non-decreasing function and* Q *be a cube of* $\mathbb{R}^N$. *Then there exists a constant* $C > 0$, *depending only on* N *and* p, *such that*

$$\Big(F(1) + \int_0^1 F(t)t^{-(p+1)}\,dt\Big) \int_Q \int_Q |f(x) - f(y)|^p \,dx\,dy$$

$$\leq C \left(|Q|^{N+p} \int_Q \int_Q \frac{F(|f(x) - f(y)|)}{|x - y|^{N+p}} \,dx\,dy + F(1)|Q|^2 \right).$$

As we will see, Theorem 6 is also a consequence of Theorem 3.

Our paper is organized as follows. In Section 2, we will prove Theorem 3. The heart of the matter is estimate (2.9). Theorems 1 and 3 will be proved in Section 3. In Section 4, we will prove Theorem 2. Finally, in Section 5, we will prove Theorems 4, 5, and 6.

2 Estimate in the spirit of the Poincaré inequality

2.1 Technical lemma

In this section, we prove a technical result which will be useful in the proof of Theorem 3. Its proof is based on the ideas of J. Bourgain and H-M. Nguyen in the proof of [5, Lemma 1].

Lemma 1 *Let* $p \geq 1$ *and* f *be a measurable function defined on a bounded interval* I *into* $\mathbb{R}$. *Suppose that there exist* $0 < \tau < \frac{1}{2}$, $c_1 < c_2$, *and two non-empty sub-intervals* I_1 *and* I_2 *of* I *such that*

$$\{x \in I_1;\, f(x) < c_1\} \geq \tau |I_1| \quad \textit{and} \quad \{x \in I_2;\, f(x) > c_2\} \geq \tau |I_2|.$$

Then there exists a positive constant C_τ, *depending only* p *and* τ, *such that*

$$\underset{|f(x)-f(y)|>\delta}{\int_I \int_I} \frac{\delta^p}{|x - y|^{p+1}} \,dx\,dy \geq C_\tau (c_2 - c_1)^p |I|^{1-p}, \quad \forall\, \delta \in (0, \delta_0),$$

where $\delta_0 = \frac{\tau^2(c_2 - c_1)}{200} \min\{\frac{|I_1|}{|I|}, \frac{|I_2|}{|I|}\}$.

Proof. By scaling and translating, we can assume as well that $I = [0,1]$, $c_1 = 0$, and $c_2 = 1$. Take $\delta \in (0, \frac{\tau^2}{200}) \min\{|I_1|, |I_2|\}$ and $K \in \mathbb{Z}_+$ such that $\delta < 2^{-K} \leq 2\delta$. Denote

$$J = \Big\{ j \in \mathbb{Z}_+ ; \frac{1}{4} < j 2^{-K} < \frac{3}{4} \Big\}.$$

Then

$$|J| \geq 2^{K-1} - 2 \approx \frac{1}{\delta}. \tag{2.1}$$

For each j, define the following sets

$$A_j = \left\{ x \in [0,1];\ (j-1)2^{-K} \leq f(x) < j 2^{-K} \right\},$$

$$B_j = \bigcup_{j' < j} A_{j'}, \text{ and } C_j = \bigcup_{j' > j} A_{j'},$$

so that $B_j \times C_j \subset \left[|f(x) - f(y)| \geq 2^{-K} \right] \subset [|f(x) - f(y)| > \delta]$.
Since the collection (A_j) is disjoint, it follows from (2.1) that

$$\operatorname{card}(G) \geq 2^{K-2} - 3 \approx \frac{1}{\delta}, \tag{2.2}$$

where G is defined by

$$G = \{ j \in J;\ |A_j| < 2^{-K+2} \}.$$

For each $j \in G$, set $\lambda_{1,j} = |A_j| > 0$ (see [13, Lemma 3]) and consider the function $\psi_j(t)$ defined as follows

$$\psi_j(t) = \left| [t - \frac{4}{\tau}\lambda_{1,j}, t + \frac{4}{\tau}\lambda_{1,j}] \cap B_j \right|, \quad \forall\, t \in [\frac{4}{\tau}\lambda_{1,j}, 1 - \frac{4}{\tau}\lambda_{1,j}].$$

We claim that there exist $s_{1,j}$ and $s_{2,j}$ in $[\frac{4}{\tau}\lambda_{1,j}, 1 - \frac{4}{\tau}\lambda_{1,j}]$ such that

$$\frac{\tau \psi_j(s_{1,j})}{8\lambda_{1,j}} > \tau/2 \quad \text{and} \quad \frac{\tau \psi_j(s_{2,j})}{8\lambda_{1,j}} < 1 - \tau/2. \tag{2.3}$$

We prove this by contradiction. Suppose that

$$\frac{\tau \psi_j(s)}{8\lambda_{1,j}} \leq \frac{\tau}{2}, \quad \forall\, s \in [\frac{4}{\tau}\lambda_{1,j}, 1 - \frac{4}{\tau}\lambda_{1,j}].$$

Then

$$|I_1 \cap B_j| \leq \tau |I_1|/2 + \frac{8}{\tau}\lambda_{1,j}.$$

However,

$$\frac{8}{\tau}\lambda_{1,j} \leq \frac{8.2^{-K+2}}{\tau} = \frac{32.2^{-K}}{\tau} \leq \frac{64\delta}{\tau} < \tau |I_1|/2.$$

This implies

$$|I_1 \cap B_j| < \tau |I_1|.$$

This contradicts to $|I_1 \cap B_j| \geq |I_1 \cap B_1| > \tau |I_1|$.

Thus there exists $s_{1,j} \in [\frac{4}{\tau}\lambda_{1,j}, 1-\frac{4}{\tau}\lambda_{1,j}]$ such that $\frac{\tau\psi_j(s_{1,j})}{8\lambda_{1,j}} > \tau/2$. Similarly, there exists $s_{2,j} \in [\frac{4}{\tau}\lambda_{1,j}, 1-\frac{4}{\tau}\lambda_{1,j}]$ such that $\frac{\tau\psi_j(s_{2,j})}{8\lambda_{1,j}} < 1-\tau/2$. Since ψ_j is a continuous function on $[\frac{4}{\tau}\lambda_{1,j}, 1-\frac{4}{\tau}\lambda_{1,j}]$, it follows from (2.3) that there exists $t_{1,j} \in [\frac{4}{\tau}\lambda_{1,j}, 1-\frac{4}{\tau}\lambda_{1,j}]$ such that

$$\tau/2 \leq \frac{\tau\psi_j(t_{1,j})}{8\lambda_{1,j}} \leq 1-\tau/2.$$

By the same method used as in [5, Lemma 1] (see also [13, Lemma 4]), there exists t_j and λ_j such that

$$\tau/2 \leq \frac{\tau|[t_j - \frac{4}{\tau}\lambda_j, t_j + \frac{4}{\tau}\lambda_j] \cap B_j|}{8\lambda_j} \leq 1-\tau/2, \tag{2.4}$$

and

$$\tau/64 \leq \frac{\tau|[t_j - \frac{4}{\tau}\lambda_j, t_j + \frac{4}{\tau}\lambda_j] \cap A_j|}{8\lambda_j} \leq \tau/8. \tag{2.5}$$

The rest of the proof now follows from the one of [5, Lemma 1].

Set $\lambda = \inf_{j\in G} \lambda_j$ ($\lambda > 0$ since G is finite). Suppose $G = \bigcup_{i=1}^n I_m$, where I_m is defined as follows

$$I_m = \{j \in G;\, 2^{m-1}\lambda \leq \lambda_j < 2^m\lambda\}, \quad \forall\, m \geq 1.$$

Then, from (2.2),

$$\sum_{m=1}^n \operatorname{card}(I_m) \gtrsim \frac{1}{\delta}. \tag{2.6}$$

For each m ($1 \leq m \leq n$), since $A_j \cap A_k = \emptyset$ for $j \neq k$, it follows from (2.5) that there exists $J_m \subset I_m$ such that

$$a)\, \operatorname{card}(J_m) \gtrsim \operatorname{card}(I_m) \quad \text{and} \quad b)\, |t_i - t_j| > \frac{2^{m+3}\lambda}{\tau}, \quad \forall\, i, j \in J_m. \tag{2.7}$$

Then, from (2.7-b) and the definition of I_m,

$$[t_i - \frac{4}{\tau}\lambda_i, t_i + \frac{4}{\tau}\lambda_i] \cap [t_j - \frac{4}{\tau}\lambda_j, t_j + \frac{4}{\tau}\lambda_j] = \emptyset, \quad \forall\, i, j \in J_m, \tag{2.8}$$

and

$$\sum_{m=1}^n \operatorname{card}(J_m) \gtrsim \frac{1}{\delta}.$$

Set $U_0 := \emptyset$ and, for $m = 1, 2, \dots, n$,

$$\begin{cases} L_m = \big\{ j \in J_m;\ \big|[t_j - \frac{4}{\tau}\lambda_j, t_j + \frac{4}{\tau}\lambda_j] \setminus U_{m-1}\big| \geq \dfrac{8(1-\tau/4)\lambda_j}{\tau} \big\}, \\ U_m = \big(\bigcup\limits_{j \in L_m} [t_j - \frac{4}{\tau}\lambda_j, t_j + \frac{4}{\tau}\lambda_j] \big) \cup U_{m-1}, \\ a_m = \operatorname{card}(J_m) \quad \text{and} \quad b_m = \operatorname{card}(L_m). \end{cases}$$

From (2.8) and the definitions of J_m and L_m,

$$\frac{\tau}{4} 2^{m-1}(a_m - b_m) \leq \sum_{i=1}^{m-1} 2^i b_i$$

which shows that

$$a_m \leq b_m + \frac{8}{\tau} \sum_{i=1}^{m-1} 2^{(i-m)} b_i.$$

Consequently,

$$\sum_{m=1}^{n} a_m \leq \sum_{m=1}^{n} b_m + \frac{8}{\tau} \sum_{m=1}^{n} \sum_{i=1}^{m-1} 2^{(i-m)} b_i = \sum_{m=1}^{n} b_m + \frac{8}{\tau} \sum_{i=1}^{n} b_i \sum_{m=i+1}^{n} 2^{(i-m)}.$$

Since $\sum_{i=1}^{\infty} 2^{-i} = 1$, it follows from from (2.6) and (2.7-a) that

$$\sum_{m=1}^{n} b_m \gtrsim \sum_{m=1}^{n} a_m \gtrsim \frac{1}{\delta}.$$

Therefore, it is easy to see that

$$\iint\limits_{\substack{I \times I \\ |f(x)-f(y)|>\delta}} \frac{1}{|x-y|^{p+1}}\, dx\, dy$$

$$\geq \sum_{m=1}^{n} \sum_{j \in L_m} \iint\limits_{\substack{([t_j - 4\lambda_j, t_j + 4\lambda_j] \setminus U_{m-1})^2 \\ x \in B_j,\, y \in C_j}} \frac{1}{|x-y|^{p+1}}\, dx\, dy$$

$$\gtrsim \sum_{m=1}^{n} \frac{b_m}{\delta^{p-1}} \gtrsim \frac{1}{\delta^p},$$

which yields the conclusion of Lemma 1. □

2.2 Proof of Theorem 3 in the one dimensional case

In this section, we prove the following estimate

$$\int_I \int_I |f(x) - f(y)|^p\, dx\, dy \leq C_p \Big(|I|^{p+1} \int_I \int_I\limits_{|f(x)-f(y)|>\delta} \frac{\delta^p}{|x-y|^{p+1}}\, dx\, dy + \delta^p |I|^2 \Big), \tag{2.9}$$

for any $p \geq 1$ and for any measurable function f defined on a bounded interval I into $\mathbb{R}$. Here C_p denotes a positive constant depending only on p.

Observe that if we define the function

$$f_A = (f \vee (-A)) \wedge A,$$

then

$$|f_A(x) - f_A(y)| \leq |f(x) - f(y)|.$$

Hence, for all $A \geq 0$ and $\delta > 0$,

$$\underset{|f_A(x)-f_A(y)|>\delta}{\int_I \int_I} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy \leq \underset{|f(x)-f(y)|>\delta}{\int_I \int_I} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy.$$

Thus without loss of generality, one may assume that $f \in L^\infty(I)$.

By scaling, one may assume that $I = [0,1]$ and

$$\int_I \int_I |f(x) - f(y)|^p\,dx\,dy = 6a_p, \tag{2.10}$$

where $a_p = \max\{a_{1,p}, a_{2,p}, 4\}$, $a_{1,p}$ and $a_{2,p}$ are positive constants which will be defined later.

Set

$$\alpha = \frac{2}{3}.$$

Consider $\tilde{f} : [0,2] :\rightarrow \mathbb{R}$ an extension by reflection of f, i.e.,

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in [0,1], \\ f(2-x) & \text{if } x \in (1,2], \end{cases}$$

and define $g : [0,2] \rightarrow \mathbb{S}^1$ by

$$g(x) = e^{i\tilde{f}(x)}, \quad \forall\, x \in [0,2].$$

Case 1: There exist $x \in I$ and $0 < r < 1$ such that $\left| \fint_x^{x+r} g(s)\,ds \right| \leq \alpha_1 := \dfrac{\alpha+1}{2}$. Then there exist z_1 and z_2 in $\mathbb{S}^1$ such that $|z_1 - z_2| \gtrsim 1$,

$$\left\{ y \in (x, x+r);\ |g(y) - z_1| \leq \frac{|z_1 - z_2|}{3} \right\} \gtrsim 1,$$

and

$$\left\{ y \in (x, x+r);\ |g(y) - z_2| \leq \frac{|z_1 - z_2|}{3} \right\} \gtrsim 1.$$

This was observed in [4] and [12]. Thus, by Lemma 1, there exists $\delta_0 \approx 1$ such that

$$\mathop{\int_0^2\int_0^2}_{|g(x)-g(y)|\geq\delta} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy \gtrsim 1, \quad \forall\, 0<\delta<\delta_0.$$

Since

$$\mathop{\int_0^2\int_0^2}_{|g(x)-g(y)|\geq\delta} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy \leq 4 \mathop{\int_0^1\int_0^1}_{|g(x)-g(y)|\geq\delta} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy, \quad \forall\,\delta>0,$$

this implies

$$\mathop{\int_0^1\int_0^1}_{|g(x)-g(y)|\geq\delta} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy \gtrsim 1, \quad \forall\, 0<\delta<\delta_0. \tag{2.11}$$

Case 2: $\left|\fint_x^{x+r} g(s)\,ds\right| \geq \alpha_1$, for all x, $r\in I$.

Case 2.1: f is continuous.

We claim that

$$\mathop{\int_I\int_I}_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy \geq b_p, \quad \forall\, 0<\delta<1. \tag{2.12}$$

where $b_p = C_\tau\alpha^p$, C_τ is the positive constant less than 1 in Lemma 1 corresponding to $\tau = \frac{\alpha}{2-\alpha}$.

We will prove this by contradiction. Suppose that there exists $0<\delta_0<1$ such that

$$\mathop{\int_I\int_I}_{|g(x)-g(y)|>\delta_0} \frac{\delta_0^p}{|x-y|^{p+1}}\,dx\,dy < b_p. \tag{2.13}$$

Define $\tilde{u}: Q\to\mathbb{R}^2$, where $Q=I^2$ as follows

$$\tilde{u}(x,r) = \fint_x^{x+r} g(s)\,ds, \quad \forall\,(x,r)\in Q.$$

Then

$$|\tilde{u}(X)| \geq \alpha, \quad \forall\, X\in Q. \tag{2.14}$$

Set

$$u(X) = \frac{\tilde{u}(X)}{|\tilde{u}(X)|}, \quad \forall\, X\in Q.$$

Let ψ be the continuous lifting of u in Q, i.e. $e^{i\psi}=u$, such that

$$\psi(x,0) = f(x), \quad \forall\, x\in I.$$

Then, for all $(x, y) \in I^2$,

$$|f(x)-f(y)| \le |\psi(x,0)-\psi(x,1)|+|\psi(x,1)-\psi(y,1)|+|\psi(y,1)-\psi(y,0)|. \quad (2.15)$$

We first remark that

$$|\psi(x,1) - \psi(y,1)| \lesssim 1. \quad (2.16)$$

We now estimate $|\psi(x,0) - \psi(x,1)|$. Take $k \in \mathbb{Z}$ such that

$$2k\pi \le \psi(x,1) - \psi(x,0) < 2k\pi + 2\pi. \quad (2.17)$$

Without loss of generality, assume that $k \ge 0$ and $\psi(x,0) = 0$. Thus it follows from (2.29) that there exist $0 < t_1 < t_2 < \cdots < t_{2k-1} < t_{2k} \le 1$ such that

$$\begin{cases} \psi(x, t_{2m-1}) & = & 2m\pi - \pi, \\ \psi(x, t_{2m}) & = & 2m\pi, \end{cases} \qquad \forall\, 1 \le m \le k.$$

Hence it follows from (2.14) that

$$\begin{cases} ⨍_x^{x+t_{2m-1}} g_1(s)\, ds \le -\alpha, \\ ⨍_x^{x+t_{2m}} g_1(s)\, ds \ge \alpha, \end{cases} \qquad \forall\, 1 \le m \le k. \quad (2.18)$$

We claim that

$$\delta_0 \ge \frac{\alpha^3}{200(2-\alpha)^2} t_{2k-1}.$$

From (2.18), one has

$$\begin{cases} |\{s \in (x + t_{2m-2}, x + t_{2m-1});\, g_1(s) \le -\alpha/2\}| \ge \dfrac{\alpha}{2-\alpha} t_{2m-1}, \\ |\{s \in (x + t_{2m-1}, x + t_{2m});\, g_1(s) \ge \alpha/2\}| \ge \dfrac{\alpha}{2-\alpha} t_{2m}. \end{cases} \quad (2.19)$$

In fact, set $\mu = |\{s \in (x + t_{2m-1}, x + t_{2m});\, g_1(s) \ge \alpha/2\}|/(t_{2m} - t_{2m-1})$. Then since

$$\alpha t_{2m} = \int_x^{x+t_{2m}} g\, ds = \int_x^{x+t_{2m-1}} g\, ds + \int_{x+t_{2m-1}}^{x+t_{2m}} g\, ds = -\alpha t_{2m-1} + \int_{x+t_{2m-1}}^{x+t_{2m}} g\, ds,$$

it follows that

$$\mu(t_{2m} - t_{2m-1}) + \frac{\alpha}{2}(1-\mu)(t_{2m} - t_{2m-1}) - \alpha t_{2m-1} \ge \alpha t_{2m}.$$

Hence

$$\mu \ge \frac{\alpha(t_{2m} + t_{2m-1})}{(2-\alpha)(t_{2m} - t_{2m-1})},$$

which yields

$$|\{s \in (x+t_{2m-1}, x+t_{2m});\ g_1(s) \le -\alpha/2\}| = \mu(t_{2m}-t_{2m-1}) \ge \frac{\alpha}{2-\alpha} t_{2m}.$$

This is the second inequality of (2.19). The first inequality of (2.19) is proved by the same manner.

Thus if $\delta_0 \le \dfrac{\alpha^3}{200(2-\alpha)^2} t_{2k-1}$, applying Lemma 1 for the interval $(x, x+t_{2m})$ with $I_1 = (x, x+t_{2m-1})$ and $I_2 = (x, x+t_{2m})$, one gets

$$\underset{|g_1(x)-g_1(y)|>\delta_0}{\int_I \int_I} \frac{\delta_0^p}{|x-y|^{p+1}}\, dx\, dy \ge C_\tau \alpha^p = b_p.$$

This contradicts (2.13).

Hence,

$$t_1 < t_2 < \dots < t_{2k-1} \le \frac{\alpha^3}{200(2-\alpha)^2} \delta_0. \tag{2.20}$$

Also since

$$\int_x^{x+t_{2m+1}} g\, ds = \int_x^{x+t_{2m}} g\, ds + \int_{x+t_{2m}}^{x+t_{2m+1}} g\, ds$$

it follows from (2.18) that

$$-\alpha t_{2m} + (t_{2m+1} - t_{2m}) \ge \alpha t_{2m+1}.$$

This implies

$$t_{2m+1} \ge \frac{1+\alpha}{1-\alpha} t_{2m} \ge 2 t_{2m} \tag{2.21}$$

Since $0 < \delta_0 < 1$, it follows from (2.19) and (2.21) that

$$\begin{aligned} \underset{|g(\xi)-g(x)|\ge\delta_0}{\int_x^{x+t_{2k-1}}} \frac{\delta_0^p}{|\xi-x|^{p+1}}\, d\xi &\ge \sum_{m=1}^{k-1} \underset{|g(\xi)-g(x)|\ge\delta_0}{\int_{x+t_{2m}}^{x+t_{2m+1}}} \frac{\delta_0^p}{|\xi-x|^{p+1}}\, d\xi \\ &\gtrsim \sum_{m=1}^{k-1} 2^{mp} \gtrsim 2^{kp} - 1. \end{aligned}$$

However, there exists a positive constant c_p such that $2^m \ge c_p m^p$ for all $m \ge 1$. Thus, it follows that

$$\underset{|g(\xi)-g(x)|\ge\delta_0}{\int_0^2} \frac{\delta_0^p}{|\xi-x|^{p+1}}\, d\xi + 1 \ge c_p |\psi(x,1) - \psi(x,0)|^p. \tag{2.22}$$

Similarly,

$$\int_{\substack{0\\|g(\xi)-g(y)|\geq\delta_0}}^{2} \frac{\delta_0^p}{|\xi-y|^{p+1}}\,d\xi+1\geq c_p|\psi(y,1)-\psi(y,0)|^p. \tag{2.23}$$

Combining (2.28), (2.20), (2.22), and (2.23) yields

$$|f(x)-f(y)|^p\lesssim \int_{\substack{0\\|g(\xi)-g(x)|\geq\delta_0}}^{2} \frac{\delta_0^p}{|\xi-x|^{p+1}}\,d\xi+\int_{\substack{0\\|g(\xi)-g(y)|\geq\delta_0}}^{2} \frac{\delta_0^p}{|\xi-y|^{p+1}}\,d\xi+1.$$

This implies for some positive constant $a_{1,p}>0$,

$$\int_I\int_I|f(x)-f(y)|^p\,dx\,dy\leq a_{1,p}\Big(\int_{\substack{0\\|g(x)-g(y)|\geq\delta_0}}^{2}\int_0^2 \frac{\delta_0^p}{|x-y|^{p+1}}\,dx\,dy+1\Big)$$

Thus it follows from (2.10) that

$$\int_{\substack{0\\|g(x)-g(y)|\geq\delta_0}}^{1}\int_0^1 \frac{\delta_0^p}{|x-y|^{p+1}}\,dx\,dy\geq 1,$$

since

$$\int_{\substack{0\\|g(x)-g(y)|\geq\delta_0}}^{2}\int_0^2 \frac{\delta_0^p}{|x-y|^{p+1}}\,dx\,dy\leq 4\int_{\substack{0\\|g(x)-g(y)|\geq\delta_0}}^{1}\int_0^1 \frac{\delta_0^p}{|x-y|^{p+1}}\,dx\,dy.$$

This contradicts (2.13).

Case 2.2: $f\in L^\infty(I)$. Take $0<\delta\leq\min\{\delta_1,\delta_2,\frac{1}{4}\}$ such that

$$\int_{\substack{I\\|f(x)-f(y)|>\delta}}\int_I \frac{1}{|x-y|^{p+1}}\,dx\,dy<+\infty.$$

Here δ_1 and δ_2 are positive constants which will be defined later. We claim that

$$\int_{\substack{I\\|g(x)-g(y)|>\delta}}\int_I \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy\geq b_p. \tag{2.24}$$

We recall here that $b_p=C_\tau\alpha^p$, C_τ is the positive constant less than 1 in Lemma 1 corresponding to $\tau=\frac{\alpha}{2-\alpha}$. We prove this by contradiction. Suppose that

$$\int_{\substack{I\\|g(x)-g(y)|>\delta}}\int_I \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy<b_p. \tag{2.25}$$

For each $\varepsilon \in (0,1)$, define

$$f_\varepsilon(x) = \fint_x^{x+\varepsilon} \tilde{f}(s)\, ds, \quad \forall\, x \in I.$$

Consider $\tilde{f}_\varepsilon : [0,2] :\to \mathbb{R}$ an extension by reflection of f_ε, i.e.,

$$\tilde{f}_\varepsilon(x) = \begin{cases} f_\varepsilon(x) & \text{if } x \in [0,1], \\ f_\varepsilon(2-x) & \text{if } x \in (1,2], \end{cases}$$

and define $g_\varepsilon : [0,2] \to \mathbb{S}^1$ by

$$g_\varepsilon(x) = e^{i\tilde{f}_\varepsilon(x)}, \quad \forall\, x \in [0,2].$$

Let $A > 1$ be such that $\|f\|_{L^\infty(I)} \leq A$. Then there exists $0 < r_0 < 1$ such that

$$\iint_{\substack{B \\ |f(x)-f(y)|>\delta}} \frac{1}{|x-y|^{p+1}}\, dx\, dy < \frac{c}{A^2}, \tag{2.26}$$

for all measurable subset B of I^2 with $|B| \leq r_0^2$. Here $c = \min\{c_1, c_2, c_3\}$, where c_1, c_2 and c_3 are positive constants which will be defined later.

Take $\varepsilon_0 \leq r_0/2$ such that for all $\varepsilon \leq \varepsilon_0$,

$$\left| \fint_x^{x+r} g_\varepsilon(s)\, ds - \fint_x^{x+r} g(s)\, ds \right| \leq \frac{\alpha_1 - \alpha}{2},$$

for all $r \geq r_0/2$. Then

$$\left| \fint_x^{x+r} g_\varepsilon(s)\, ds \right| \geq \alpha_2 = \alpha + \frac{\alpha_1 - \alpha}{2}, \quad \forall\, r \geq r_0/2$$

We claim that

$$\left| \fint_x^{x+r} g_\varepsilon(s)\, ds \right| \geq \alpha_2,$$

for all $\varepsilon \leq \varepsilon_0$.

We prove this by contradiction. Suppose that there exists x, r, and ε ($r \leq r_0/2$ and $\varepsilon \leq \varepsilon_0 \leq r_0/2$) such that

$$\left| \fint_x^{x+r} g_\varepsilon(s)\, ds \right| < \alpha_2. \tag{2.27}$$

If $r \leq \varepsilon \leq r_0/2$, from (2.27), there exist s and t in $[x, x+r)$ such that $|f_\varepsilon(s) - f_\varepsilon(t)| \gtrsim 1$. It follows that

$$\fint_x^{x+2\varepsilon} \fint_x^{x+2\varepsilon} |f(\xi) - f(\eta)|\, d\xi\, d\eta \geq 2\delta_1.$$

Thus since $\|f\|_{L^\infty(I)} \leq A$, one has

$$\{(\xi,\eta) \in (x, x+2\varepsilon)^2;\, |f(\xi) - f(\eta)| \geq \delta_1\} \gtrsim \frac{\varepsilon^2}{A}.$$

This implies

$$\int_x^{x+2\varepsilon} \int_{\substack{x \\ |f(x)-f(y)|>\delta}}^{x+2\varepsilon} \frac{1}{|x-y|^{p+1}}\, dx\, dy \geq \frac{c_1}{A}.$$

This contradicts to (2.26) since $2\varepsilon \leq r_0$.

If $\varepsilon \leq r \leq r_0/2$, from (2.27), there exist z_1 and z_2 such that $|z_1 - z_2| \gtrsim 1$,

$\{s \in (x, x+r); |e^{if_\varepsilon} - z_1| \leq |z_1 - z_2|/3\} \gtrsim 1$ and $\{s \in (x, x+r); |e^{if_\varepsilon} - z_2| \leq |z_1 - z_2|/3\} \gtrsim 1$

Consequently, there exist $I_1 \subset (x, x+2r)$ et $I_2 \subset (x, x+2r)$ such that $|I_1| \gtrsim r$, $|I_2| \gtrsim r$, and

$$\left| ⨍_{I_1} f(s)\, ds - ⨍_{I_2} f(s)\, ds \right| \gtrsim 1.$$

Thus

$$⨍_x^{x+2r} ⨍_x^{x+2r} |f(\xi) - f(\eta)| \geq 2\delta_2.$$

Thus since $\|f\|_{L^\infty(I)} \leq A$, one has

$$\{(\xi,\eta) \in (x, x+2r)^2;\, |f(\xi) - f(\eta)| \geq \delta_2\} \gtrsim \frac{r^2}{A}.$$

This implies

$$\int_x^{x+2r} \int_{\substack{x \\ |f(x)-f(y)|>\delta}}^{x+2r} \frac{1}{|x-y|^{p+1}}\, dx\, dy \geq \frac{c_2}{A}.$$

This contradicts to (2.26) since $2r \leq r_0$. Thus

$$\left| ⨍_x^{x+r} e^{if_\varepsilon(s)}\, ds \right| \geq \alpha_2,$$

for all $\varepsilon \leq \varepsilon_0$.

For each $\varepsilon \leq \min\{\varepsilon_0, r_0^2\}/2$, consider

$$\mathcal{A}_\varepsilon = \left\{ x \in I;\, \left| ⨍_x^{x+r} e^{if_\varepsilon(s)}\, ds - e^{if(x)} \right| \geq \frac{1}{2} \text{ for some } r < r_0^2/2 \right\}.$$

Take $x \in \mathcal{A}_\varepsilon$ and $r < r_0^2/2$ such that $\left| ⨍_x^{x+r} e^{if_\varepsilon(s)}\, ds - e^{if(x)} \right| \geq 1/2$. Then there exist $J \subset (x, x+r+\varepsilon)$ such that $|J| \geq r + \varepsilon$

$$\left| ⨍_J f(s)\, ds - f(x) \right| \geq 1/2.$$

Consequently,

$$\int_{\substack{x \\ |f(x)-f(y)|>\frac{1}{4}}}^{x+r+\varepsilon} \frac{1}{|\xi - x|^{p+1}}\, d\xi \geq \frac{c_3}{A}.$$

Therefore,

$$c_3|\mathcal{A}_\varepsilon|/A \leq c/A^2.$$

This implies

$$|\mathcal{A}_\varepsilon| \leq 1/A.$$

Take $\varepsilon \leq \min\{\varepsilon_0, r_0^2\}/2$ sufficiently small such that $|\mathcal{B}_\varepsilon| \leq 1/A$, where

$$\mathcal{B}_\varepsilon = \{x \in I; |f_\varepsilon(x) - f(x)| \geq \alpha_1 - \alpha\}.$$

Define $\tilde{u}_\varepsilon : Q \to \mathbb{R}^2$, where $Q = I^2$ as follows

$$\tilde{u}_\varepsilon(x,r) = \fint_x^{x+r} e^{if_\varepsilon(s)}\, ds, \quad \forall\, (x,r) \in Q.$$

Set

$$u_\varepsilon(X) = \frac{\tilde{u}_\varepsilon(X)}{|\tilde{u}_\varepsilon(X)|}, \quad \forall\, X \in Q.$$

Let ψ_ε be the lifting of u_ε in Q such that

$$\psi_\varepsilon(x,0) = f_\varepsilon(x), \quad \forall\, x \in I.$$

We have

$$\begin{aligned} |f(x) - f(y)| \leq & |f(x) - \psi_\varepsilon(x,1)| + |\psi_\varepsilon(x,1) - \psi_\varepsilon(y,1)| + |\psi_\varepsilon(y,1) - f(y)| \\ & + |f_\varepsilon(x) - f(x)| + |f_\varepsilon(y) - f(y)|, \end{aligned} \tag{2.28}$$

for all $(x,y) \in I^2$.

Take $x \in \mathcal{B}_\varepsilon \setminus \mathcal{A}_\varepsilon$ and $k \in \mathbb{Z}$ such that

$$2k\pi \leq \psi_\varepsilon(x,1) - f(x) < 2k\pi + 2\pi. \tag{2.29}$$

Without loss of generality, one can assume as well that $k \geq 0$, $f(x) = 0$ and there exist $0 < s_1 < s_2 < \cdots < s_{2k-1} < s_{2k} \leq 1$ such that

$$\left\{ \begin{array}{lcl} \psi_\varepsilon(x, s_{2m-1}) & = & 2m\pi - \pi - \pi/4, \\ \psi_\varepsilon(x, s_{2m}) & = & 2m\pi - \pi/4, \end{array} \right. \quad \forall\, 1 \leq m \leq k,$$

Then $s_1 \geq r_0^2/2$. Thus there exist $t_1, \ldots, t_{2k-1}$, such that $0 < s_1 < t_1 < s_2 < \cdots < s_{2k-1} < t_{2k-1} < s_{2k} \leq 1$ such that

$$\left\{ \begin{array}{lcl} \arg(u(x, t_{2m-1}) & = & \pi, \\ \arg(u(x, t_{2m})) & = & 0, \end{array} \right. \quad \forall\, 1 \leq m \leq k-1$$

(if ε is sufficiently small). Applying the method used in the case f is continuous, one has, for each $(x,y) \in [\mathcal{B}_\varepsilon \setminus \mathcal{A}_\varepsilon]^2$,

$$|f(x) - f(y)|^p \lesssim \int_{\substack{0 \\ |g(\xi)-g(x)|\geq\delta}}^2 \frac{\delta^p}{|\xi - x|^{p+1}}\, d\xi + \int_{\substack{0 \\ |g(\xi)-g(y)|\geq\delta}}^2 \frac{\delta^p}{|\xi - y|^{p+1}}\, d\xi + 1.$$

Thus

$$\iint_{[\mathcal{B}_\varepsilon \setminus \mathcal{A}_\varepsilon]^2} |f(x) - f(y)|^p \, dx\, dy \leq a_{2,p}\Big(\int_{\substack{0 \\ |g(x)-g(y)|\geq\delta}}^2 \int_0^2 \frac{\delta^p}{|x-y|^{p+1}}\, dx\, dy + 1\Big)$$

This implies that

$$\int_{\substack{0 \\ |g(x)-g(y)|\geq\delta}}^2 \int_0^2 \frac{\delta^p}{|x-y|^{p+1}}\, dx\, dy \geq 1,$$

since

$$\iint_{I^2 \setminus [\mathcal{B}_\varepsilon \setminus \mathcal{A}_\varepsilon]^2} |f(x) - f(y)|^p \, dx\, dy \leq \frac{4A}{A} = 4 \leq a_p.$$

This contradicts (2.25).

Therefore, (2.9) follows from (2.11), (2.12) and (2.24). $\square$

2.3 Proof of Theorem 3 for arbitrary dimensions

Theorem 3 in the case $N \geq 2$ is a consequence of itself in the one dimensional case and the following

Lemma 2 *Let g be a measurable function on $Q = I^N$, where I is a bounded interval of $\mathbb{R}$, and $1 \leq p < +\infty$. Then*

$$\int_{\substack{I^{N-1} \\ |g(x',x_N)-g(x',y_N)|>2\delta}} \int_I \int_I \frac{1}{|x_N - y_N|^{p+1}}\, dx_N\, dy_N\, dx'$$

$$\leq C_{N,p} \int_{\substack{Q \\ |g(x)-g(y)|>\delta}} \int_Q \frac{1}{|x-y|^{N+p}}\, dx\, dy, \quad \forall\, \delta > 0,$$

for some $C_{N,p} > 0$ depending only on N and p.

Remark 9 Technique which will be used in the proof of Lemma 2 was appeared in [1]. A similar estimate was mentioned in [5] ([5, Lemma 3]).

Proof. One has

$$\underset{|f(x',x_N)-f(x',y_N)|>2\delta}{\int_{I^{N-1}}\int_I\int_I} \frac{1}{|x_N-y_N|^{p+1}}\,dx_N\,dy_N\,dx'$$

$$\lesssim \underset{|f(x',x_N)-f(x',y_N)|>2\delta}{\int_{I^{N-1}}\int_I\int_I} \int_{B_{N-1}\left(x',\frac{|x_N-y_N|}{2}\right)\cap I^{N-1}} \frac{1}{|x_N-y_N|^{N+p}}\,dy'\,dx_N\,dy_N\,dx', \tag{2.30}$$

where $B_{N-1}(x',r)$ denotes the set $\{y'\in\mathbb{R}^{N-1};\ |y'-x'|\le r\}$.

On the other hand, if $|g(x',x_N)-g(x',y_N)|>2\delta$ then, for all $y'\in\mathbb{R}^{N-1}$,

$$\left|g(x',x_N)-g\left(y',\frac{x_N+y_N}{2}\right)\right|>\delta \quad\text{or}\quad \left|g\left(y',\frac{x_N+y_N}{2}\right)-g(x',x_N)\right|>\delta.$$

Hence

$$\underset{|f(x',x_N)-f(x',y_N)|>2\delta}{\int_{I^{N-1}}\int_I\int_I} \int_{B_{N-1}\left(x',\frac{|x_N-y_N|}{2}\right)\cap I^{N-1}} \frac{1}{|x_N-y_N|^{N+p}}\,dy'\,dx_N\,dy_N\,dx'$$

$$\lesssim \underset{\left|f(x',x_N)-f\left(y',\frac{x_N+y_N}{2}\right)\right|>\delta}{\int_{I^{N-1}}\int_I\int_I} \int_{B_{N-1}\left(x',\frac{|x_N-y_N|}{2}\right)\cap I^{N-1}} \frac{1}{|x_N-y_N|^{N+p}}\,dy'\,dx_N\,dy_N\,dx'$$

$$+ \underset{\left|f\left(y',\frac{x_N+y_N}{2}\right)-f(y',y_N)\right|>\delta}{\int_{I^{N-1}}\int_I\int_I} \int_{B_{N-1}\left(x',\frac{|x_N-y_N|}{2}\right)\cap I^{N-1}} \frac{1}{|x_N-y_N|^{N+p}}\,dy'\,dx_N\,dy_N\,dx'.$$

However,

$$\underset{\left|f(x',x_N)-f\left(y',\frac{x_N+y_N}{2}\right)\right|>\delta}{\int_{I^{N-1}}\int_I\int_I} \int_{B_{N-1}\left(x',\frac{|x_N-y_N|}{2}\right)\cap I^{N-1}} \frac{1}{|x_N-y_N|^{N+p}}\,dy'\,dx_N\,dy_N\,dx'$$

$$\lesssim \underset{|f(x)-f(y)|>\delta}{\int_{I^N}\int_{I^N}} \frac{1}{|x-y|^{N+p}}\,dx\,dy.$$

and

$$\underset{\left|f\left(y',\frac{x_N+y_N}{2}\right)-f(y',y_N)\right|>\delta}{\int_{I^{N-1}}\int_I\int_I} \int_{B_{N-1}\left(x',\frac{|x_N-y_N|}{2}\right)\cap I^{N-1}} \frac{1}{|x_N-y_N|^{N+p}}\,dy'\,dx_N\,dy_N\,dx'$$

$$\lesssim \underset{|f(x)-f(y)|>\delta}{\int_{I^N}\int_{I^N}} \frac{1}{|x-y|^{N+p}}\,dx\,dy.$$

Thus

$$\mathop{\int_{I^{N-1}}\int_I\int_I}_{|f(x',x_N)-f(x',y_N)|>2\delta} \int_{B_{N-1}\left(x',\frac{|x_N-y_N|}{2}\right)\cap I^{N-1}} \frac{1}{|x_N-y_N|^{N+p}}\,dy'\,dx_N\,dy_N\,dx'$$

$$\lesssim \mathop{\int_{I^N}\int_{I^N}}_{|f(x)-f(y)|>\delta} \frac{1}{|x-y|^{N+p}}\,dx\,dy. \tag{2.31}$$

Combining (2.30) and (2.3) yields

$$\mathop{\int_{I^{N-1}}\int_I\int_I}_{|f(x',x_N)-f(x',y_N)|>2\delta} \frac{1}{|x_N-y_N|^{p+1}}\,dx_N\,dy_N\,dx'$$

$$\lesssim \mathop{\int_{I^N}\int_{I^N}}_{|f(x)-f(y)|>\delta} \frac{1}{|x-y|^{N+p}}\,dx\,dy.$$

□

Proof of Theorem 3. For simplifying the notation, one may assume that $N=2$ and $I=[0,1]$. Then

$$\begin{aligned}\int_{I^2}\int_{I^2}|f(x_1,x_2)-f(y_1,y_2)|^p\,dx\,dy &\lesssim \int_{I^2}\int_{I^2}|f(x_1,x_2)-f(x_1,y_2)|^p\,dx\,dy\\ &\quad+\int_{I^2}\int_{I^2}|f(x_1,y_2)-f(y_1,y_2)|^p\,dx\,dy.\end{aligned} \tag{2.32}$$

Using (2.9), one has

$$\iint_{I^2}|f(x_1,x_2)-f(x_1,y_2)|^p\,dx_2dy_2$$

$$\lesssim \mathop{\int_I\int_I}_{|f(x_1,x_2)-f(x_1,y_2)|>2\delta} \frac{\delta^p}{|x_2-y_2|^{1+p}}\,dx_2\,dy_2+\delta^p.$$

However, by Lemma 2, one has

$$\mathop{\int_I\int_I\int_I}_{|f(x_1,x_2)-f(x_1,y_2)|>2\delta} \frac{1}{|x_2-y_2|^{1+p}}\,dx_2\,dy_2\,dx_1 \lesssim \mathop{\int_{I^2}\int_{I^2}}_{|f(x)-f(y)|>\delta} \frac{1}{|x-y|^{2+p}}\,dx\,dy.$$

Hence

$$\int_{I^2}\int_{I^2}|f(x_1,x_2)-f(x_1,y_2)|^p\,dx\,dy \lesssim \mathop{\int_{I^2}\int_{I^2}}_{|f(x)-f(y)|>\delta} \frac{\delta^p}{|x-y|^{2+p}}\,dx\,dy+\delta^p. \tag{2.33}$$

Similarly,

$$\int_{I^2}\int_{I^2} |f(x_1,y_2)-f(y_1,y_2)|^p \,dx\,dy \lesssim \underset{|f(x)-f(y)|>\delta}{\int_{I^2}\int_{I^2}} \frac{\delta^p}{|x-y|^{2+p}}\,dx\,dy+\delta^p. \quad (2.34)$$

Combining (2.32), (2.33), and (2.34) yields

$$\int_{I^2}\int_{I^2} |f(x)-f(y)|^p \,dx\,dy \lesssim \underset{|f(x)-f(y)|>\delta}{\int_{I^2}\int_{I^2}} \frac{\delta^p}{|x-y|^{2+p}}\,dx\,dy+\delta^p.$$

□

3 Remarks on the Sobolev inequality

One of manners to prove the Sobolev inequality is to use the representation formula and Riesz potential theory (see e.g. [15]). Recently, P. Hajlasz and P. Koskela adapted and extended this method to show the relation between the Poincaré inequality and the Sobolev inequality in [9]. Their typical result is as follows: If $u \in L^1(\mathbb{R}^N)$ and $g \in L^1(\mathbb{R}^N)$ such that

$$⨍_Q |u - ⨍_Q u|\,dx \le C|Q|^{1/N} ⨍_Q |g|\,dx,$$

for all cubes Q of $\mathbb{R}^N$. Then $u \in L_{\mathrm{w}}^{\frac{N}{N-1}}(\mathbb{R}^N)$, i.e.,

$$|\{|u| \ge t\}| \le \frac{C}{t^{\frac{N}{N-1}}},$$

for some positive constant C. This method does not seem to work in our situation where (1.7) holds. However, it is clear to see that (1.7) gives an estimate of the sharp function $f^{\sharp,\Delta} \in L^p(\mathbb{R}^N)$ (see the definition of $f^{\sharp,\Delta}$ in (3.1)). This is the source of our approach.

We first recall the definition of dyadic maximal operator and dyadic sharp function (see [16]). For each locally integrable function f defined on $\mathbb{R}^N$, one defines $M^\Delta f$ and $f^{\sharp,\Delta}$ as follows

$$(M^\Delta f)(x) = \sup_Q ⨍_Q |f|\,dy,$$

and

$$f^{\sharp,\Delta}(x) = \sup_Q ⨍_Q |f - f_Q|\,dy, \quad (3.1)$$

where the supremum is taken over all dyadic cubes Q containing x and $f_Q = ⨍_Q f\,dy$.

The following estimate plays an important role in this section (see [16, Estimate (22), page 153]).

Proposition 4 *For a locally integrable function f on $\mathbb{R}^N$, and for b and c positive with $b<1$, we have the following inequality*

$$|\{M^{\Delta}f>\alpha,\, f^{\sharp,\Delta}\leq c\alpha\}|\leq a|\{M^{\Delta}f>b\alpha\}|.$$

for all $\alpha>0$ and $a=2^N c/(1-b)$.

The key of this section is the following

Lemma 3 *Let $v\in L^p(\mathbb{R}^N)$ $(p\geq 1)$, $g\in L^1(\mathbb{R}^N)$, $\delta>0$, $q\geq 1$ and $r\geq 1$ such that*

$$\left|\left\{u_k^{\sharp,\Delta}>t\right\}\right|\leq c\frac{\|g_k\|_{L^1}^r}{(t-\delta)^q},\quad \forall\, t>\delta,\, k\in\mathbb{Z}, \tag{3.2}$$

for some positive constant c, where $g_k=g\chi_{2^k<|v|<2^{k+2}}$ and $u_k=T(2^k,2^{k+2},v)$. Here $T(l,m,u)$ $(0<l<m)$ is defined as follows

$$T(l,m,u)(x)=\begin{cases}\frac{m+l}{m-l}(u(x)-l), & \textit{if } l<u(x)\leq\frac{m+l}{2},\\ -\frac{m+l}{m-l}(u(x)-m), & \textit{if } \frac{m+l}{2}<u(x)\leq m,\\ 0, & \textit{if } 0\leq u(x)\leq l \textit{ or } u(x)>m,\\ -u(-x), & \textit{if } u(x)<0,\end{cases} \tag{3.3}$$

for each function u defined on $\mathbb{R}^N$. Then there exist two positive constants C and λ such that

$$\int\limits_{|v|>\delta}|v|^q\,dx\leq C\left(\|g\|_{L^1}^r+\delta^q|\{\delta<|v|<\lambda\delta\}|\right).$$

Remark 10 This lemma is motivated by the truncation method presented in [10] (see also [9]).

Proof. Applying Proposition 4 with $b=\frac{1}{2}$, $\alpha=2^k$, one has

$$|\{M^{\Delta}u_k>2^k\}|\leq a|\{M^{\Delta}u_k>2^{k-1}\}|+|\{u_k^{\sharp,\Delta}>(1-c)2^k\}|,$$

with $a=2^{N+1}c$, where $0<c<\frac{1}{2}$ is a constant defined later. Consequently,

$$2^{kq}|\{M^{\Delta}u_k>2^k\}|\leq a2^{kq}|\{M^{\Delta}u_k>2^{k-1}\}|+2^{kq}|\{u_k^{\sharp,\Delta}>2^{k-1}\}|. \tag{3.4}$$

Take $k_0\in\mathbb{Z}$ such that $2^{k_0-3}\leq\delta<2^{k_0-2}$. Then, from the definition of $M^{\Delta}u_k$, one has

$$\sum_{k_0}^{m}2^{kq}|\{M^{\Delta}u_k>2^k\}|\gtrsim\sum_{k_0}^{m}2^{kq}|\{u_k>2^k\}|\gtrsim\int_{2^{k_0+1}}^{2^{m+1}}t^{q-1}|\{|v|>t\}|\,dt, \tag{3.5}$$

for all $m \geq k_0 + 1$. We claim that

$$2^{kq}|\{M^{\Delta}u_k > 2^{k-1}\}| \lesssim \int_{\mathbb{R}^N} |u_k|^q \, dx. \tag{3.6}$$

If $q = 1$, this inequality follows directly from the property of maximal functions (see e.g. [16]). As $q > 1$, by the theory of maximal functions, one also has

$$2^{kq}|\{M^{\Delta}u_k > 2^{k-1}\}| \lesssim \int_{\mathbb{R}^N} |M^{\Delta}u_k|^q \, dx \lesssim \int_{\mathbb{R}^N} |u_k|^q \, dx.$$

Thus (3.6) holds.

As a consequence of (3.6), one has

$$\sum_{k_0}^{m} 2^{kq}|\{M^{\Delta}u_k > 2^{k-1}\}| \lesssim \sum_{k_0}^{m} \int_{\mathbb{R}^N} |u_k|^q \, dx \lesssim \int_{2^{k_0}}^{\infty} t^{q-1}|\{|v| > t\}| \, dt, \tag{3.7}$$

for all $m \geq k_0 + 1$. Combining (3.4), (3.5) and (3.7) yields

$$\int_{2^{k_0+1}}^{2^{m+1}} t^{q-1}|\{|v| > t\}| \, dt \lesssim a \int_{2^{k_0}}^{\infty} t^{q-1}|\{|v| > t\}| \, dt + \sum_{k_0}^{m} 2^{kq}|\{u_k^{\sharp,\Delta} > 2^{k-1}\}|. \tag{3.8}$$

However, since $r \geq 1$, using (3.2), one gets

$$\sum_{k_0}^{m} 2^{kq}\{u_k^{\sharp,\Delta} > 2^{k-1}\} \lesssim \sum_{k_0}^{m} \|g_k\|_{L^1}^r \leq \|g\|_{L^1}^r. \tag{3.9}$$

Combining (3.8) and (3.9) implies

$$\int_{2^{k_0+1}}^{\infty} t^{q-1}|\{|v| > t\}| \, dt \leq C \left(a \int_{2^{k_0}}^{\infty} t^{q-1}|\{|v| > t\}| \, dt + \|g\|_{L^1}^r \right).$$

Take c such that $Cc2^{N+1} = 1/2$, then $Ca = 1/2$. Then

$$\int_{|v|>\delta} |v|^q \, dx \lesssim \|g\|_{L^1}^r + \delta^q |\{\delta < |v| < \lambda\delta\}|,$$

for some $\lambda > 1$. □

We now present here some applications.

3.1 Proof of Theorem 1

Define

$$u_k = T(2^k, 2^{k+2}, v),$$

where $T(l, m, u)$ is defined in (3.3). Then using (1.7), one has

$$\int_Q \int_Q |u_k(x) - u_k(y)|^p \, dx \, dy \lesssim |Q|^{\frac{N+p}{N}} \underset{|u_k(x)-u_k(y)|>\delta}{\int_Q \int_Q} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy + \delta^p |Q|^2. \tag{3.10}$$

However,

$$\underset{|u_k(x)-u_k(y)|>\delta}{\int_Q \int_Q} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy \lesssim \int_Q \chi_{2^k<|v|\leq 2^{k+2}} \underset{|v(x)-v(y)|>\delta}{\int_Q} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy.$$

Then it follows from (3.10) that

$$u_k^{\sharp,\Delta}(x) \leq C \sup_Q |Q|^{1/N} \left(\fint_Q |h_k| \, dx \right)^{1/p} + C\delta,$$

where supremum is taken over all cubes Q containing x. Here by definition

$$h(x) = \underset{|v(y)-v(x)|>\delta}{\int_{\mathbb{R}^N}} \frac{\delta^p}{|y-x|^{N+p}} \, dy,$$

and

$$h_k(x) = h(x) \chi_{2^k<|v|\leq 2^{k+2}}(x).$$

Applying Vitali's covering theorem (see e.g. [8]), one obtains

$$\{u_k^{\sharp,\Delta} > t\} \lesssim \frac{||h_k||^{\frac{N}{N-p}}}{(t - C\delta)^{\frac{Np}{N-p}}}, \quad \forall \, t > C\delta.$$

Theorem 1 now follows from Lemma 3. □

3.2 Proof of Theorem 3

Define

$$u_k = T(2^k, 2^{k+2}, v),$$

(see (3.3)) and set

$$h(x) = \underset{|v(y)-v(x)|>\delta}{\int} \frac{|v(y) - v(x)|^p}{|x-y|^{N+sp}} \, dy.$$

Then

$$u_k^{\sharp,\Delta}(x) \leq \sup_Q |Q|^{s/N} (\fint_Q |h_k| \, dx)^{1/p} + \delta$$

where supremum is taken over all cubes Q containing x. Here by definition

$$h_k(x) = h(x) \chi_{2^k<|v|\leq 2^{k+2}}(x).$$

Applying Vitali's covering theorem, one obtains

$$\{u_k^{\sharp,\Delta} > t\} \lesssim \frac{||h_k||^{\frac{N}{N-sp}}}{(t-\delta)^{\frac{Np}{N-sp}}}.$$

Theorem 3 now follows from Lemma 3. □

4 Remark on the Rellich-Kondrachov theorem

In this section, we prove Theorem 2. The following lemma is the key of this section.

Lemma 4 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a measurable function and Q be a cube of $\mathbb{R}^N$. Then*

$$\int_Q |f(x) - f_\varepsilon(x)|^p \, dx \lesssim \varepsilon^p \underset{|f(x)-f(y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy + \delta^p |Q|,$$

where

$$f_\varepsilon = \frac{1}{|\varepsilon Q|} f * \chi_\varepsilon$$

with χ_ε is the characteristic function of εQ. Here aQ denotes the cube with the same center as Q but the lenght of each side equal to a times the one of Q.

Proof. Let $(Q_i)_{i\in I}$ be the collection of open cubes whose the length of each side is ε such that

$$Q_i \cap Q_j = \emptyset, \quad \forall\, i \neq j, \quad Q \subset \cup_{i\in I}\overline{Q_i} \quad \text{and} \quad Q \not\subset \cup_{i\in I\setminus\{i_0\}}\overline{Q_i}, \forall\, i_0 \in I.$$

Then

$$\begin{aligned}\int_Q |f(x) - f_\varepsilon(x)|^p \, dx &\le \sum_{i\in I} \int_{Q_i} |f(x) - f_\varepsilon(x)|^p \, dx \\ &\lesssim \frac{1}{\varepsilon^N} \sum_i \int_{3Q_i}\int_{3Q_i} |f(x) - f(y)|^p \, dx \, dy.\end{aligned}$$

On the other hand, by Theorem 3, one has

$$\int_{3Q_i}\int_{3Q_i} |f(x) - f(y)|^p \, dx \, dy \lesssim \varepsilon^{N+p} \underset{|f(x)-f(y)|>\delta}{\int_{3Q_i}\int_{3Q_i}} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy + \varepsilon^N \delta^p.$$

Hence

$$\int_Q |f(x) - f_\varepsilon(x)|^p \, dx \lesssim \varepsilon^p \underset{|f(x)-f(y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+p}} \, dx \, dy + \delta^p |Q|.$$

□ .

Remark 11 The technique used in the proof of Lemma 4 appeared in the one of [5, Theorem 1].

Proof of Theorem 2. Applying Lemma 4, one has for each cube Q of $\mathbb{R}^N$,

$$\int_Q |f_n(x) - f_{n,\varepsilon}(x)|^p \, dx \lesssim \varepsilon^p \underset{|f_n(x)-f_n(y)|>\delta_n}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta_n^p}{|x-y|^{N+p}} \, dx \, dy + \delta_n^p |Q|,$$

where

$$f_{n,\varepsilon} = \frac{1}{|\varepsilon Q|} f_n * \chi_\varepsilon$$

with χ_ε is the characteristic function of εQ. Here aQ denotes the cube with the same center as Q but the length of each side equal to a times the one of Q.

Hence

$$\lim_{\varepsilon \to 0} \left(\limsup_{n\to\infty} \int_Q |f_n(x) - f_{n,\varepsilon}(x)|^p \, dx \right) = 0.$$

By the theorem of Riesz-Frechet-Kolmogorov (see e.g. [6, Théorème IV.25]) and [6, Corollaire IV.27], there exists a sub-sequence (f_{n_k}) of (f_n) and $f \in L^p(\mathbb{R}^N)$ such that f_{n_k} converges to f in $L^p_{\text{loc}}(\mathbb{R}^N)$. The second assertion of Theorem 2 follows from [13, Theorem 3]. □

Remark 12 When $p > 1$, Theorem 2 implies the well-known theorem of Rellich-Kondrachov, since $I_\delta(f) \le C_{N,p} \int_{\mathbb{R}^N} |\nabla f|^p \, dx$.

Remark 13 Theorem 2 is motivated by the works of J. Bourgain, H. Brezis, and P. Mironescu in [2] (see also [14]).

5 Generalizations

In this section, we will prove Theorems 4, 5, and 6. We first give

5.1 Proof of Theorem 6

Without loss of generality, one suppose that $Q = [0,1]^N$. Since F is a non-decreasing function,

$$\iint_{Q^2} \frac{F(|f(x)-f(y)|)}{|x-y|^{N+p}} \, dx \, dy \\ \ge \sum_{n\ge 0} F(2^{-n}) \underset{2^{-n}<|f(x)-f(y)|\le 2^{-n+1}}{\int_Q \int_Q} \frac{1}{|x-y|^{N+p}} \, dx \, dy. \quad (5.1)$$

On the other hand,

$$\sum_{n\geq 0} F(2^{-n}) \int_Q \int_Q\limits_{2^{-n}<|f(x)-f(y)|\leq 2^{-n+1}} \frac{1}{|x-y|^{N+p}}\,dx\,dy$$

$$= \sum_{n\geq 0} F(2^{-n}) \int_Q \int_Q\limits_{|f(x)-f(y)|>2^{-n}} \frac{1}{|x-y|^{N+p}}\,dx\,dy$$

$$- \sum_{n\geq 0} F(2^{-n-1}) \int_Q \int_Q\limits_{|f(x)-f(y)|>2^{-n}} \frac{1}{|x-y|^{N+p}}\,dx\,dy$$

$$- F(1) \int_Q \int_Q\limits_{|f(x)-f(y)|>2} \frac{1}{|x-y|^{N+p}}\,dx\,dy$$

$$= \sum_{n\geq 0} [F(2^{-n}) - F(2^{-n-1})] \int_Q \int_Q\limits_{|f(x)-f(y)|>2^{-n}} \frac{1}{|x-y|^{N+p}}\,dx\,dy$$

$$- F(1) \int_Q \int_Q\limits_{|f(x)-f(y)|>2} \frac{1}{|x-y|^{N+p}}\,dx\,dy.$$

Hence, from (5.1),

$$\int_Q \int_Q \frac{F(|f(x)-f(y)|)}{|x-y|^{N+p}}\,dx\,dy + F(1) \int_Q \int_Q\limits_{|f(x)-f(y)|>2} \frac{1}{|x-y|^{N+p}}\,dx\,dy$$

$$\geq \sum_{n\geq 0} [F(2^{-n}) - F(2^{-n-1})] \int_Q \int_Q\limits_{|f(x)-f(y)|>2^{-n}} \frac{1}{|x-y|^{N+p}}\,dx\,dy. \quad (5.2)$$

Applying Theorem 3, one has

$$\int_Q \int_Q\limits_{|f(x)-f(y)|>2^{-n}} \frac{1}{|x-y|^{N+p}}\,dx\,dy + 1 \gtrsim 2^{np} \int_Q \int_Q |f(x)-f(y)|^p\,dx\,dy.$$

This implies

$$\sum_{n\ge 0}[F(2^{-n})-F(2^{-n-1})]\underset{|f(x)-f(y)|>2^{-n}}{\int_Q\int_Q}\frac{1}{|x-y|^{N+p}}\,dx\,dy$$

$$+\sum_{n\ge 0}[F(2^{-n})-F(2^{-n-1})]$$

$$\gtrsim \sum_{n\ge 0}2^{np}[F(2^{-n})-F(2^{-n-1})]\int_Q\int_Q|f(x)-f(y)|^p\,dx\,dy. \tag{5.3}$$

However,

$$\sum_{n\ge 0}2^{np}[F(2^{-n})-F(2^{-n-1})]=F(1)+\sum_{n\ge 1}2^{np}(1-2^{-p})F(2^{-n})$$

$$\gtrsim F(1)+\int_0^1 F(t)t^{-(p+1)}\,dt,$$

and

$$\sum_{n\ge 0}[F(2^{-n})-F(2^{-n-1})]=F(1).$$

Thus it follows from (5.3) that

$$\sum_{n\ge 0}[F(2^{-n})-F(2^{-n-1})]\underset{|f(x)-f(y)|>2^{-n}}{\int_Q\int_Q}\frac{1}{|x-y|^{N+p}}\,dx\,dy+F(1)$$

$$\gtrsim \left(F(1)+\int_0^1 F(t)t^{-(p+1)}\,dt\right)\underset{|f(x)-f(y)|>2}{\int_Q\int_Q}\frac{1}{|x-y|^{N+p}}\,dx\,dy. \tag{5.4}$$

Combining (5.2) and (5.4) yields

$$\Big(F(1)+\int_0^1 F(t)t^{-(p+1)}\,dt\Big)\int_Q\int_Q|f(x)-f(y)|^p\,dx\,dy$$

$$\lesssim \int_Q\int_Q\frac{F(|f(x)-f(y)|)}{|x-y|^{N+p}}\,dx\,dy+F(1)\underset{|f(x)-f(y)|>2}{\int_Q\int_Q}\frac{1}{|x-y|^{N+p}}\,dx\,dy+F(1).$$

Since F is non-decreasing, this implies

$$\left(F(1)+\int_0^1 F(t)t^{-(p+1)}\,dt\right)\int_Q\int_Q|f(x)-f(y)|^p\,dx\,dy$$

$$\lesssim \int_Q\int_Q\frac{F(|f(x)-f(y)|)}{|x-y|^{N+p}}\,dx\,dy+F(1).$$

□

Remark 14 The techniques used in the proof of Theorem 6 were appeared in the proof of Assertion (b) of [13, Theorem 1].

5.2 Proof of Theorems 4 and 5

Proof of Theorem 4. The conclusion of Theorem 4 follows from the same method used as in the proof of Theorem 1. However, in place of Theorem 3, one uses Theorem 6. The details of the proof are left for the reader. $\square$

Proof of Theorem 5. Applying the same method used in Lemma 4, However, in place of Theorem 3, using Theorem 6 one has

$$\int_Q |f_n - f_{n,\varepsilon}|^p\,dx \lesssim \varepsilon^p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|f_n(x)-f_n(y)|)}{|x-y|^{N+p}}\,dx\,dy + F_n(1)|Q|.$$

for any cube Q of $\mathbb{R}^N$. Hence

$$\lim_{\varepsilon\to 0}\left(\limsup_{n\to\infty}\int_Q |f_n(x) - f_{n,\varepsilon}(x)|^p\,dx\right) = 0.$$

Therefore, there exists a sub-sequence (f_{n_k}) of (f_n) and $f \in L^p(\mathbb{R}^N)$ such that f_{n_k} converges to f in $L^p_{\text{loc}}(\mathbb{R}^N)$ (see the proof of Theorem 2). $\square$

5.3 Some applications

In this section, we give an application of Theorems 4 and 5.

Set

$$F_\varepsilon(t) = \begin{cases} \varepsilon t^{p+\varepsilon} & \text{if } 0 \le t \le 1,\\ \varepsilon & \text{otherwise.}\end{cases}$$

As a consequence of Theorem 4, one has

Corollary 1 *Let $g \in L^p(\mathbb{R}^N)$ $(1 \le p < N)$ and $0 < \varepsilon < 1$. Assume that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_\varepsilon(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

Then $g \in L^q(\mathbb{R}^N)$ with $q = \frac{Np}{N-p}$ and there exist there positive constants C, C_1, C_2 depending only on N and p, such that

$$C\int_{|g|>\varepsilon} |g|^q\,dx \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_\varepsilon(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy + \varepsilon^q|\{C_1\varepsilon < |g| < C_2\varepsilon\}|.$$

Theorem 5 has the following consequence.

Corollary 2 *Let* (g_n) *be a sequence of functions in* $L^p(\mathbb{R}^N)$ $(1 \le p < N)$ *and* (ε_n) *be a positive sequence converging to 0. Assume that*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_{\varepsilon_n}(|g_n(x)-g_n(y)|)}{|x-y|^{N+p}}\,dx\,dy < +\infty.$$

Then there exists a subsequence g_{n_k} *of* g_n *and* $g \in L^p(\mathbb{R}^N)$ *such that* (g_{n_k}) *converges to* g *in* $L^p_{\rm loc}(\mathbb{R}^N)$.

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