A representation formula for the voltage perturbations caused by diametrically small conductivity inhomogeneities. Proof of uniform validity.

Hoai-Minh Nguyen and Michael S. Vogelius *

September 4, 2008

Abstract

We revisit the asymptotic formulas originally derived in [5] and [7]. These formulas concern the perturbation in the voltage potential caused by the presence of diametrically small conductivity inhomogeneities. We significantly extend the validity of the previously derived formulas, by showing that they are asymptotically correct, uniformly with respect to the conductivity of the inhomogeneities. We also extend the earlier formulas by allowing the conductivities of the inhomogeneities to be completely arbitrary L^∞, positive definite, symmetric matrix-valued functions. We briefly discuss the relevance of the uniform asymptotic validity, and the admission of arbitrary anisotropically conducting inhomogeneities, as far as applications of the perturbation formulas to "approximate cloaking" are concerned.

1 Introduction

Asymptotic formulas that quantify the effect of small conductivity inhomogeneities on the voltage potential of an electrical conductor have recently received quite a bit of attention, see for instance [1, 4, 5] and references therein. One important application of such formulas has been the approximate solution of the Electrical Impedance Tomography problem, namely : "to determine the location and

*Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

(some) geometric properties of the inhomogeneities from boundary measurements of voltages and current fluxes" [3]. Another more recent application is the precise estimation of the degree of near-invisibility associated with approximate cloaks obtained by so-called "mapping techniques", see [10]. Let Ω be a connected, bounded, smooth domain in $I\!R^d$, $d = 2, 3$, let γ_0 be a smooth background conductivity, and let $f \in H^{-1/2}(\partial\Omega)$ be a prescribed normal boundary flux (with $\int_{\partial\Omega} f \; d\sigma = 0$). u_0 denotes the background voltage potential, *i.e.*, the solution to

$$\nabla \cdot (\gamma_0 \nabla u_0) = 0 \quad \text{in } \Omega \quad , \quad (\gamma_0 \nabla u_0) \cdot n = f \quad \text{on } \partial\Omega \quad , \tag{1.1}$$

say, with $\int_{\partial\Omega} u_0 \; d\sigma = 0$. We could allow any finite number of (well separated) conductivity inhomogeneities inside Ω – but for simplicity let us assume there is only one (the principal effect of a finite number would simply be the sum of the individual effects). This conductivity inhomogeneity has small diameter (say, of magnitude of $0 < \rho < 1$). We shall denote the open set occupied by the inhomogeneity D_ρ. The conductivity inside D_ρ is given by the symmetric, positive definite matrix-valued function $\gamma_{1,\rho}$. We define γ_ρ to be the conductivity

$$\gamma_\rho = \begin{cases} \gamma_0 \text{ in } \Omega \setminus D_\rho \\ \gamma_{1,\rho} \text{ in } D_\rho \end{cases} \quad . \tag{1.2}$$

u_ρ denotes the voltage potential corresponding to the conductivity distribution γ_ρ, *i.e.*, the solution to

$$\nabla \cdot (\gamma_\rho \nabla u_\rho) = 0 \quad \text{in } \Omega \quad , \quad (\gamma_\rho \nabla u_\rho) \cdot n = f \quad \text{on } \partial\Omega \quad , \tag{1.3}$$

with $\int_{\partial\Omega} u_\rho \; d\sigma = 0$. Initially we shall just assume that D_ρ is contained in a small ball, *i.e.*, in a set of the form $x_0 + \rho B_1$, where x_0 is a point in Ω, B_1 is the unit ball, centered at the origin, and ρ is taken sufficiently small that $x_0 + \rho B_1 \subset\subset \Omega$. For simplicity let us assume Ω contains the origin, and that $x_0 = 0$. By B_δ we shall denote the ball of radius δ, centered at the origin. The first result in this paper (Theorem 1 in section 2) asserts that, given any positive s and δ

$$\|u_\rho - u_0\|_{H^s(\Omega \setminus \overline{B_\delta})} \leq C \rho^d \|f\|_{H^{-1/2}(\partial\Omega)} \quad , \tag{1.4}$$

with a constant C that is *independent* of $\gamma_{1,\rho}$, ρ and f, but depends on Ω, δ, s, and the background conductivity γ_0. The novelty here is that the constant C is independent of $\gamma_{1,\rho}$, the (ρ dependent) conductivity of the inhomogeneity. If we only consider a conductivity γ_1, that is isotropic and independent of ρ, and we do not insist that the constant C be independent of γ_1, then the estimate (1.4) follows immediately from the representation formula(s) proven in [4, 5]. However, it is exactly the dependence of $\gamma_{1,\rho}$ on ρ, and the independence of the constant C of $\gamma_{1.\rho}$ that is important for applications to cloaking. More precisely: to obtain meaningful estimates of the degree of near-invisibility associated with certain approximate cloaks constructed by "mapping techniques" it is most convenient to have an estimate for $u_\rho - u_0$ that is uniform in $\gamma_{1,\rho}$. For instance, in [10] we could have used the fact that C is independent of $\gamma_{1,\rho}$ to give a much more direct proof of our near-invisibility estimate – as it were (without recognizing this uniformity) we had to use a somewhat indirect argument based on monotonicity and the validity of the estimate in the two degenerate isotropic cases, $\gamma_1 = 0$, and $\gamma_1 = \infty$.

Having proven the uniform estimate (1.4) we then return to consider the question of uniform validity of asymptotic formulas such as that derived in [5]. For that purpose we consider D_ρ of the form $D_\rho = \rho D \subset\subset \Omega$, where D is a bounded, simply connected, smooth domain. For simplicity we take γ_0 to be a constant positive definite symmetric matrix, whereas we permit $\gamma_{1,\rho}$ to be an arbitrary positive definite symmetric matrix-valued L^∞ function. The representation formula from [5] asserts that for constant, isotropic γ_0 and γ_1 (ρ independent)

$$u_\rho(x) = u_0(x) + \rho^d |D| \nabla_y G(x,0) \cdot (\gamma_0 - \gamma_1) M \nabla u_0(0) + o(\rho^d) \quad , \tag{1.5}$$

where $o(\rho^d)/\rho^d \to 0$ as $\rho \to 0$. The function $G(x,y)$ is a particular Green's function (the so-called Neumann function) for the Laplacian on the domain Ω, namely the solution to

$$\Delta_x G(x,y) = -\delta_{\{x=y\}} \quad , \qquad \text{with } \frac{\partial}{\partial n_x} G(x,y) = -\frac{1}{|\partial\Omega|} \quad , \tag{1.6}$$

normalized by $\int_{\partial\Omega} G(x,y)\ d\sigma_x = 0$. In order to calculate the matrix M let ϕ_k denote the solution to

$$\nabla \cdot (\gamma \nabla \phi_k) = 0 \quad \text{in } I\!R^d \quad , \quad \phi_k(z) - z_k \to 0 \quad \text{as } |z| \to \infty \quad , \tag{1.7}$$

with γ denoting the rescaled conductivity function

$$\gamma = \begin{cases} \gamma_0 \text{ in } I\!R^d \setminus D \\ \gamma_1 \text{ in } D \end{cases} \quad .$$

The symmetric, positive definite matrix M from [5] is given by

$$M_{jk} = \frac{1}{|D|} \int_D \frac{\partial \phi_k}{\partial z_j}\ dz \quad .$$

In [5] it is shown that the matrix M (corresponding to constant isotropic γ_0 and γ_1) is a function of the single scalar variable $c = \gamma_1/\gamma_0$, and it is shown that $(1 - \frac{\gamma_1}{\gamma_0})M = (1-c)M(c)$ has finite limits as $c \to 0$ and $c \to \infty$. These limits are exactly the symmetric, positive definite matrices that appear in the asymptotic formulas for the two degenerate cases, $\gamma_1 = 0$, and $\gamma_1 = \infty$ (see [7]). The second term in the right hand side of (1.5) is therefore of order ρ^d uniformly in γ_1 – furthermore it will also, uniformly in γ_1, represent the leading term of $u_\rho - u_0$ provided the remainder term $o(\rho^d)$ can be shown to have the property that $o(\rho^d)/\rho^d \to 0$, uniformly in γ_1, as $\rho \to 0$. This uniform smallness assertion (for variable, ρ dependent, anisotropic $\gamma_{1,\rho}$) is exactly the content of Theorem 2 in section 3.

It is well known that a formula similar to (1.5) (and a bound similar to (1.4)) holds for arbitrarily shaped, volumetrically small inhomogeneities, with $\rho^d|D|$ replaced by $|D_\rho|$, provided $\gamma_{1,\rho}$ stays bounded and bounded away from zero (*cf.* [4]). However, for these results to be valid uniformly in $\gamma_{1,\rho}$ a condition of the type $D_\rho = \rho D$ (or $D_\rho \subset \rho D$) is absolutely essential. To see this consider D_ρ in the form of a thin, "square" sheet $(-1,1)^{d-1} \times (-\rho, \rho)$ (or a smoothed-out version of this). For a fixed ρ the solutions corresponding to a sequence of conductivity problems with (isotropic) conductivities approaching $+\infty$ will converge to a solution to the conductivity problem in $\Omega \setminus \overline{D_\rho}$ that is constant on ∂D_ρ. It is therefore not very difficult to see that we may pick a sequence $\rho_n \to 0$ and a sequence of isotropic (constant) conductivities $\gamma_{1,n} \to \infty$ such that the corresponding sequence of solutions to the conductivity problems approaches a function that solves $\nabla \cdot (\gamma_0 \nabla w_0) = 0$ in $\Omega \setminus ([-1,1]^{d-1} \times \{0\})$, and is constant on $[-1,1]^{d-1} \times \{0\}$. Since this limit generically is not γ_0-harmonic in all of Ω, and thus not equal to u_0, it follows that the estimate (1.4), or a formula like (1.5) (with $\rho^d|D|$ replaced by $|D_\rho|$) cannot hold uniformly in γ_1 for D_ρ of the form $D_\rho = (-1,1)^{d-1} \times (-\rho, \rho)$.

In section 4 we show that the condition $D_\rho = \rho D$ may be slightly relaxed without affecting the uniform validity of the principal two terms of the asymptotic expansion of u_ρ. To be precise, Theorem 3 asserts that the result in Theorem 2 still remains valid for domains D_ρ that satisfy $(1 - r_\rho)\rho D \subset D_\rho \subset (1 + r_\rho)\rho D$ with $r_\rho \to 0$ as $\rho \to 0$.

We conclude the main part of this paper with a brief discussion of potential applications of our results to approximate cloaking. The final appendix of this paper contains a number of results concerning solvability, uniquess and representation formulas for exterior problems, that were crucial for the analysis in section 3.

2 A preliminary uniform estimate

In this section, γ_0 denotes a smooth (say, C^∞) symmetric, positive definite matrix-valued function, defined on $\overline{\Omega}$, and $\gamma_{1,\rho}$ denotes a symmetric (uniformly) positive definite matrix-valued L^∞ function defined on D_ρ. The conductivity γ_ρ is given by (1.2). To simplify notation concerning we introduce

$$C^\infty_+(\overline{\Omega}) = \left(C^\infty(\overline{\Omega})\right)^{d\times d} \cap \{ \ \gamma(x) \ \text{ symmetric, positive definite, } \ \min \ \gamma > 0 \ \} \ ,$$

and

$$L^\infty_+(D_\rho) = (L^\infty(D_\rho))^{d\times d} \cap \{ \ \gamma(x) \ \text{ symmetric, positive definite, ess inf}\, \gamma > 0 \ \} \ .$$

Here $\min \gamma$ signifies the largest real number m, such that $\xi^t \gamma(x) \xi \geq m|\xi|^2$ for all $\xi \in I\!R^d$, and all $x \in \overline{\Omega}$, and ess inf γ denotes the supremum of the set of real numbers m for which $\xi^t \gamma(x)\xi \geq m|\xi|^2$ for all $\xi \in I\!R^d$, and almost all $x \in \Omega$. Let F be an element of $L^2(\Omega)$, with support inside $\Omega \setminus \overline{B_\delta}$, for some $\delta > 0$, and let f be an element of $H^{-1/2}(\partial\Omega)$, with $\int_\Omega F\,dx + \int_{\partial\Omega} f\,d\sigma = 0$. Consider the standard weak solution, $v_\rho \in H^1(\Omega)$, to the boundary value problem

$$\nabla \cdot (\gamma_\rho \nabla v_\rho) = F \quad \text{in } \Omega \quad , \quad (\gamma_\rho \nabla v_\rho) \cdot n = f \quad \text{on } \partial\Omega \quad , \tag{2.1}$$

normalized by $\int_{\partial\Omega} v_\rho \, d\sigma = 0$. Let $v_0 \in H^1(\Omega)$ denote the solution to the corresponding problem with γ_ρ replaced by the background conductivity γ_0,

$$\nabla \cdot (\gamma_0 \nabla v_0) = F \quad \text{in } \Omega \quad , \quad (\gamma_0 \nabla v_0) \cdot n = f \quad \text{on } \partial\Omega \quad , \tag{2.2}$$

normalized by $\int_{\partial\Omega} v_0 \, d\sigma = 0$. These two solutions are also the minimizers of the corresponding energies

$$E_\rho(v) = \frac{1}{2}\int_\Omega < \gamma_\rho \nabla v, \nabla v > \ dx + \int_\Omega Fv\ dx - \int_{\partial\Omega} fv\ d\sigma \quad ,$$

and

$$E_0(v) = \frac{1}{2}\int_\Omega < \gamma_0 \nabla v, \nabla v > \ dx + \int_\Omega Fv\ dx - \int_{\partial\Omega} fv\ d\sigma \quad ,$$

in $H^1(\Omega) \cap \{\int_{\partial\Omega} v\ d\sigma = 0\}$. The smoothness of γ_0 is needed to insure that v_0 be smooth, and that all the "error" norms are equivalent by elliptic regularity estimates. As a first result in this section we shall prove

Lemma 1. *Suppose $D_\rho \subset B_{K\rho}$ for some positive constant K. Let γ_ρ be given by (1.2), with $\gamma_0 \in C^\infty_+(\overline{\Omega})$ and $\gamma_{1,\rho} \in L^\infty_+(D_\rho)$. Suppose $\delta > 0$ is sufficiently small that $\overline{B_\delta} \subset \Omega$. Let F be an element of $L^2(\Omega)$, with support inside $\Omega \setminus \overline{B_\delta}$, and let f be an element of $H^{-1/2}(\partial\Omega)$, with $\int_\Omega F\ dx + \int_{\partial\Omega} f\ d\sigma = 0$. There exists a constant ρ_0, independent of $\gamma_{1,\rho}$, F and f, and a constant C, independent of $\gamma_{1,\rho}$, ρ, F and f, such that*

$$|E_\rho(v_\rho) - E_0(v_0)| \leq C\rho^d \left(\|F\|^2_{L^2(\Omega\setminus\overline{B_\delta})} + \|f\|^2_{H^{-1/2}(\partial\Omega)} \right) \quad , \quad \textit{for} \ \ \rho < \rho_0 \quad . \tag{2.3}$$

The constants ρ_0 and C depend on γ_0, Ω, δ and K.

Proof. Pick $\rho_0 < \delta/2K$ so that $B_{2K\rho}$ is contained in B_δ for $\rho < \rho_0$. We divide the proof of the estimate (2.3) into two separate cases.

The case $E_\rho(v_\rho) \geq E_0(v_0)$. In this case

$$\begin{aligned} |E_\rho(v_\rho) - E_0(v_0)| &= E_\rho(v_\rho) - E_0(v_0) \\ &\leq E_\rho(v^*) - E_0(v_0) \qquad \forall v^* \in H^1(\Omega) \quad , \end{aligned} \tag{2.4}$$

and we proceed to construct an appropriate v^*. Let $0 \leq \chi \leq 1$ denote a smooth cut-off function with

$$\chi \equiv 1 \quad \text{in } B_{K\rho} \quad , \quad \chi \equiv 0 \quad \text{in } \Omega \setminus \overline{B_{2K\rho}} \quad , \quad \text{and} \quad |\nabla \chi| \leq \frac{C}{\rho} \quad \text{everywhere} \quad .$$

Using this χ we define

$$v^* = \chi(x)v_0(0) + (1-\chi(x))v_0(x) \quad .$$

Then

$$\nabla v^* \equiv 0 \text{ in } B_{K\rho} \ \text{(which contains } D_\rho) \quad , \quad \nabla v^* = \nabla v_0 \ \ \forall x \in \Omega \setminus \overline{B_{2K\rho}} \quad , \tag{2.5}$$

and furthermore

$$\begin{aligned} |\nabla v^*(x)|^2 &= |\nabla \chi(x)\,(v_0(0)-v_0(x)) + (1-\chi(x))\nabla v_0(x)|^2 \\ &\le 2\left(|\nabla\chi(x)|^2|v_0(0)-v_0(x)|^2 + |1-\chi(x)|^2|\nabla v_0(x)|^2\right) \\ &\le 2\left(C\|\nabla v_0\|^2_{C^0(\overline{B_{2K\rho}})} + \|\nabla v_0\|^2_{C^0(\overline{B_{2K\rho}})}\right) \quad , \end{aligned} \tag{2.6}$$

for all $x \in B_{2K\rho}$. The constant C is independent of ρ and $\gamma_{1,\rho}$. We note that $v^*(x) = v_0(x)$ for $x \in \partial\Omega$, and for $x \in \Omega \setminus \overline{B_\delta}$, and as a consequence of this and (2.5), (2.6),

$$\begin{aligned} &2\left(E_\rho(v^*) - E_0(v_0)\right) \\ &\quad = \int_\Omega < \gamma_\rho \nabla v^*, \nabla v^* > \ dx - \int_\Omega < \gamma_0 \nabla v_0, \nabla v_0 > \ dx \\ &\quad = \int_{B_{2K\rho}\setminus B_{K\rho}} < \gamma_0 \nabla v^*, \nabla v^* > \ dx - \int_{B_{2K\rho}} < \gamma_0 \nabla v_0, \nabla v_0 > \ dx \\ &\quad \le \int_{B_{2K\rho}\setminus B_{K\rho}} < \gamma_0 \nabla v^*, \nabla v^* > \ dx \\ &\quad \le C\rho^d \|\nabla v_0\|^2_{C^0(\overline{B_{2K\rho}})} \le C\rho^d\left(\|F\|^2_{L^2(\Omega\setminus\overline{B_\delta})} + \|f\|^2_{H^{-1/2}(\partial\Omega)}\right) \quad , \end{aligned}$$

with C independent of ρ and $\gamma_{1,\rho}$. This verifies (2.3) in case $E_\rho(u_\rho) > E_0(v_0)$.

The case $E_\rho(v_\rho) < E_0(v_0)$. In this case

$$|E_\rho(v_\rho) - E_0(v_0)| = -E_\rho(v_\rho) + E_0(v_0) \quad , \tag{2.7}$$

and to get an estimate of the type $C\rho^d\left(\|F\|^2_{L^2(\Omega\setminus\overline{B_\delta})} + \|f\|^2_{H^{-1/2}(\partial\Omega)}\right)$ that is independent of $\gamma_{1,\rho}$, we shall introduce the dual variational principle. Let V denote the set

$$V = \{\ \sigma \in (L^2(\Omega))^d \ : \ \nabla\cdot\sigma = F \quad \text{in } \Omega \quad , \quad \sigma\cdot n = f \quad \text{on } \partial\Omega\ \} \quad .$$

Then it is well known that

$$\begin{aligned} E_\rho(v_\rho) &= -\frac{1}{2}\int_\Omega < \gamma_\rho \nabla v_\rho, \nabla v_\rho > \ dx \\ &= \max_{\sigma\in V} -\frac{1}{2}\int_\Omega < \gamma_\rho^{-1}\sigma, \sigma > \ dx \\ &\ge -\frac{1}{2}\int_\Omega < \gamma_\rho^{-1}\sigma^*, \sigma^* > \ dx \qquad \forall \sigma^* \in V \quad . \end{aligned} \tag{2.8}$$

Since

$$E_0(v_0) = -\frac{1}{2}\int_\Omega < \gamma_0 \nabla v_0, \nabla v_0 > \ dx$$

it follows from (2.7) and (2.8) that

$$|E_\rho(v_\rho) - E_0(v_0)| \leq \frac{1}{2}\int_\Omega < \gamma_\rho^{-1}\sigma^*, \sigma^* > \ dx - \frac{1}{2}\int_\Omega < \gamma_0 \nabla v_0, \nabla v_0 > \ dx \ \ , \quad (2.9)$$

for any $\sigma^* \in V$. We proceed to construct $\sigma^* \in V$ for which $\int_\Omega < \gamma_\rho^{-1}\sigma^*, \sigma^* > \ dx$ is near $\int_\Omega < \gamma_0 \nabla v_0, \nabla v_0 > \ dx$. Let W_ρ denote the solution to

$$\begin{aligned} \Delta W_\rho &= 0 \quad \text{in } B_{2K} \setminus \overline{B_K} \ \ , \\ \frac{\partial W_\rho}{\partial n} &= 0 \quad \text{on } \partial B_K \ \ , \\ \frac{\partial W_\rho}{\partial n} &= (\gamma_0 \nabla v_0(\rho x)) \cdot n \quad \text{on } \partial(B_{2K}) \ \ . \end{aligned}$$

This problem has a solution, since $\int_{\partial(B_{2K\rho})}(\gamma_0 \nabla v_0) \cdot n \ d\sigma = 0$. The solution is unique up to a constant, and it satisfies the estimate

$$\begin{aligned} \|\nabla W_\rho\|^2_{L^2(B_{2K}\setminus \overline{B_K})} &\leq C\|\nabla v_0\|^2_{C^0(\partial(B_{2K\rho}))} \\ &\leq C\left(\|F\|^2_{L^2(\Omega\setminus\overline{B_\delta})} + \|f\|^2_{H^{-1/2}(\partial\Omega)}\right) \ \ . \end{aligned}$$

It follows immediately by rescaling that $w_\rho = \rho W_\rho(\frac{x}{\rho})$ satisfies

$$\begin{aligned} \Delta w_\rho &= 0 \quad \text{in } B_{2K\rho} \setminus \overline{B_{K\rho}} \ \ , \\ \frac{\partial w_\rho}{\partial n} &= 0 \quad \text{on } \partial(B_{K\rho}) \ \ , \\ \frac{\partial w_\rho}{\partial n} &= (\gamma_0 \nabla v_0) \cdot n \quad \text{on } \partial(B_{2K\rho}) \ \ , \end{aligned}$$

and

$$\|\nabla w_\rho\|^2_{L^2(B_{2K\rho}\setminus\overline{B_{K\rho}})} \leq C\rho^d \left(\|F\|^2_{L^2(\Omega\setminus\overline{B_\delta})} + \|f\|^2_{H^{-1/2}(\partial\Omega)}\right) \ \ , \quad (2.10)$$

with a constant C that is independent of ρ and $\gamma_{1,\rho}$. We now define the field σ^* by the formula

$$\sigma^* = \begin{cases} 0 & \text{in} \quad B_{K\rho} \\ \nabla w_\rho & \text{in} \quad B_{2K\rho} \setminus \overline{B_{K\rho}} \\ \gamma_0 \nabla v_0 & \text{in} \quad \Omega \setminus \overline{B_{2K\rho}} \end{cases} \ \ .$$

This field is clearly in $[L^2(\Omega)]^d$, and it satisfies $\nabla \cdot \sigma^* = F$ in Ω as well as $\sigma^* \cdot n = (\gamma_0 \nabla v_0) \cdot n = f$ on $\partial\Omega$, *i.e.*, σ^* is an element of V. By using σ^* as a test field in (2.9) we get

$$
\begin{aligned}
&|E_\rho(v_\rho) - E_0(v_0)| \\
&\quad \le \frac{1}{2}\int_\Omega < \gamma_\rho^{-1}\sigma^*, \sigma^* > \; dx - \frac{1}{2}\int_\Omega < \gamma_0 \nabla v_0, \nabla v_0 > \; dx \\
&\quad = \frac{1}{2}\int_{B_{2K\rho}\setminus B_{K\rho}} < \gamma_0^{-1}\nabla w_\rho, \nabla w_\rho > \; dx - \frac{1}{2}\int_{B_{2K\rho}} < \gamma_0 \nabla v_0, \nabla v_0 > \; dx \\
&\quad \le \int_{B_{2K\rho}\setminus B_{K\rho}} < \gamma_0^{-1}\nabla w_\rho, \nabla w_\rho > \; dx \\
&\quad \le C\rho^d \left(\|F\|^2_{L^2(\Omega\setminus\overline{B_\delta})} + \|f\|^2_{H^{-1/2}(\partial\Omega)} \right) \quad ,
\end{aligned}
$$

with C independent of ρ and the conductivity $\gamma_{1,\rho}$ of the inhomogeneity D_ρ. For the last inequality we have used the estimate (2.10). This verifies (2.3) in case $E_\rho(u_\rho) < E_0(v_0)$, and thus completes the proof of Lemma 1. □

Let F, f and γ_ρ be as in the preceding lemma. We easily calculate that

$$
\begin{aligned}
&-\int_{\Omega\setminus\overline{B_\delta}} F v_\rho \; dx + \int_{\partial\Omega} f v_\rho \; d\sigma \\
&\quad = \int_\Omega < \gamma_\rho \nabla v_\rho, \nabla v_\rho > \; dx \\
&\quad = -\int_\Omega < \gamma_\rho \nabla v_\rho, \nabla v_\rho > \; dx - 2\int_\Omega F v_\rho \; dx + 2\int_{\partial\Omega} f v_\rho \; d\sigma \\
&\quad = -2E_\rho(v_\rho) \quad ,
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&-\int_{\Omega\setminus\overline{B_\delta}} F v_0 \; dx + \int_{\partial\Omega} f v_0 \; d\sigma \\
&\quad = \int_\Omega < \gamma_0 \nabla v_0, \nabla v_0 > \; dx \\
&\quad = -\int_\Omega < \gamma_0 \nabla v_0, \nabla v_0 > \; dx - 2\int_\Omega F v_0 \; dx + 2\int_{\partial\Omega} f v_0 \; d\sigma \\
&\quad = -2E_0(v_0) \quad .
\end{aligned}
$$

As a consequence

$$
\int_{\Omega\setminus\overline{B_\delta}} F\,(v_\rho - v_0) \; dx - \int_{\partial\Omega} f\,(v_\rho - v_0) \; d\sigma = 2\,(E_\rho(v_\rho) - E_0(v_0)) \quad ,
$$

and so, due to Lemma 1

$$
\begin{aligned}
&\left| \int_{\Omega\setminus\overline{B_\delta}} F\,(v_\rho - v_0) \; dx - \int_{\partial\Omega} f\,(v_\rho - v_0) \; d\sigma \right| \\
&\qquad \le C\rho^d \left(\|F\|^2_{L^2(\Omega\setminus\overline{B_\delta})} + \|f\|^2_{H^{-1/2}(\partial\Omega)} \right) \quad , \qquad (2.11)
\end{aligned}
$$

with C independent of ρ, F, f and $\gamma_{1,\rho}$. If we define the bounded linear operator $A_\rho : (F, f) \to \Big((v_\rho - v_0)|_{\Omega\setminus\overline{B_\delta}}, -(v_\rho - v_0)|_{\partial\Omega}\Big)$ from $L^2(\Omega \setminus \overline{B_\delta}) \times H^{-1/2}(\partial\Omega)$ into $L^2(\Omega \setminus \overline{B_\delta}) \times H^{1/2}(\partial\Omega)$, then (2.11) simply asserts that

$$| < A_\rho(F, f), (F, f) > | \le C\rho^d \left(\|F\|^2_{L^2(\Omega\setminus\overline{B_\delta})} + \|f\|^2_{H^{-1/2}(\partial\Omega)}\right) \quad ,$$

where $< \cdot , \cdot >$ denotes the natural duality between $L^2(\Omega \setminus \overline{B_\delta}) \times H^{1/2}(\partial\Omega)$ and $L^2(\Omega \setminus \overline{B_\delta}) \times H^{-1/2}(\partial\Omega)$. We note that A_ρ is selfadjoint, and by "polarization" it now follows that

$$\sup_{|||(F,f)|||\le 1} \sup_{|||(G,g)|||\le 1} | < A_\rho(F, f), (G, g) > | \le C\rho^d \quad ,$$

with $|||(F, f)||| = \left(\|F\|^2_{L^2(\Omega\setminus\overline{B_\delta})} + \|f\|^2_{H^{-1/2}(\partial\Omega)}\right)^{1/2}$. In other words

$$\begin{aligned} \left(\|v_\rho - v_0\|^2_{L^2(\Omega\setminus\overline{B_\delta})} + \|v_\rho - v_0\|^2_{H^{1/2}(\partial\Omega)}\right)^{1/2} &= \sup_{|||(G,g)|||\le 1} | < A_\rho(F, f), (G, g) > | \\ &\le C\rho^d \quad , \end{aligned}$$

for all (F, f) with $\|F\|^2_{L^2(\Omega\setminus\overline{B_\delta})} + \|f\|^2_{H^{-1/2}(\partial\Omega)} \le 1$, or

$$\begin{aligned} &\left(\|v_\rho - v_0\|^2_{L^2(\Omega\setminus\overline{B_\delta})} + \|v_\rho - v_0\|^2_{H^{1/2}(\partial\Omega)}\right)^{1/2} \\ &\qquad \le C\rho^d \left(\|F\|^2_{L^2(\Omega\setminus\overline{B_\delta})} + \|f\|^2_{H^{-1/2}(\partial\Omega)}\right)^{1/2} \quad , \end{aligned}$$

with C independent of ρ, F, f and $\gamma_{1,\rho}$. Since $v_\rho - v_0$ solves the equation

$$\nabla \cdot (\gamma_0 \nabla(v_\rho - v_0)) = 0 \quad \text{in } \Omega \setminus \overline{B_\delta} \quad , \quad \text{with} \quad (\gamma_0 \nabla(v_\rho - v_0)) \cdot n = 0 \quad \text{on } \partial\Omega \quad ,$$

elliptic regularity theory implies that the above estimate also holds for the Sobolev norm $\left(\|v_\rho - v_0\|^2_{H^s(\Omega\setminus\overline{B_{2\delta}})} + \|v_\rho - v_0\|^2_{H^s(\partial\Omega)}\right)^{1/2}$, *i.e.*,

$$\begin{aligned} &\left(\|v_\rho - v_0\|^2_{H^s(\Omega\setminus\overline{B_{2\delta}})} + \|v_\rho - v_0\|^2_{H^{1/2}(\partial\Omega)}\right)^{1/2} \\ &\qquad \le C\rho^d \left(\|F\|^2_{L^2(\Omega\setminus\overline{B_\delta})} + \|f\|^2_{H^{-1/2}(\partial\Omega)}\right)^{1/2} \\ &\qquad \le C\rho^d \left(\|F\|^2_{L^2(\Omega\setminus\overline{B_{2\delta}})} + \|f\|^2_{H^{-1/2}(\partial\Omega)}\right)^{1/2} \quad , \end{aligned}$$

for any $s \in I\!R_+$ and any F having support inside $\Omega \setminus \overline{B_{2\delta}}$. Replacing δ with $\delta/2$ we have therefore established

Theorem 1. *Suppose $D_\rho \subset B_{K\rho}$ for some positive constant K. Let γ_ρ be given by* (1.2)*, with $\gamma_0 \in C^\infty_+(\overline{\Omega})$ and $\gamma_{1,\rho} \in L^\infty_+(D_\rho)$. Suppose $\delta > 0$ is sufficiently small* *that $\overline{B_\delta} \subset \Omega$. Let F be an element of $L^2(\Omega)$, with support inside $\Omega \backslash \overline{B_\delta}$, and let f be an element of $H^{-1/2}(\partial\Omega)$, with $\int_\Omega F\ dx + \int_{\partial\Omega} f\ d\sigma = 0$. Let v_ρ and v_0 denote the solutions to (2.1) and (2.2) respectively, normalized by $\int_{\partial\Omega} v_\rho\ d\sigma = \int_{\partial\Omega} v_0\ d\sigma = 0$. Given any $s \in I\!\!R_+$ there exists a constant ρ_0, independent of $\gamma_{1,\rho}$, F and f, and a constant C, independent of $\gamma_{1,\rho}$, ρ, F and f, such that*

$$\left(\|v_\rho - v_0\|^2_{H^s(\Omega\backslash\overline{B_\delta})} + \|v_\rho - v_0\|^2_{H^s(\partial\Omega)}\right)^{1/2} \leq C\rho^d \left(\|F\|^2_{L^2(\Omega\backslash\overline{B_\delta})} + \|f\|^2_{H^{-1/2}(\partial\Omega)}\right)^{1/2}$$

for all $\rho < \rho_0$.

Theorem 1 is of independent importance. However for the purpose of this paper it is two corollaries, both pertaining to the special case $F = 0$, that are particularly relevant. The first corollary is of direct relevance to the estimation of the effectivity of approximate cloaks, as briefly discussed in section 5.

Corollary 1. *Suppose $D_\rho \subset B_{K\rho}$ for some positive constant K. Let γ_ρ be given by (1.2), with $\gamma_0 \in C^\infty_+(\overline{\Omega})$ and $\gamma_{1,\rho} \in L^\infty_+(D_\rho)$. Let f be an element of $H^{-1/2}(\partial\Omega)$, with $\int_{\partial\Omega} f\ d\sigma = 0$. Let u_ρ and u_0 denote the solutions to (1.3) and (1.1) respectively, normalized by $\int_{\partial\Omega} u_\rho\ d\sigma = \int_{\partial\Omega} u_0\ d\sigma = 0$. Given any $s \in I\!\!R_+$, and any $\delta \in I\!\!R_+$, there exists a constant ρ_0, independent of $\gamma_{1,\rho}$ and f, and a constant C, independent of $\gamma_{1,\rho}$, f, and ρ, such that*

$$\|u_\rho - u_0\|_{H^s(\Omega\backslash\overline{B_\delta})} + \|u_\rho - u_0\|_{H^s(\partial\Omega)} \leq C\rho^d \|f\|_{H^{-1/2}(\partial\Omega)} \qquad \forall \rho < \rho_0 \quad .$$

The second corollary, which shall prove essential for our analysis in section 3.2, estimates the combined perturbation caused by the small inhomogeneity and a change in the normal flux. In order to formulate this corollary we need some additional notation. Let w_ρ be the solution to

$$\begin{cases} \nabla \cdot (\gamma_\rho \nabla w_\rho) &= 0 \quad \text{in } \Omega \quad , \\ (\gamma_\rho \nabla w_\rho) \cdot n &= g \quad \text{on } \partial\Omega \quad , \end{cases} \tag{2.12}$$

normalized by $\int_{\partial\Omega} w_\rho\ d\sigma = 0$. Here we assume that g is an element of $H^{-1/2}(\partial\Omega)$, with $\int_{\partial\Omega} g\ d\sigma = 0$. It follows immediately that

$$\begin{aligned} &\int_{\Omega\backslash D_\rho} < \gamma_0 \nabla(w_\rho - u_\rho), \nabla(w_\rho - u_\rho) > \ dx \\ &\qquad \leq \int_{\partial\Omega} (g - f)(w_\rho - u_\rho)\ d\sigma \\ &\qquad \leq \|g - f\|_{H^{-1/2}(\partial\Omega)} \|w_\rho - u_\rho\|_{H^{1/2}(\partial\Omega)} \\ &\qquad \leq C\|g - f\|_{H^{-1/2}(\partial\Omega)} \|w_\rho - u_\rho\|_{H^1(\Omega\backslash\overline{B_\delta})} \quad . \end{aligned} \tag{2.13}$$

Due to the fact that $\int_{\partial\Omega} w_\rho\ d\sigma = \int_{\partial\Omega} u_\rho\ d\sigma = 0$ we have

$$\|w_\rho - u_\rho\|_{H^1(\Omega\backslash\overline{B_\delta})} \leq C\|\nabla(w_\rho - u_\rho)\|_{L^2(\Omega\backslash\overline{B_\delta})} \quad .$$

A combination of this with (2.13) immediately gives that there exist constants ρ_0 and C such that

$$\|w_\rho - u_\rho\|_{H^1(\Omega\setminus\overline{B_\delta})} \leq C\|g-f\|_{H^{-1/2}(\partial\Omega)} \quad , \qquad \text{for} \quad \rho < \rho_0 \quad . \tag{2.14}$$

The constants ρ_0 and C are independent of $\gamma_{1,\rho}$,f and g. If we decompose

$$w_\rho - u_0 = (w_\rho - u_\rho) + (u_\rho - u_0) \quad ,$$

and combine Corollary 1 with (2.14) we obtain

Corollary 2. *Suppose $D_\rho \subset B_{K\rho}$ for some positive constant K. Let γ_ρ be given by (1.2), with $\gamma_0 \in C_+^\infty(\overline{\Omega})$ and $\gamma_{1,\rho} \in L_+^\infty(D_\rho)$. Let f , $g \in H^{-\frac{1}{2}}(\partial\Omega)$ with $\int_{\partial\Omega} f d\sigma = \int_{\partial\Omega} g = 0$, and let u_0 and w_ρ in $H^1(\Omega)$ denote the solutions to (1.1) and (2.12), normalized by $\int_{\partial\Omega} u_0 \ d\sigma = \int_{\partial\Omega} w_\rho \ d\sigma = 0$. Given any $\delta > 0$ there exists a constant ρ_0, independent of $\gamma_{1,\rho}$, f and g, and a constant C, independent of $\gamma_{1,\rho}$, f, g and ρ, such that*

$$\|w_\rho - u_0\|_{H^1(\Omega\setminus\overline{B_\delta})} \leq C\left(\|g-f\|_{H^{-1/2}(\partial\Omega)} + \rho^d\|f\|_{H^{-1/2}(\partial\Omega)}\right) \quad ,$$

for all $\rho < \rho_0$.

3 The leading order asymptotics when $D_\rho = \rho D$

We shall now consider the case when the inhomogeneity D_ρ is of the form $D_\rho = \rho D$, for some bounded, simply connected, smooth domain D. We shall examine issues related to the principal term of the expression

$$\frac{1}{\rho^d}(u_\rho - u_0)$$

as $\rho \to 0$. We briefly describe the structure of the expression

$$\frac{1}{\rho^d}(u_\rho - u_0) \quad , \quad \rho \to 0 \quad ,$$

a structure that is already well known, *cf.* [5]. What is not at all known, and what we shall prove here is that for inhomogeneities that are dilatations of a fixed set D, we can describe the expression $(u_\rho - u_0)/\rho^d$ by an explicit (bounded) formula that is asymptotically correct *uniformly* in $\gamma_{1,\rho}$. As pointed out in the introduction such a formula is not available for general D_ρ. For simplicity we shall restrict attention to the case

$$\gamma_\rho(x) = \begin{cases} \gamma_0 & \text{if } x \in \Omega \setminus \rho D \quad , \\ \gamma_{1,\rho}(x) & \text{if } x \in \rho D \quad , \end{cases}$$

where γ_0 is a constant, symmetric, positive definite matrix, and $\gamma_{1,\rho}$ is an arbitrary symmetric (uniformly) positive definite matrix-valued function defined on ρD, in other words

$$\begin{aligned} &\gamma_{1,\rho} \in L^\infty_+(\rho D) \\ &\quad = (L^\infty(\rho D))^{d\times d} \cap \{ \ \gamma(x) \ \text{ symmetric, positive definite, } \ \operatorname{ess\,inf} \gamma > 0 \ \} \ . \end{aligned}$$

By a simple linear change of variables $x' = Lx$, γ_ρ changes into $L\gamma_\rho L^T \circ L^{-1}$, D changes into $L(D)$, and Ω into $L(\Omega)$. Since we can choose L such that $L\gamma_0 L^T = I$ we may thus, without loss of generality, assume that γ_ρ is of the form

$$\gamma_\rho(x) = \begin{cases} \quad I & \text{if } x \in \Omega \setminus \rho D \ , \\ \gamma_{1,\rho}(x) & \text{if } x \in \rho D \ , \end{cases} \tag{3.1}$$

where $\gamma_{1,\rho}$ is an arbitrary matrix-valued function in $L^\infty_+(\rho D)$. The novelty of the present results is the fact that $\gamma_{1,\rho}$ is an arbitrary matrix valued function in $L^\infty_+(\rho D)$, and the fact that all the estimates and convergence (approximation) statements are uniform with respect to $\gamma_{1,\rho} \in L^\infty_+(\rho D)$.
Let γ^*_ρ denote the rescaled coefficient

$$\gamma^*_\rho(z) = \begin{cases} \quad I & \text{if } z \in I\!R^d \setminus D \ , \\ \gamma_{1,\rho}(\rho z) & \text{if } z \in D \ , \end{cases}$$

and let ϕ_k be the solution to

$$\nabla \cdot \left(\gamma^*_\rho \nabla \phi_k\right) = 0 \text{ in } I\!R^d \quad , \quad \text{with } \phi_k(z) - z_k \to 0 \ \text{ as } |z| \to \infty \ . \tag{3.2}$$

We note that $\phi_k(z) = \psi_k(z) + z_k$, where ψ_k satisfies

$$\begin{cases} \Delta\psi_k &= 0 & \text{in } I\!R^d \setminus \overline{D} \ , \\ \nabla \cdot (\gamma_{1,\rho}(\rho\,\cdot)\nabla\psi_k) &= -\nabla \cdot (\gamma_{1,\rho}(\rho\,\cdot)\nabla z_k) & \text{in } D \ , \\ \frac{\partial\psi_k}{\partial n}\Big|_{\text{ext}} - (\gamma_{1,\rho}(\rho\,\cdot)\nabla\psi_k)\cdot n\Big|_{\text{int}} &= (\gamma_{1,\rho}(\rho\,\cdot)n)_k - n_k & \text{on } \partial D \ , \\ [\psi_k] &= 0 & \text{on } \partial D \ , \end{cases} \tag{3.3}$$

and $\psi_k(z) \to 0$ as $|z| \to \infty$. Here $[\psi] = \psi\Big|_{\text{ext}} - \psi\Big|_{\text{int}}$ denotes the jump of the function ψ across ∂D. It is easy to see that the solution ψ_k (up to a constant, for $d = 2$) coincides with the unique solution to (3.3) in $W^1(I\!R^d)$, the existence of which is guaranteed by Proposition 4 in the appendix. We define the function L_ρ:

$$L_\rho(x) = \nabla_y \Phi(x, 0) \cdot \int_D (I - \gamma_{1,\rho}(\rho z))\nabla\phi_k(z)\, dz \,\frac{\partial}{\partial x_k} u_0(0) \ , \tag{3.4}$$

and the function W_ρ:

$$\begin{cases} \Delta W_\rho &= 0 & \text{in } \Omega \ , \\ \frac{\partial W_\rho}{\partial n} &= -\frac{\partial L_\rho}{\partial n} & \text{on } \partial\Omega \ , \end{cases} \tag{3.5}$$

normalized by $\int_{\partial\Omega} W_\rho \, d\sigma = -\int_{\partial\Omega} L_\rho \, d\sigma$. Here $\Phi(x,y)$ is the free space fundamental solution for the Laplacian

$$\Phi(x,y) = \begin{cases} -\frac{1}{2\pi} \ln|x-y| & \text{if } d=2 \quad , \\ \frac{1}{4\pi|x-y|} & \text{if } d=3 \quad . \end{cases}$$

We note that ψ_k, ϕ_k, L_ρ and W_ρ generically all depend on ρ; they would be independent of ρ if we only considered $\gamma_{1,\rho} \in L^\infty_+(\rho D)$ of the form $\gamma_{1,\rho}(x) = \gamma_1(x/\rho)$ for a given fixed $\gamma_1 \in L^\infty_+(D)$ (for instance if we considered inhomogeneities ρD of fixed constant conductivity γ_1). In the case $\gamma_{1,\rho}$ is constant (but possibly dependent on ρ) we define the positive definite matrix M:

$$M_{jk} = \frac{1}{|D|} \int_D \frac{\partial \phi_k}{\partial z_j} \, dz \quad . \tag{3.6}$$

We note that M may depend on ρ. Then

$$L_\rho(x) = |D|(I - \gamma_{1,\rho})_{ij} M_{jk} \frac{\partial}{\partial x_k} u_0(0) \frac{\partial}{\partial y_i} \Phi(x,0) \quad ,$$

and it is easy to see that

$$L_\rho(x) + W_\rho(x) = |D|(I - \gamma_{1,\rho})_{ij} M_{jk} \frac{\partial}{\partial x_k} u_0(0) \frac{\partial}{\partial y_i} G(x,0) \quad , \tag{3.7}$$

where G is the special Green's function introduced in (1.6). Here and in the future we use the Einstein summation convention, *i.e.*, repeated indices (representing integers) in a single term implies summation from 1 to d.

Theorem 2. *Suppose f is in $H^{-1/2}(\partial\Omega)$ with $\int_{\partial\Omega} f \, d\sigma = 0$, and γ_ρ is given by (3.1) with $\gamma_{1,\rho} \in L^\infty_+(\rho D)$. Let u_0 and u_ρ denote the solutions to (1.1) and (1.3), normalized by $\int_{\partial\Omega} u_0 \, d\sigma = \int_{\partial\Omega} u_\rho \, d\sigma = 0$. Let $L_\rho(x)$ and $W_\rho(x)$ be given by (3.4) and (3.5). Then for any fixed $\delta > 0$,*

$$\lim_{\rho \to 0} \int_{\Omega \setminus B_\delta} \left| \frac{1}{\rho^d} \nabla(u_\rho - u_0) - \nabla L_\rho - \nabla W_\rho \right|^2 dx = 0 \quad .$$

The limiting process is uniform in $\gamma_{1,\rho} \in L^\infty_+(\rho D)$ and in $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \quad , \int_{\partial\Omega} f \, d\sigma = 0 \right\}$. That is, for any $\epsilon > 0$ and $\delta > 0$, there exists a positive constant $\tau(\epsilon,\delta)$, such that

$$\int_{\Omega \setminus B_\delta} \left| \frac{1}{\rho^d} \nabla[u_\rho - u_0] - \nabla L_\rho - \nabla W_\rho \right|^2 dx < \epsilon,$$

for all $\rho < \tau$, all $\gamma_{1,\rho} \in L^\infty_+(\rho D)$ and all $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \quad , \int_{\partial\Omega} f \, d\sigma = 0 \right\}$. The term $\nabla L_\rho + \nabla W_\rho$ is bounded in $L^2(\Omega \setminus \overline{B_\delta})$, uniformly with respect to ρ, $\gamma_{1,\rho} \in L^\infty_+(\rho D)$, and $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \quad , \int_{\partial\Omega} f \, d\sigma = 0 \right\}$.

We note that that the function $\frac{1}{\rho^d}(u_\rho - u_0) - (L_\rho + W_\rho)$ is harmonic in $\Omega \setminus \overline{B_{\delta/2}}$, with $\frac{\partial}{\partial n}((u_\rho - u_0) - (L_\rho + W_\rho)) = 0$ on $\partial\Omega$. A combination of standard elliptic theory and the energy estimate of Theorem 2 therefore yields similar estimates in any $H^s(\Omega \setminus \overline{B_\delta})$ (or $H^s(\partial\Omega)$) norm, as stated in the following corollary.

Corollary 3. *Suppose f is in $H^{-1/2}(\partial\Omega)$ with $\int_{\partial\Omega} f \, d\sigma = 0$, and γ_ρ is given by (3.1) with $\gamma_{1,\rho} \in L^\infty_+(\rho D)$. Let u_0 and u_ρ denote the solutions to (1.1) and (1.3), normalized by $\int_{\partial\Omega} u_0 \, d\sigma = \int_{\partial\Omega} u_\rho \, d\sigma = 0$. Let $L_\rho(x)$ and $W_\rho(x)$ be given by (3.4) and (3.5), and normalized by $\int_{\partial\Omega}(L_\rho + W_\rho) \, d\sigma = 0$. Then for any fixed $\delta > 0$, and any $s \in I\!R_+$*

$$\|u_\rho - u_0 - \rho^d(L_\rho + W_\rho)\|_{H^s(\Omega\setminus\overline{B_\delta})} = \rho^d o(1) \quad ,$$

where the term $o(1)$ tends to zero uniformly in $\gamma_{1,\rho} \in L^\infty_+(\rho D)$ and in $f \in \left\{\|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ \ , \int_{\partial\Omega} f \, d\sigma = 0\right\}$, as $\rho \to 0$. As a consequence we also have

$$\|u_\rho - u_0 - \rho^d(L_\rho + W_\rho)\|_{H^s(\partial\Omega)} = \rho^d o(1) \quad ,$$

for any $s \in I\!R_+$. The term $L_\rho + W_\rho$ is bounded in $H^s(\Omega\setminus\overline{B_\delta})$ (and in $H^s(\partial\Omega)$) uniformly with respect to ρ, $\gamma_{1,\rho} \in L^\infty_+(\rho D)$, and $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ , \int_{\partial\Omega} f \, d\sigma = 0 \right\}$.

As stated the terms L_ρ and W_ρ in general depend on ρ; Theorem 2 and Corollary 3 therefore do not assert the existence of a limit of $(u_\rho - u_0)/\rho^d$, as $\rho \to 0$. Due to the boundedness of the term $L_\rho + W_\rho$ we may arrive at a limit by extraction of a subsequence, much as was the case with the representation formulas in [4]. The boundedness of the term $L_\rho + W_\rho$ as stated in Theorem 2 and Corollary 3 is equivalent to the boundedness of the matrix $\int_D (I - \gamma_{1,\rho}(\rho z))_{ij} \frac{\partial}{\partial z_j} \phi_k(z) \, dz$ as a function of ρ and $\gamma_{1,\rho}$. The assumption that γ_0 be constant (the identity, after a linear change of variables) may also be relaxed. For smooth γ_0 we may carry out a "freezing of the coefficient"-arguement, much like in [5]. The formulation of Theorem 2 and Corollary 3 would not change, but the identity matrix appearing in the definition of γ^*_ρ and in the formula (3.4) would be replaced by $\gamma_0(0)$, the first equation of (3.3) would become $\nabla \cdot (\gamma_0(0)\nabla\psi_k) = 0$ in $I\!R^d \setminus \overline{D}$, and the transmission condition of (3.3) would be replaced by

$$(\gamma_0(0)\nabla\psi_k) \cdot n\Big|_{\text{ext}} - (\gamma_{1,\rho}(\rho\,\cdot)\nabla\psi_k) \cdot n\Big|_{\text{int}} = (\gamma_{1,\rho}(\rho\,\cdot)n)_k - (\gamma_0(0)n)_k \quad .$$

Furthermore Φ would be replaced by a Green's function for the operator $\nabla \cdot (\gamma_0 \nabla \cdot)$, and W_ρ would satisfy $\nabla \cdot (\gamma_0 \nabla W_\rho) = 0$ in Ω, $\nabla W_\rho \cdot n = -\nabla L_\rho \cdot n$ on $\partial\Omega$. As mentioned earlier, the terms L_ρ and W_ρ are independent of ρ if $\gamma_{1,\rho}$ is of the form $\gamma_{1,\rho}(x) = \gamma_1(x/\rho)$, with γ_1 independent of ρ. In that case it would indeed be possible to extend the results proven here to any order in ρ, in other words to prove that (for a single inhomogeneity of the form ρD) one has an asymptotic expansion to any order (as already established in [1]) which is uniform in γ_1 and $f \in \{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ , \ \int_{\partial\Omega} f \ d\sigma = 0 \}$.

Before proceeding with the proof of Theorem 2 we introduce some auxiliary functions. The function J is defined as follows: if $d = 3$, then J is the $W^1(I\!R^3 \setminus \overline{D})$ solution of

$$\begin{cases} \Delta J &= 0, \quad \text{in } I\!R^3 \setminus \overline{D} \ , \\ \dfrac{\partial J}{\partial n} &= 1 \quad \text{on } \partial D \ , \end{cases}$$

if $d = 2$, then $J = 0$. The existence and uniqueness of J is assured by Proposition 3 in the appendix. The function $\hat{H}_{1,\rho}$ is the $W^1(I\!R^d \setminus \overline{D})$ solution of

$$\begin{cases} \Delta \hat{H}_{1,\rho} &= 0, \qquad \text{in } I\!R^d \setminus \overline{D} \ , \\ \hat{H}_{1,\rho}(x) &= \dfrac{1}{\rho}(u_0(\rho x) - u_0(0)), \quad \text{on } \partial D \ , \end{cases}$$

for $d = 2$ as well as $d = 3$. The existence and uniqueness of $\hat{H}_{1,\rho}$ is assured by Proposition 2 in the appendix. The function $\hat{H}_1$ is the $W^1(I\!R^d \setminus \overline{D})$ solution of of

$$\begin{cases} \Delta \hat{H}_1 &= 0 \qquad \text{in } I\!R^d \setminus \overline{D} \ , \\ \hat{H}_1(x) &= \nabla u_0(0) \cdot x \quad \text{on } \partial D \ , \end{cases}$$

for $d = 2$ as well as $d = 3$. The existence and uniqueness of $\hat{H}_1$ is again assured by Proposition 2 in the appendix. Since $(u_0(\rho x) - u_0(0))/\rho \to \nabla u_0(0) \cdot x$ in $H^{1/2}(\partial D)$, as $\rho \to 0$, the estimate (6.1) from Proposition 2 gives that

$$\hat{H}_{1,\rho} \to \hat{H}_1 \quad \text{in } W^1(I\!R^d \setminus \overline{D}) \text{ as } \rho \to 0 \ . \tag{3.8}$$

We also define

$$\lambda_\rho = \frac{1}{|D|} \int_{\partial D} \frac{\partial \hat{H}_{1,\rho}}{\partial n} \, d\sigma \quad , \quad \text{and} \quad \lambda_0 = \frac{1}{|D|} \int_{\partial D} \frac{\partial \hat{H}_1}{\partial n} \, d\sigma \ . \tag{3.9}$$

Due to the convergence (3.8) and the fact that both $\hat{H}_{1,\rho}$ and $\hat{H}_1$ are harmonic in $I\!R^d \setminus \overline{D}$ it follows that

$$\frac{\partial \hat{H}_{1,\rho}}{\partial n} \to \frac{\partial \hat{H}_1}{\partial n} \quad \text{in } H^{-\frac{1}{2}}(\partial D) \ , \tag{3.10}$$

and therefore, in particular

$$\lambda_\rho \to \lambda_0 \ , \text{ as } \rho \to 0 \ . \tag{3.11}$$

The convergences in (3.8), (3.10) and (3.11) are uniform in $f \in \{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ , \ \int_{\partial\Omega} f \ d\sigma = 0 \}$, since all the involved quantities only depend on values of u_0

near $x = 0$. Finally we define $u_{1,\rho} \in H^1(\Omega \setminus \overline{\rho D})$ to be the unique solution to

$$\begin{cases} \Delta u_{1,\rho} &= 0 & \text{in } \Omega \setminus \overline{\rho D} \quad , \\ \dfrac{\partial u_{1,\rho}}{\partial n} &= f & \text{on } \partial\Omega \quad , \\ u_{1,\rho} &= u_0(0) + \rho\lambda_\rho J(\cdot/\rho) & \text{on } \partial\rho D \quad . \end{cases} \tag{3.12}$$

In order to study the behaviour of $\frac{1}{\rho^d}\nabla(u_\rho - u_0)$ as required for the proof of Theorem 2 we divide the function $\frac{1}{\rho^d}(u_\rho - u_0)$ into two terms:

$$\frac{1}{\rho^d}(u_\rho - u_0) = \frac{1}{\rho^d} w_{1,\rho} + \frac{1}{\rho^d} w_{2,\rho} \quad , \tag{3.13}$$

with

$$w_{1,\rho} = u_{1,\rho} - u_0 \quad , \tag{3.14}$$

and

$$w_{2,\rho} = u_\rho - u_{1,\rho} \quad . \tag{3.15}$$

In the following two sections we shall study the behaviour of $\frac{1}{\rho^d} w_{1,\rho}$ and $\frac{1}{\rho^d} w_{2,\rho}$.

3.1 A uniform estimate for the first remainder term

The function $w_{1,\rho} = u_{1,\rho} - u_0 \in H^1(\Omega \setminus \overline{\rho D})$ satisfies

$$\begin{cases} \Delta w_{1,\rho} &= 0 & \text{in } \Omega \setminus \overline{\rho D} \quad , \\ \dfrac{\partial w_{1,\rho}}{\partial n} &= 0 & \text{on } \partial\Omega \quad , \\ w_{1,\rho} &= \psi_{1,\rho} := u_0(0) - u_0(\cdot) + \rho\lambda_\rho J(\cdot/\rho) & \text{on } \partial(\rho D) \quad . \end{cases}$$

We define

$$H_{1,\rho} = -\hat{H}_{1,\rho} + \lambda_\rho J \in W^1(I\!R^d \setminus \overline{D}) \quad ,$$

and

$$H_1 = -\hat{H}_1 + \lambda_0 J \in W^1(I\!R^d \setminus \overline{D}) \quad .$$

The function $H_{1,\rho}$ satisfies

$$\begin{cases} \Delta H_{1,\rho} &= 0, & \text{in } I\!R^d \setminus \overline{D} \quad , \\ H_{1,\rho}(x) &= \dfrac{1}{\rho}\psi_{1,\rho}(\rho x), & \text{on } \partial D \quad , \end{cases}$$

and due to the convergence results (3.8), (3.10), and (3.11) it follows that

$$H_{1,\rho} \to H_1 \quad \text{in } W^1(I\!R^d \setminus \overline{D}) \quad , \text{ and } \quad \frac{\partial}{\partial n} H_{1,\rho} \to \frac{\partial}{\partial n} H_1 \quad \text{in } H^{-1/2}(\partial D) \quad , \tag{3.16}$$

as $\rho \to 0$. This convergence is uniform in $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ , \ \int_{\partial\Omega} f \, d\sigma = 0 \right\}$. Applying Proposition 2 we get

$$H_{1,\rho}(x) = \int_{\partial D} \frac{\partial}{\partial n_y} \Phi(x,y) H_{1,\rho}(y) \, d\sigma_y - \int_{\partial D} \Phi(x,y) \frac{\partial}{\partial n} H_{1,\rho}(y) \, d\sigma_y + C_\rho$$

for some constant C_ρ. We note that, due to the particular choice of λ_ρ (and (6.2) for $d = 2$)

$$\int_{\partial D} \frac{\partial}{\partial n} H_{1,\rho} \, d\sigma = 0 \ .$$

Let Λ_D denote the Neumann to Dirichlet map associated with the Laplacian on D. In other words: for any $\phi \in H^{-1/2}(\partial D)$ with $\int_{\partial D} \phi \ d\sigma = 0$, set $\Lambda_D(\phi) = w|_{\partial D} \in H^{1/2}(\partial D)$, where w is the solution to

$$\Delta w = 0 \quad \text{in } D \ , \quad \frac{\partial}{\partial n} w = \phi \quad \text{on } \partial D \ , \quad \int_{\partial D} w \ d\sigma = 0 \ .$$

With the use of Green's formula and this notation the above representation formula for $H_{1,\rho}$ may be rewritten

$$H_{1,\rho}(x) = \int_{\partial D} \frac{\partial}{\partial n_y} \Phi(x,y) \left[H_{1,\rho}(y) - \Lambda_D \left(\frac{\partial}{\partial n} H_{1,\rho} \right)(y) \right] d\sigma_y \ + C_\rho \ ,$$

for $x \in I\!R^d \setminus \overline{D}$. The rescaled function

$$v_{1,\rho}(x) = \rho H_{1,\rho}(x/\rho) \ , \tag{3.17}$$

may then be represented as

$$\begin{aligned} v_{1,\rho}(x) &= \rho \int_{\partial D} \frac{\partial}{\partial n_y} \Phi(x/\rho, y) \left[H_{1,\rho}(y) - \Lambda_D \left(\frac{\partial}{\partial n} H_{1,\rho} \right)(y) \right] d\sigma_y \ + \rho C_\rho \\ &= \rho^d \int_{\partial D} \frac{\partial}{\partial n_y} \Phi(x, \rho y) \left[H_{1,\rho}(y) - \Lambda_D \left(\frac{\partial}{\partial n} H_{1,\rho} \right)(y) \right] d\sigma_y \ + \rho C_\rho \ . \end{aligned}$$

Due to the convergence result (3.16) and the continuity of the operator Λ_D from $H^{-1/2}$ to $H^{1/2}$ it follows that

$$H_{1,\rho} - \Lambda_D \left(\frac{\partial}{\partial n} H_{1,\rho} \right) \to H_1 - \Lambda_D \left(\frac{\partial}{\partial n} H_1 \right) \quad \text{in } H^{1/2}(\partial D) \ .$$

Thus

$$\lim_{\rho \to 0} \int_{\Omega \setminus B_\delta} \left| \frac{1}{\rho^d} \nabla v_{1,\rho} - \nabla L_1 \right|^2 dx = 0 \ , \tag{3.18}$$

for any fixed $\delta > 0$, and

$$\lim_{\rho \to 0} \frac{1}{\rho^d} \frac{\partial v_{1,\rho}}{\partial n} = \frac{\partial L_1}{\partial n}, \quad \text{in } H^{-1/2}(\partial\Omega) \ , \tag{3.19}$$

where

$$L_1(x) := \nabla_y \Phi(x,0) \cdot \int_{\partial D} \left(H_1 - \Lambda_D \left(\frac{\partial}{\partial n} H_1 \right) \right) n_y \, d\sigma_y \quad , \qquad x \in \Omega \setminus \{0\} \quad .$$

The convergence is uniform in $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \;\;,\; \int_{\partial\Omega} f \; d\sigma = 0 \right\}$. In order to complete the analysis of the behaviour of $\frac{1}{\rho^d} w_{1,\rho}$ it only remains to examine

$$W_{1,\rho} := \frac{1}{\rho^d}(w_{1,\rho} - v_{1,\rho}) \quad . \tag{3.20}$$

For that purpose we shall make use of the following lemma.

Lemma 1. *Suppose $f_\rho \in H^{-\frac{1}{2}}(\partial\Omega)$ and $f_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ with $\int_{\partial\Omega} f_\rho = \int_{\partial\Omega} f_0 = 0$, and suppose $f_\rho \to f_0$ in $H^{-\frac{1}{2}}(\partial\Omega)$, as $\rho \to 0$. Let w_ρ and w_0 denote solutions to*

$$\begin{cases} \Delta w_\rho & = 0 \quad \text{in } \Omega \setminus \overline{\rho D} \quad , \\ \dfrac{\partial w_\rho}{\partial n} & = f_\rho \quad \text{on } \partial\Omega \quad , \\ w_\rho & = 0 \quad \text{on } \partial \rho D \quad , \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta w_0 & = 0 \quad \text{in } \Omega \quad , \\ \dfrac{\partial w_0}{\partial n} & = f_0 \quad \text{on } \partial\Omega \quad , \end{cases}$$

respectively. Then

$$\lim_{\rho \to 0} \int_{\Omega \setminus \overline{\rho D}} |\nabla w_\rho - \nabla w_0|^2 \, dx = 0.$$

The convergence is uniform in the sense that given any $\epsilon > 0$ there exists $\tau(\epsilon) > 0$ such that for any $f_0 \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \;\;,\; \int_{\partial\Omega} f \; d\sigma = 0 \right\}$

$$\rho < \tau(\epsilon) \quad , \qquad \text{and} \quad f_\rho \in H^{-1/2}(\partial\Omega) \;\; , \;\; \|f_\rho - f_0\|_{H^{-1/2}(\partial\Omega)} < \tau(\epsilon)$$

implies

$$\int_{\Omega \setminus \overline{\rho D}} |\nabla w_\rho - \nabla w_0|^2 \, dx < \epsilon \quad .$$

Proof of Lemma 1 Standard coercivity arguments (in this case, direct integration by parts) gives that

$$\|\nabla w_\rho\|_{L^2(\Omega \setminus \overline{\rho D})} \leq C \|f_\rho\|_{H^{-1/2}(\partial\Omega)} \quad .$$

Let $E_f(w)$ denote the energy

$$\frac{1}{2} \int_\Omega |\nabla w|^2 \; dx - \int_{\partial\Omega} f w \; d\sigma \quad .$$

We extend w_ρ to all of Ω by setting it to zero on ρD. For simplicity we also call this $H^1(\Omega)$ extension w_ρ. It is well known that w_ρ is the minimizer of $E_{f_\rho}(\cdot)$ in

$H^1(\Omega) \cap \{w \;:\; w \equiv 0 \text{ on } \rho D\}$, and that w_0 is the minimizer of $E_{f_0}(\cdot)$ in $H^1(\Omega)$. A simple calculation gives

$$\begin{aligned}
&\int_\Omega |\nabla(w_\rho - w_0)|^2 \; dx \\
&= 2\left[E_{f_\rho}(w_\rho) - E_{f_0}(w_0)\right] + 2\int_{\partial\Omega} (f_\rho - f_0) w_\rho \; d\sigma \\
&\le 2\left[E_{f_\rho}(w_\rho) - E_{f_0}(w_0)\right] + C\|f_\rho - f_0\|_{H^{-1/2}(\partial\Omega)} \|\nabla w_\rho\|_{L^2(\Omega)} \\
&\le 2\left[E_{f_\rho}(w^*) - E_{f_0}(w_0)\right] + C\|f_\rho - f_0\|_{H^{-1/2}(\partial\Omega)} \|f_\rho\|_{H^{-1/2}(\partial\Omega)} \qquad (3.21)
\end{aligned}$$

for any $w^* \in H^1(\Omega) \cap \{w \;:\; w \equiv 0 \text{ on } \rho D\}$. We proceed to construct an appropriate test function w^*. For simplicity assume B_1 is compactly contained in Ω and suppose $\rho D \subset B_{K\rho}$, with $K\rho < 1$. Define

$$\chi_\rho(x) = \begin{cases} 0 & \text{if } |x| < K\rho \;\;, \\ 1 - \frac{\log|x|}{\log(K\rho)} & \text{if } K\rho < |x| < 1 \;\;, \\ 1 & \text{if } 1 < |x| \;\;, \end{cases}$$

and set $\tilde{w}_\rho = \chi_\rho w_0 \in H^1(\Omega) \cap \{w \;:\; w \equiv 0 \text{ on } \rho D\}$. Then

$$\begin{aligned}
&\int_{K\rho<|x|<1} |\nabla(\tilde{w}_\rho - w_0)|^2 \; dx \\
&\quad = \frac{1}{(\log(K\rho))^2} \int_{K\rho<|x|<1} \left|\nabla\left(\log|x| \, w_0\right)\right|^2 \; dx \\
&\quad \le \frac{C}{(\log(K\rho))^2} \Bigg(\sup_{x\in B_1} |w_0(x)|^2 \int_{K\rho<|x|<1} \frac{1}{|x|^2} \; dx \\
&\qquad\qquad + \sup_{x\in B_1} |\nabla w_0(x)|^2 \int_{K\rho<|x|<1} (\log|x|)^2 \; dx \Bigg) \\
&\quad \le \frac{C}{(\log(K\rho))^2} \left(\sup_{x\in B_1} |w_0(x)|^2 + \sup_{x\in B_1} |\nabla w_0(x)|^2 \right) |\log(\rho)| \\
&\quad \to 0 \quad \text{as } \rho \to 0 \;\; . \qquad (3.22)
\end{aligned}$$

The convergence is uniform on $\left\{ \|f_0\|_{H^{-\frac{1}{2}}(\partial\Omega)} \le 1 \;\;,\;\; \int_{\partial\Omega} f_0 \; d\sigma = 0 \;\right\}$. We also have

$$\begin{aligned}
\int_\Omega |\nabla(\tilde{w}_\rho - w_0)|^2 \; dx \;\; &= \;\; \int_{|x|<K\rho} |\nabla w_0|^2 \; dx + \int_{K\rho<|x|<1} |\nabla(\tilde{w}_\rho - w_0)|^2 \; dx \\
&\le \;\; \int_{K\rho<|x|<1} |\nabla(\tilde{w}_\rho - w_0)|^2 \; dx + C\rho^d \sup_{x\in B_1} |\nabla w_0(x)|^2 \;\;,
\end{aligned}$$

and by combination with (3.22) we therefore get

$$\|\nabla(\tilde{w}_\rho - w_0)\|_{L^2(\Omega)} \to 0 \quad \text{as } \rho \to 0 \;\; .$$

The convergence is uniform on $\left\{ \|f_0\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ , \ \int_{\partial\Omega} f_0 \ d\sigma = 0 \right\}$. Since

$$\left| \|\nabla \tilde{w}_\rho\|_{L^2(\Omega)} - \|\nabla w_0\|_{L^2(\Omega)} \right| \leq \|\nabla(\tilde{w}_\rho - w_0)\|_{L^2(\Omega)} \quad ,$$

it follows that $\|\nabla \tilde{w}_\rho\|_{L^2(\Omega)}$ is bounded uniformly in ρ and in $f_0 \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ , \ \int_{\partial\Omega} f \ d\sigma = 0 \right\}$, and so

$$\begin{aligned}
&\left| \|\nabla \tilde{w}_\rho\|^2_{L^2(\Omega)} - \|\nabla w_0\|^2_{L^2(\Omega)} \right| \\
&\qquad = \left| \|\nabla \tilde{w}_\rho\|_{L^2(\Omega)} - \|\nabla w_0\|_{L^2(\Omega)} \right| \left(\|\nabla \tilde{w}_\rho\|_{L^2(\Omega)} + \|\nabla w_0\|_{L^2(\Omega)} \right) \\
&\qquad \leq \|\nabla(\tilde{w}_\rho - w_0)\|_{L^2(\Omega)} \left(\|\nabla \tilde{w}_\rho\|_{L^2(\Omega)} + \|\nabla w_0\|_{L^2(\Omega)} \right) \quad , \\
&\qquad \to 0 \quad \text{as } \rho \to 0 \quad ,
\end{aligned}$$

uniformly on $\left\{ \|f_0\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ , \ \int_{\partial\Omega} f_0 \ d\sigma = 0 \right\}$. Due to the fact that $\tilde{w}_\rho = w_0$ on $\partial\Omega$ we have the estimate

$$\begin{aligned}
2\left[E_{f_\rho}(\tilde{w}_\rho) - E_{f_0}(w_0)\right] &= \|\nabla \tilde{w}_\rho\|^2_{L^2(\Omega)} - \|\nabla w_0\|^2_{L^2(\Omega)} + \int_{\partial\Omega} (f_0 - f_\rho) w_0 \\
&\leq \|\nabla \tilde{w}_\rho\|^2_{L^2(\Omega)} - \|\nabla w_0\|^2_{L^2(\Omega)} \qquad (3.23) \\
&\qquad + C\|f_\rho - f_0\|_{H^{-1/2}(\partial\Omega)} \|f_0\|_{H^{-1/2}(\partial\Omega)} \quad .
\end{aligned}$$

From 3.21 (with $w^* = \tilde{w}_\rho$) and (3.23) we now conclude that

$$\begin{aligned}
&\int_\Omega |\nabla(w_\rho - w_0)|^2 \ dx \\
&\qquad \leq \left[E_{f_\rho}(\tilde{w}_\rho) - E_{f_0}(w_0)\right] \\
&\qquad\qquad + C\|f_\rho - f_0\|_{H^{-1/2}(\partial\Omega)} \|f_\rho\|_{H^{-1/2}(\partial\Omega)} \\
&\qquad \leq \frac{1}{2} \left(\|\nabla \tilde{w}_\rho\|^2_{L^2(\Omega)} - \|\nabla w_0\|^2_{L^2(\Omega)} \right) \\
&\qquad\qquad + C\|f_\rho - f_0\|_{H^{-1/2}(\partial\Omega)} \left(\|f_\rho\|_{H^{-1/2}(\partial\Omega)} + \|f_0\|_{H^{-1/2}(\partial\Omega)} \right) \\
&\qquad \to 0 \quad \text{as } \rho \to 0 \quad .
\end{aligned}$$

Since the term $\|\nabla \tilde{w}_\rho\|^2_{L^2(\Omega)} - \|\nabla w_0\|^2_{L^2(\Omega)}$ converges uniformly with respect to $f_0 \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ , \ \int_{\partial\Omega} f \ d\sigma = 0 \right\}$ this last convergence is clearly uniform in the sense asserted in this lemma. □

We now return to the estimation of $W_{1,\rho} := \frac{1}{\rho^d}(w_{1,\rho} - v_{1,\rho})$. This function is the unique solution to

$$\begin{cases}
\Delta W_{1,\rho} &= 0 \qquad \text{in } \Omega \setminus \overline{\rho D} \quad , \\
\frac{\partial W_{1,\rho}}{\partial n} &= -\frac{1}{\rho^d} \frac{\partial v_{1,\rho}}{\partial n}(x) \qquad \text{on } \partial\Omega \quad , \\
W_{1,\rho} &= 0 \qquad \text{on } \partial\rho D \quad .
\end{cases}$$

Let W_1 denote a solution to

$$\begin{cases} \Delta W_1 & = 0 \qquad \text{in } \Omega \ , \\ \frac{\partial W_1}{\partial n} & = -\frac{\partial L_1}{\partial n} \quad \text{on } \partial\Omega \ . \end{cases}$$

According to (3.19) $\frac{1}{\rho^d}\frac{\partial v_{1,\rho}}{\partial n}$ converges to $\frac{\partial L_1}{\partial n}$ in $H^{-1/2}(\partial\Omega)$, uniformly with respect to $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ , \ \int_{\partial\Omega} f \ d\sigma = 0 \right\}$, and so from Lemma 1 it follows that

$$\lim_{\rho\to 0} \int_{\Omega\setminus\overline{\rho D}} |\nabla W_{1,\rho} - \nabla W_1|^2 \ dx = 0 \ . \tag{3.24}$$

The limit is uniform in $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ , \ \int_{\partial\Omega} f \ d\sigma = 0 \right\}$. A combination of (3.18) with (3.24) now yields

$$\begin{aligned} &\left(\int_{\Omega\setminus B_\delta} \left|\frac{1}{\rho^d}\nabla w_{1,\rho} - \nabla L_1 - \nabla W_1\right|^2 dx\right)^{1/2} \\ &\qquad \leq \left(\int_{\Omega\setminus B_\delta} \left|\frac{1}{\rho^d}\nabla(w_{1,\rho} - v_{1,\rho}) - \nabla W_1\right|^2 dx\right)^{1/2} \\ &\qquad\quad + \left(\int_{\Omega\setminus B_\delta} \left|\frac{1}{\rho^d}\nabla v_{1,\rho} - \nabla L_1\right|^2 dx\right)^{1/2} \\ &\qquad = \left(\int_{\Omega\setminus B_\delta} \left|\nabla W_{1,\rho} - \nabla W_1\right|^2 dx\right)^{1/2} \\ &\qquad\quad + \left(\int_{\Omega\setminus B_\delta} \left|\frac{1}{\rho^d}\nabla v_{1,\rho} - \nabla L_1\right|^2 dx\right)^{1/2} \\ &\qquad \to 0 \quad \text{as } \rho \to 0 \ , \end{aligned} \tag{3.25}$$

uniformly on $\left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ , \ \int_{\partial\Omega} f \ d\sigma = 0 \right\}$. Since all the involved functions are independent of $\gamma_{1,\rho}$ it follows immediately that the limit is also uniform with respect to $\gamma_{1,\rho} \in L^\infty_+(\rho D)$. This completes the study of the behaviour of the first remainder term $\frac{1}{\rho^d} w_{1,\rho}$, as $\rho \to 0$.

3.2 A uniform estimate for the second remainder term

It remains to examine the behaviour of the term the $\frac{1}{\rho^d} w_{2,\rho} = \frac{1}{\rho^d}(u_\rho - u_{1,\rho})$ on $\Omega \setminus B_\delta$. For that purpose it will be convenient to extend the function $u_{1,\rho}$ to all of Ω by setting it equal to the constant $u_0(0)$ on ρD. From (1.3) and (3.12) it now follows that $w_{2,\rho}$ is a solution to

$$\begin{cases} \nabla\cdot(\gamma_{1,\rho}\nabla w_{2,\rho}) &= 0 & \text{in } \rho D \quad , \\ \Delta w_{2,\rho} &= 0 & \text{in } \Omega\setminus\overline{\rho D} \quad , \\ \frac{\partial w_{2,\rho}}{\partial n} &= 0 & \text{on } \partial\Omega \quad , \\ \frac{\partial w_{2,\rho}}{\partial n}\Big|_{\text{ext}} - (\gamma_{1,\rho}\nabla w_{2,\rho})\cdot n\Big|_{\text{int}} &= -\frac{\partial u_{1,\rho}}{\partial n}\Big|_{\text{ext}} & \text{on } \partial\rho D \quad , \\ [w_{2,\rho}] &= -\rho\lambda_\rho J(\cdot/\rho) & \text{on } \partial\rho D \quad . \end{cases}$$

For brevity we shall use the notation $\psi_{2,\rho} := -\frac{\partial u_{1,\rho}}{\partial n}\Big|_{\text{ext}}$ on $\partial\rho D$. Since $\psi_{2,\rho}\in H^{-\frac{1}{2}}(\partial\rho D)$, $\int_{\partial\rho D}\psi_{2,\rho}=0$, and $\lambda_\rho J\in H^{\frac{1}{2}}(\partial D)$, Proposition 4 guarantees the existence of $H_{2,\rho}\in W^1(\mathbb{R}^d\setminus\overline{D})\times H^1(D)$, a solution to

$$\begin{cases} \nabla\cdot(\gamma_{1,\rho}(\rho\,\cdot\,)\nabla H_{2,\rho}) &= 0 & \text{in } D \quad , \\ \Delta H_{2,\rho} &= 0 & \text{in } \mathbb{R}^d\setminus\overline{D} \quad , \\ \frac{\partial H_{2,\rho}}{\partial n}\Big|_{\text{ext}} - (\gamma_{1,\rho}(\rho\,\cdot\,)\nabla H_{2,\rho})\cdot n\Big|_{\text{int}} &= \psi_{2,\rho}(\rho\,\cdot\,) & \text{on } \partial D \quad , \\ [H_{2,\rho}] &= -\lambda_\rho J & \text{on } \partial D \quad . \end{cases}$$

This solution is unique for $d=3$, and it is unique modulo a constant for $d=2$. Moreover, $H_{2,\rho}$ may be represented as

$$H_{2,\rho}(x) = \int_{\partial D}\frac{\partial}{\partial n_y}\Phi(x,y)H_{2,\rho}\Big|_{\text{ext}}\,d\sigma_y - \int_{\partial D}\Phi(x,y)\frac{\partial H_{2,\rho}}{\partial n}(y)\Big|_{\text{ext}}\,d\sigma_y \;+C_\rho \quad .$$

The rescaled function, $v_{2,\rho}=\rho H_{2,\rho}(x/\rho)$, may then be represented as

$$\begin{aligned} v_{2,\rho}(x) &= \rho\int_{\partial D}\frac{\partial}{\partial n_y}\Phi(x/\rho,y)H_{2,\rho}\Big|_{\text{ext}}\,d\sigma_y \\ &\qquad -\rho\int_{\partial D}\Phi(x/\rho,y)\frac{\partial H_{2,\rho}}{\partial n}(y)\Big|_{\text{ext}}\,d\sigma_y \;+\rho C_\rho \\ &= \rho^d\int_{\partial D}\left(\frac{\partial}{\partial n_y}\Phi\right)(x,\rho y)H_{2,\rho}\Big|_{\text{ext}}\,d\sigma_y \\ &\qquad -\rho^{d-1}\int_{\partial D}\Phi(x,\rho y)\frac{\partial H_{2,\rho}}{\partial n}(y)\Big|_{\text{ext}}\,d\sigma_y \;+\rho C_\rho \quad . \end{aligned} \tag{3.26}$$

For the last identity we have (at least in the case $d=2$) used that

$$\int_{\partial D}\frac{\partial H_{2,\rho}}{\partial n}\Big|_{\text{ext}}\,d\sigma = 0 \quad , \tag{3.27}$$

which in turn (for $d=2$ as well as $d=3$) follows from the fact that

$$\int_{\partial\rho D}\psi_{2,\rho} = -\int_{\partial\rho D}\frac{\partial u_{1,\rho}}{\partial n}\Big|_{\text{ext}} = 0 \quad , \quad \text{and} \quad \int_{\partial D}(\gamma_{1,\rho}(\rho\,\cdot)\nabla H_{2,\rho})\cdot n\Big|_{\text{int}} = 0 \quad .$$

Using (3.27) once more we may rewrite (3.26) as

$$\begin{aligned} v_{2,\rho}(x) &= \rho^d \int_{\partial D} \left(\frac{\partial}{\partial n_y}\Phi\right)(x,\rho y) H_{2,\rho}\Big|_{\text{ext}} d\sigma_y \\ &\quad - \rho^{d-1} \int_{\partial D} \left(\Phi(x,\rho y) - \Phi(x,0)\right) \frac{\partial H_{2,\rho}}{\partial n}(y)\Big|_{\text{ext}} d\sigma_y \; + \rho C_\rho \quad . \end{aligned} \tag{3.28}$$

We now study the asymptotic behaviour of $H_{2,\rho}\Big|_{\text{ext}}$ and $\frac{\partial H_{2,\rho}}{\partial n}\Big|_{\text{ext}}$ on ∂D. To this end the definitions of $w_{1,\rho}$ and $W_{1,\rho}$, (3.14) and (3.20) respectively, yield that

$$u_{1,\rho} = u_0 + w_{1,\rho} = u_0 + \rho^d W_{1,\rho} + v_{1,\rho}, \qquad \text{in } \Omega \setminus \overline{\rho D} \quad . \tag{3.29}$$

Since $v_{1,\rho}(x) = \rho H_{1,\rho}(x/\rho)$ (see (3.17))

$$\frac{\partial v_{1,\rho}}{\partial n}(x) = \left(\frac{\partial H_{1,\rho}}{\partial n}\right)(x/\rho) \quad ,$$

and so

$$\left(\frac{\partial v_{1,\rho}}{\partial n}\right)(\rho x) = \frac{\partial H_{1,\rho}}{\partial n}(x) \quad , \tag{3.30}$$

for $x \in \partial D$. Choose K such that $\rho D \subset B_{\frac{K}{2}\rho}$ (*i.e.*, $D \subset B_{\frac{K}{2}}$). We may without loss of generality suppose ρ is sufficiently small that $\overline{B_{K\rho}} \subset \Omega$. From (3.24)

$$\lim_{\rho \to 0} \int_{B_{K\rho} \setminus \rho D} |\nabla W_{1,\rho} - \nabla W_1|^2 \, dx = 0 \quad ,$$

which implies that

$$\lim_{\rho \to 0} \left(\int_{B_{K\rho} \setminus \rho D} |\nabla W_{1,\rho}|^2 \, dx - \int_{B_{K\rho} \setminus \rho D} |\nabla W_1|^2 \, dx \right) = 0 \quad .$$

In particular

$$\int_{B_{K\rho} \setminus \rho D} |\nabla W_{1,\rho}|^2 \, dx \to 0 \quad \text{as } \rho \to 0 \quad .$$

The rescaled function

$$\tilde{W}_{1,\rho}(x) = W_{1,\rho}(\rho x)$$

satisfies $\Delta \tilde{W}_{1,\rho} = 0$ in $B_K \setminus \overline{D}$, $\tilde{W}_{1,\rho} = 0$ on ∂D,

$$\rho^{d-2} \int_{B_K \setminus D} |\nabla \tilde{W}_{1,\rho}|^2 \, dx = \int_{B_{K\rho} \setminus \rho D} |\nabla W_{1,\rho}|^2 \, dx \to 0 \quad \text{as} \quad \rho \to 0 \quad ,$$

and

$$\frac{\partial \tilde{W}_{1,\rho}}{\partial n}(x) = \rho \left(\frac{\partial W_{1,\rho}}{\partial n}\right)(\rho x) \qquad \text{on } \partial D \quad .$$

Therefore

$$\begin{aligned}\rho^d \left\| \left(\frac{\partial W_{1,\rho}}{\partial n}\right)(\rho x)\right\|^2_{H^{-\frac{1}{2}}(\partial D)} &= \rho^{d-2}\left\|\frac{\partial \tilde{W}_{1,\rho}}{\partial n}\right\|^2_{H^{-\frac{1}{2}}(\partial D)} \\ &\leq C\rho^{d-2}\left\|\nabla \tilde{W}_{1,\rho}\right\|^2_{L^2(B_K\setminus\overline{D})} \to 0 \quad ,\end{aligned}$$

as $\rho \to 0$. The convergence is uniform with respect to $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ \ , \int_{\partial\Omega} f \ d\sigma = 0 \right\}$, and since $W_{1,\rho}$ is independent of $\gamma_{1,\rho}$, this convergence is also uniform in $\gamma_{1,\rho} \in L^\infty_+(\rho D)$. It immediately follows (after multiplication by ρ^d) that

$$\lim_{\rho\to 0}\left\|\rho^d\left(\frac{\partial W_{1,\rho}}{\partial n}\right)(\rho\,\cdot)\right\|_{H^{-\frac{1}{2}}(\partial D)} = 0 \quad . \tag{3.31}$$

By a combination of (3.29), (3.30),(3.31), and (3.16)

$$\begin{aligned}\lim_{\rho\to 0}\psi_{2,\rho}(\rho x) &= -\lim_{\rho\to 0}\left(\frac{\partial u_{1,\rho}}{\partial n}\Big|_{\text{ext}}\right)(\rho x) \\ &= -\lim_{\rho\to 0}\nabla u_0(\rho x)\cdot n - \lim_{\rho\to 0}\frac{\partial H_{1,\rho}}{\partial n}(x) \\ &= -\nabla u_0(0)\cdot n - \frac{\partial H_1}{\partial n}(x) \quad ,\end{aligned} \tag{3.32}$$

for $x \in \partial D$; the convergence takes place in $H^{-1/2}(\partial D)$ and is uniform in $\gamma_{1,\rho} \in L^\infty_+(\rho D)$ and $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ \ , \int_{\partial\Omega} f \ d\sigma = 0 \right\}$. Let $\tilde{H}_{2,\rho} \in W^1(\mathbb{R}^d \setminus \overline{D}) \times H^1(D)$ denote the solution to

$$\left\{\begin{array}{rll} \nabla\cdot(\gamma_{1,\rho}(\rho\,\cdot)\nabla\tilde{H}_{2,\rho}) &= 0 & \text{in } D \ \ , \\ \Delta\tilde{H}_{2,\rho} &= 0 & \text{in } I\!R^d\setminus\overline{D} \ \ , \\ \frac{\partial\tilde{H}_{2,\rho}}{\partial n}\Big|_{\text{ext}} - (\gamma_{1,\rho}(\rho\,\cdot)\nabla\tilde{H}_{2,\rho})\cdot n\Big|_{\text{int}} &= -\nabla u_0(0)\cdot n - \frac{\partial H_1}{\partial n} & \text{on } \partial D \ \ , \\ [\tilde{H}_{2,\rho}] &= -\lambda_0 J & \text{on } \partial D \ \ . \end{array}\right. \tag{3.33}$$

We note that

$$\int_{\partial D}\left(-\nabla u_0(0)\cdot n - \frac{\partial H_1}{\partial n}\right)\,d\sigma = \lim_{\rho\to 0}\rho^{-d+1}\int_{\partial\rho D}\psi_{2,\rho}\,d\sigma = 0 \quad ,$$

and so the existence of $\tilde{H}_{2,\rho}$ is guaranteed by Proposition 4. $\tilde{H}_{2,\rho}$ is unique for $d = 3$, and it is unique modulo a constant for $d = 2$. We note that $\tilde{H}_{2,\rho}$ is independent of ρ if $\gamma_{1,\rho}(x)$ is of the form $\gamma_{1,\rho}(x) = \gamma_1(x/\rho)$. Proposition 4 also yields that

$$\int_{I\!R^d\setminus D}|\nabla\tilde{H}_{2,\rho}|^2\,dx \text{ is bounded}$$

independently of ρ, $\gamma_{1,\rho} \in L^\infty_+(\rho D)$ and $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \;\; , \; \int_{\partial\Omega} f \; d\sigma = 0 \right\}$, and in combination with (3.32) and (3.11) it guarantees that

$$\begin{aligned} &\int_{\mathbb{R}^d \setminus D} |\nabla(H_{2,\rho} - \tilde{H}_{2,\rho})|^2 \; dx \\ &\qquad \leq C \left(\|\psi_{2,\rho}(\rho\cdot) + \nabla u_0(0) \cdot n + \frac{\partial H_1}{\partial n}\|_{H^{-1/2}(\partial\Omega)} + |\lambda_\rho - \lambda_0| \right) \\ &\qquad \to 0 \quad \text{as } \rho \to 0 \;\; . \end{aligned}$$

Here the constant C is independent of $\gamma_{1,\rho}$, and so the convergence is uniform in $\gamma_{1,\rho} \in L^\infty_+(\rho D)$ and $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \;\; , \; \int_{\partial\Omega} f \; d\sigma = 0 \right\}$. In particular we get that

$$\begin{aligned} &H_{2,\rho}\Big|_{\text{ext}} - \tilde{H}_{2,\rho}\Big|_{\text{ext}} \to 0 \quad \text{in } H^{1/2}(\partial D) \;\; , \quad \text{and} \\ &\frac{\partial H_{2,\rho}}{\partial n}\Big|_{\text{ext}} - \frac{\partial \tilde{H}_{2,\rho}}{\partial n}\Big|_{\text{ext}} \to 0 \quad \text{in } H^{-1/2}(\partial D) \;\; , \end{aligned}$$

uniformly in $\gamma_{1,\rho} \in L^\infty_+(\rho D)$ and $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \;\; , \; \int_{\partial\Omega} f \; d\sigma = 0 \right\}$, with $H_{2,\rho}\Big|_{\text{ext}}$, $\tilde{H}_{2,\rho}\Big|_{\text{ext}}$ and $\frac{\partial}{\partial n} H_{2,\rho}\Big|_{\text{ext}}$, $\frac{\partial}{\partial n} \tilde{H}_{2,\rho}\Big|_{\text{ext}}$ uniformly bounded in $H^{1/2}(\partial D)$ and $H^{-1/2}(\partial D)$ respectively. The $H^{1/2}$ convergence and the boundedness of the functions $H_{2,\rho}\Big|_{\text{ext}}$, $\tilde{H}_{2,\rho}\Big|_{\text{ext}}$ should be interpreted modulo constants. From (3.28) it now follows that

$$\lim_{\rho \to 0} \int_{\Omega \setminus B_\delta} \left| \frac{1}{\rho^d} \nabla v_{2,\rho} - \nabla L_{2,\rho} \right|^2 dx = 0 \;\; , \tag{3.34}$$

where $L_{2,\rho}$ denotes the function

$$L_{2,\rho}(x) = \nabla_y \Phi(x, 0) \cdot \int_{\partial D} \left(-\frac{\partial \tilde{H}_{2,\rho}}{\partial n}(y)\Big|_{\text{ext}} y + \tilde{H}_{2,\rho}(y)\Big|_{\text{ext}} n_y \right) d\sigma_y \quad \forall\, x \in \Omega \setminus \{0\} \;\; .$$

As a consequence of (3.34) and the fact that $v_{2,\rho}$ and $L_{2,\rho}$ are both harmonic in $\Omega \setminus \overline{B_\delta}$

$$\lim_{\rho \to 0} \left\| \frac{1}{\rho^d} \frac{\partial v_{2,\rho}}{\partial n} - \frac{\partial L_{2,\rho}}{\partial n} \right\|_{H^{-\frac{1}{2}}(\partial\Omega)} = 0 \;\; . \tag{3.35}$$

The convergence is uniform in $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \;\; , \; \int_{\partial\Omega} f \; d\sigma = 0 \right\}$ and $\gamma_{1,\rho} \in L^\infty_+(\rho D)$. We now consider $W_{2,\rho} = \frac{1}{\rho^d}\,(w_{2,\rho} - v_{2,\rho})$; this is a solution to

$$\begin{cases} \nabla \cdot (\gamma_\rho W_{2,\rho}) &= 0 \qquad \text{in } \Omega \;\; , \\ \frac{\partial W_{2,\rho}}{\partial n} &= -\frac{1}{\rho^d} \frac{\partial v_{2,\rho}}{\partial n} \qquad \text{on } \partial\Omega \;\; . \end{cases} \tag{3.36}$$

Problem (3.36) has a solution since, due to (3.27),

$$\int_{\partial\Omega} \frac{\partial v_{2,\rho}}{\partial n} \, d\sigma = \int_{\partial\rho D} \frac{\partial v_{2,\rho}}{\partial n}\Big|_{\text{ext}} \, d\sigma = \rho^{d-1} \int_{\partial D} \frac{\partial H_{2,\rho}}{\partial n}\Big|_{\text{ext}} \, d\sigma = 0 \quad .$$

$W_{2,\rho}$ is unique modulo a constant. A combination of (3.35) and Corollary 2, the latter with $-\frac{1}{\rho^d}\frac{\partial}{\partial n} v_{2,\rho}$ and $-\frac{\partial}{\partial n} L_{2,\rho}$ in place of g and f, yields that

$$\lim_{\rho \to 0} \int_{\Omega \setminus B_\delta} \left| \nabla W_{2,\rho} - \nabla \tilde{W}_{2,\rho} \right|^2 dx = 0 \quad , \tag{3.37}$$

uniformly with respect to $\gamma_{1,\rho} \in L^\infty_+(\rho D)$ and $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ , \ \int_{\partial\Omega} f \, d\sigma = 0 \right\}$. Here $\tilde{W}_{2,\rho}$ is a solution to

$$\begin{cases} \Delta \tilde{W}_{2,\rho} &= 0 \qquad \text{in } \Omega \quad , \\ \frac{\partial \tilde{W}_{2,\rho}}{\partial n} &= -\frac{\partial L_{2,\rho}}{\partial n} \qquad \text{on } \partial\Omega \quad . \end{cases}$$

In order to obtain (3.37) from a combination of (3.35) and Corollary 2 we have also used that $\frac{\partial L_{2,\rho}}{\partial n}$ is bounded in $H^{-1/2}(\partial\Omega)$, uniformly in ρ, $\gamma_{1,\rho} \in L^\infty_+(\rho D)$ and $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ , \ \int_{\partial\Omega} f \, d\sigma = 0 \right\}$. The convergence statements (3.34) and (3.37) now imply

$$\begin{aligned} &\left(\int_{\Omega \setminus B_\delta} \left| \frac{1}{\rho^d} \nabla w_{2,\rho} - \nabla L_{2,\rho} - \nabla \tilde{W}_{2,\rho} \right|^2 dx \right)^{1/2} \\ &\leq \left(\int_{\Omega \setminus B_\delta} \left| \frac{1}{\rho^d} \nabla (w_{2,\rho} - v_{2,\rho}) - \nabla \tilde{W}_{2,\rho} \right|^2 dx \right)^{1/2} \\ &\qquad\qquad + \left(\int_{\Omega \setminus B_\delta} \left| \frac{1}{\rho^d} \nabla v_{2,\rho} - \nabla L_{2,\rho} \right|^2 dx \right)^{1/2} \\ &= \left(\int_{\Omega \setminus B_\delta} \left| \nabla W_{2,\rho} - \nabla \tilde{W}_{2,\rho} \right|^2 dx \right)^{1/2} + \left(\int_{\Omega \setminus B_\delta} \left| \frac{1}{\rho^d} \nabla v_{2,\rho} - \nabla L_{2,\rho} \right|^2 dx \right)^{1/2} \\ &\to 0 \quad \text{as } \rho \to 0 \quad , \end{aligned} \tag{3.38}$$

uniformly in $\gamma_{1,\rho} \in L^\infty_+(\rho D)$ and $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ , \ \int_{\partial\Omega} f \, d\sigma = 0 \right\}$. This completes the study of the asymptotic behaviour of the second remainder term $\frac{1}{\rho^d} w_{2,\rho}$, as $\rho \to 0$.

3.3 Proof of the main theorem

It follows directly from the decomposition (3.13) and the estimates (3.25) and (3.38) in the two preceeding sections that

$$\begin{aligned}\left(\int_{\Omega\setminus B_\delta}\Big|\frac{1}{\rho^d}\nabla(u_\rho-u_0)-\nabla(L_1+L_{2,\rho}+W_1+\tilde W_{2,\rho})\Big|^2\,dx\right)^{1/2}\\
\le\left(\int_{\Omega\setminus B_\delta}\Big|\frac{1}{\rho^d}\nabla w_{1,\rho}-\nabla(L_1+W_1)\Big|^2\,dx\right)^{1/2}\\
+\left(\int_{\Omega\setminus B_\delta}\Big|\frac{1}{\rho^d}\nabla w_{2,\rho}-\nabla(L_{2,\rho}+\tilde W_{2,\rho})\Big|^2\,dx\right)^{1/2}\\
\to 0\quad\text{as }\rho\to 0\ .\end{aligned}\tag{3.39}$$

As a consequence of this and the fact that $\int_{\partial\Omega}u_\rho\,d\sigma=\int_{\partial\Omega}u_0\,d\sigma=0$ we also get that

$$\|\frac{1}{\rho^d}(u_\rho-u_0)-(L_1+L_{2,\rho}+W_1+\tilde W_{2,\rho}+C)\|_{H^1(\Omega\setminus\overline{B_\delta})}\to 0\quad\text{as }\rho\to 0\ ,$$

with the constant C given by

$$C=-\frac{1}{|\partial\Omega|}\int_{\partial\Omega}(L_1+L_{2,\rho}+W_1+\tilde W_{2,\rho})\,d\sigma\ .$$

Note that C generically depends on ρ. Since the function $\frac{1}{\rho^d}(u_\rho-u_0)-(L_1+L_{2,\rho}+W_1+\tilde W_{2,\rho}+C)$ is harmonic near $\partial\Omega$, and since its normal derivative vanishes on $\partial\Omega$, it follows by local elliptic regularity theory that all norms tend to zero, *i.e*,

$$\|\frac{1}{\rho^d}(u_\rho-u_0)-(L_1+L_{2,\rho}+W_1+\tilde W_{2,\rho}+C)\|_{H^s(\Omega\setminus\overline{B_\delta})}\to 0\quad\text{as }\rho\to 0\ ,$$

for any real positive s. Just as in (3.39) the convergence is uniform in $\gamma_{1,\rho}\in L^\infty_+(\rho D)$ and $f\in\left\{\|f\|_{H^{-\frac12}(\partial\Omega)}\le 1\ ,\ \int_{\partial\Omega}f\,d\sigma=0\right\}$. From continuity of the trace operator it follows that

$$\|\frac{1}{\rho^d}(u_\rho-u_0)-(L_1+L_{2,\rho}+W_1+\tilde W_{2,\rho}+C)\|_{H^s(\partial\Omega)}\to 0\quad\text{as }\rho\to 0\ ,$$

for any real positive number s, in the same uniform sense as above. Let L_ρ denote the term $L_\rho=L_1+L_{2,\rho}$ and let $W_\rho=W_1+\tilde W_{2,\rho}+C$. Then

$$\begin{aligned}L_\rho(x)\quad&=\nabla_y\Phi(x,0)\cdot\left(\int_{\partial D}\left(H_1-\Lambda_D\left(\frac{\partial}{\partial n}H_1\right)\right)n\,d\sigma\right.\\
&\left.+\int_{\partial D}\left(-\frac{\partial\tilde H_{2,\rho}}{\partial n}(z)\Big|_{\text{ext}}z+\tilde H_{2,\rho}(z)\Big|_{\text{ext}}n_z\right)\,d\sigma_z\right)\ ,\end{aligned}$$

and W_ρ is the harmonic function in Ω, uniquely determined by the boundary conditions

$$\frac{\partial}{\partial n}(W_\rho + L_\rho) = 0 \quad \text{on } \partial\Omega \ , \quad \text{and} \quad \int_{\partial\Omega} (W_\rho + L_\rho) \, d\sigma = 0 \ \ .$$

There are several ways to deduce the $L^2(\Omega \setminus \overline{B_\delta})$ boundedness of $\nabla(L_\rho + W_\rho)$. On the one hand it follows from (3.39) and the boundednes of $(u_\rho - u_0)/\rho^d$ asserted in Theorem 1; on the other hand, it also follows immediately from the formulas for L_ρ and W_ρ, and the boundednes of H_1 and that of $\tilde{H}_{2,\rho}$, stated just before (3.34) in the previous section. In order to prove our main theorem, Theorem 2, it thus only remains to verify that

$$\int_{\partial D} \left(H_1 - \Lambda_D \left(\frac{\partial}{\partial n} H_1 \right) \right) n \, d\sigma + \int_{\partial D} \left(- \frac{\partial \tilde{H}_{2,\rho}}{\partial n}(z)\Big|_{\text{ext}} z + \tilde{H}_{2,\rho}(z)\Big|_{\text{ext}} n_z \right) d\sigma_z$$
$$= \int_D (I - \gamma_{1,\rho}(\rho z)) \nabla \phi_k(z) \; dz \, \frac{\partial}{\partial x_k} u_0(0) \ \ , \tag{3.40}$$

where ϕ_k is the function defined by (3.2). In the following lemma we collect a number of identities that will be useful in order to establish this relationship.

Lemma 2. *With notation as above,*

$$\int_{\partial D} \frac{\partial \tilde{H}_{2,\rho}}{\partial n}(z)\Big|_{\text{ext}} z \, d\sigma_z \;=\; \int_D \gamma_{1,\rho}(\rho \,\cdot\,) \nabla \tilde{H}_{2,\rho} \, dz \tag{3.41}$$
$$- \int_{\partial D} \left(\nabla u_0(0) \cdot n_z + \frac{\partial H_1}{\partial n} \right) z \, d\sigma_z \ \ ,$$

$$\int_{\partial D} \tilde{H}_{2,\rho}(z)\Big|_{\text{ext}} n_z \, d\sigma_z \;=\; \int_D \nabla \tilde{H}_{2,\rho} \, dz - \lambda_0 \int_{\partial D} J n \, d\sigma \ \ , \tag{3.42}$$

and ,

$$\int_{\partial D} \left(H_1 - \Lambda_D \left(\frac{\partial}{\partial n} H_1 \right) \right) n \, d\sigma \;=\; - \int_D \nabla u_0(0) \cdot n_z z \, d\sigma_z + \lambda_0 \int_{\partial D} J n \; d\sigma$$
$$- \int_{\partial D} \frac{\partial H_1}{\partial n}(z) z \; d\sigma_z \ \ . \tag{3.43}$$

Proof. From (3.33), one has

$$\int_{\partial D} \frac{\partial \tilde{H}_{2,\rho}}{\partial n}(z)\Big|_{\text{ext}} z \, d\sigma_z = \int_{\partial D} (\gamma_1(\rho \,\cdot\,) \nabla \tilde{H}_{2,\rho}) \ \cdot \ n_z\Big|_{\text{int}} z \, d\sigma_z$$
$$- \int_{\partial D} \left(\nabla u_0(0) \ \cdot \ n_z + \frac{\partial H_1}{\partial n} \right) z \, d\sigma_z \ \ .$$

Integration by parts and use of (3.33) yields

$$\int_{\partial D} (\gamma_{1,\rho}(\rho\,\cdot\,)\nabla \tilde{H}_{2,\rho})\cdot n_z\Big|_{\text{int}} z\,d\sigma_z = \int_D \nabla\cdot(\gamma_{1,\rho}(\rho\,\cdot\,)\nabla\tilde{H}_{2,\rho}) z\,dz + \int_D \gamma_{1,\rho}(\rho\,\cdot\,)\nabla \tilde{H}_{2,\rho}\,dz = \int_D \gamma_{1,\rho}(\rho\,\cdot\,)\nabla\tilde{H}_{2,\rho}\,dz \quad .$$

Thus it follows that

$$\int_{\partial D} \frac{\partial \tilde{H}_{2,\rho}}{\partial n}(z)\Big|_{\text{ext}} z\,d\sigma_z = \int_D \gamma_{1,\rho}(\rho\,\cdot\,)\nabla\tilde{H}_{2,\rho}\,dz - \int_{\partial D}\Big(\nabla u_0(0)\cdot n_z + \frac{\partial H_1}{\partial n}\Big) z\,d\sigma_z \quad ,$$

as stated in (3.41). From (3.33) one has

$$\int_{\partial D} \tilde{H}_{2,\rho}(z)\Big|_{\text{ext}} n_z\,d\sigma_z = \int_{\partial D} \tilde{H}_{2,\rho}(z)\Big|_{\text{int}} \frac{\partial z}{\partial n_z}\,d\sigma_z - \lambda_0 \int_{\partial D} J n_z\,d\sigma_z \quad .$$

At the same time

$$\int_{\partial D} \tilde{H}_{2,\rho}(z)\Big|_{\text{int}} \frac{\partial z}{\partial n_z}\,d\sigma_z = \int_D \tilde{H}_{2,\rho}\Delta z\,dz + \int_D \nabla\tilde{H}_{2,\rho}\,dz = \int_D \nabla \tilde{H}_{2,\rho}\,dz \quad ,$$

and so

$$\int_{\partial D} \tilde{H}_{2,\rho}(z)\Big|_{\text{ext}} n_z\,d\sigma_z = \int_D \nabla\tilde{H}_{2,\rho}\,dz - \lambda_0\int_{\partial D} J n_z\,d\sigma_z \quad .$$

In order to complete the proof of this lemma it only remains to verify (3.43). To that end we let w denote the solution to

$$\Delta w = 0 \quad \text{in } D \quad , \quad \frac{\partial}{\partial n} w = \frac{\partial}{\partial n} H_1 \quad \text{on } \partial D \quad , \quad \int_{\partial\Omega} w\ d\sigma = 0 \quad .$$

With this notation $\Lambda_D(\frac{\partial}{\partial n}H_1) = w\Big|_{\partial D}$, and thus

$$\begin{aligned}
\int_{\partial D}\left(H_1 - \Lambda_D\left(\frac{\partial}{\partial n}H_1\right)\right) n\,d\sigma &= -\int_{\partial D}\hat{H}_1 n\ d\sigma + \lambda_0\int_{\partial D} Jn\ d\sigma \\
&\quad - \int_{\partial D} w(z)\frac{\partial}{\partial n_z} z\ d\sigma_z \\
&= -\int_{\partial D}\nabla u_0(0)\cdot z\ n_z\ d\sigma_z + \lambda_0\int_{\partial D} Jn\ d\sigma \\
&\quad - \int_{\partial D}\frac{\partial}{\partial n} w(z)\ z\ d\sigma_z \\
&= -\int_{\partial D}\nabla u_0(0)\cdot n_z\ z\ dz + \lambda_0\int_{\partial D} Jn\ d\sigma \\
&\quad - \int_{\partial D}\frac{\partial H_1}{\partial n}(z)\ z\ d\sigma_z \quad ,
\end{aligned}$$

which is exactly (3.43). □

Now the final step of the proof of Theorem 2. As a direct consequence of Lemma 2

$$\int_{\partial D} \left(H_1 - \Lambda_D \left(\frac{\partial}{\partial n} H_1 \right) \right) n \, d\sigma + \int_{\partial D} \left(-\frac{\partial \tilde{H}_{2,\rho}}{\partial n}(z)\Big|_{\text{ext}} z + \tilde{H}_{2,\rho}(z)\Big|_{\text{ext}} n_z \right) d\sigma_z$$
$$= \int_D (I - \gamma_{1,\rho}(\rho z)) \nabla \tilde{H}_{2,\rho}(z) \; dz \quad . \tag{3.44}$$

Furthermore, the function $\psi_k(z) \frac{\partial u_0}{\partial x_k}(0)$ (with ψ_k defined by (3.3)) may, up to a constant, be expressed compactly in terms of the functions H_1 and $\tilde{H}_{2,\rho}$

$$\psi_k(z) \frac{\partial u_0}{\partial x_k}(0) + \text{constant} = \begin{cases} H_1(z) + \tilde{H}_{2,\rho}(z) & z \in I\!R^d \setminus \overline{D} \quad , \\ -\nabla u_0(0) \cdot z + \tilde{H}_{2,\rho}(z) & z \in D \quad . \end{cases}$$

In terms of $\phi_k(z) = \psi_k(z) + z_k$ the second equation asserts that

$$\nabla \phi_k(z) \frac{\partial u_0}{\partial x_k}(0) = \nabla \left(\psi_k(z) \frac{\partial u_0}{\partial x_k}(0) + \nabla u_0(0) \cdot z \right) = \nabla \tilde{H}_{2,\rho}(z) \qquad \text{for } z \in D \quad ,$$

and therefore

$$\int_D (I - \gamma_{1,\rho}(\rho z)) \nabla \phi_k(z) \; dz \, \frac{\partial}{\partial x_k} u_0(0) = \int_D (I - \gamma_{1,\rho}(\rho z)) \nabla \tilde{H}_{2,\rho}(z) \; dz \quad .$$

A combination of the last identity and (3.44) leads to (3.40), and this completes the proof of Theorem 2.

4 Other inhomogeneities D_ρ

In this section we prove an analogue of Theorem 2 for inhomogeneities that are not exactly of the form ρD, but close. As already pointed out in the introduction a result like Theorem 2 cannot hold for volumetrically small D_ρ of arbitrary shape. We suppose D_ρ contains the origin, and that there exists a smooth, bounded domain D, star-shaped with respect to the origin, and such that

$$(1 - r_\rho)\rho D \subset D_\rho \subset (1 + r_\rho)\rho D \quad , \tag{4.1}$$

with $r_\rho > 0$, and $r_\rho \to 0$ as $\rho \to 0$. Let γ_ρ be defined as in (3.1), but suppose furthermore $\gamma_{1,\rho}$ is constant and isotropic, *i.e.*,

$$\gamma_\rho = \begin{cases} I \text{ in } \Omega \setminus D_\rho \\ cI \text{ in } D_\rho \end{cases} \quad , \tag{4.2}$$

for some scalar constant $c > 0$. You may think of c as varying with ρ, but for simplicity of notation we call it c, as opposed to c_ρ. If $E_{D_\rho}(v)$ denotes the energy expression

$$E_{D_\rho}(v) = \frac{1}{2} \int_\Omega < \gamma_\rho \nabla v, \nabla v > \; dx - \int_{\partial\Omega} f v \; d\sigma \quad ,$$

then

$$E_{(1-r_\rho)\rho D}(v) \le E_{D_\rho}(v) \le E_{(1+r_\rho)\rho D}(v) \quad \forall v \in H^1(\Omega) \ , \tag{4.3}$$

for $c \ge 1$, and

$$E_{(1+r_\rho)\rho D}(v) \le E_{D_\rho}(v) \le E_{(1-r_\rho)\rho D}(v) \quad \forall v \in H^1(\Omega) \ , \tag{4.4}$$

for $0 < c < 1$. Let u_ρ denote the solution to

$$\nabla \cdot (\gamma_\rho \nabla u_\rho) = 0 \quad \text{in } \Omega \ , \quad \frac{\partial u_\rho}{\partial n} = f \quad \text{on } \partial\Omega \ ,$$

with $\int_{\partial\Omega} u_\rho \, d\sigma = 0$, and let $u_\rho^{(\pm)}$ denote the solutions to the same problem, when D_ρ in the definition of γ_ρ is replaced by $(1 \pm r_\rho)\rho D$. The function u_ρ is the minimizer of $E_{D_\rho}(v)$ in $H^1(\Omega) \cap \{ \int_{\partial\Omega} v \, d\sigma = 0 \}$, and $u_\rho^{(\pm)}$ are the $H^1(\Omega) \cap \{ \int_{\partial\Omega} v \, d\sigma = 0 \}$ minimizers of $E_{(1\pm r_\rho)\rho D}(v)$, respectively. From (4.3) and (4.4) we conlude that

$$E_{(1-r_\rho)\rho D}(u_\rho^{(-)}) \le E_{D_\rho}(u_\rho) \le E_{(1+r_\rho)\rho D}(u_\rho^{(+)}) \ ,$$

for $c \ge 1$, and

$$E_{(1+r_\rho)\rho D}(u_\rho^{(+)}) \le E_{D_\rho}(u_\rho) \le E_{(1-r_\rho)\rho D}(u_\rho^{(-)}) \ ,$$

for $0 < c < 1$. As a consequence

$$|E_{D_\rho}(u_\rho) - E_{(1-r_\rho)\rho D}(u_\rho^{(-)})| \le |E_{(1+r_\rho)\rho D}(u_\rho^{(+)}) - E_{(1-r_\rho)\rho D}(u_\rho^{(-)})| \ , \tag{4.5}$$

for any $c > 0$. It is well know that

$$2E_{D_\rho}(u_\rho) = -\int_{\partial\Omega} f u_\rho \, d\sigma \ , \tag{4.6}$$

and similarly

$$2E_{(1\pm r_\rho)\rho D}(u_\rho^{(\pm)}) = -\int_{\partial\Omega} f u_\rho^{(\pm)} \, d\sigma \ . \tag{4.7}$$

From Theorem 2 (or rather, Corollary 3) and the formula (3.7) we have the following asymptotic information about $u_\rho^{(-)} - u_0$ and $u_\rho^{(+)} - u_\rho^{(-)}$

$$\begin{aligned} \frac{1}{\rho^d}(u_\rho^{(-)} - u_0)(x) &= (1-r_\rho)^d |D| \nabla_y G(x,0) \cdot (1-c) M \nabla u_0(0) + o(1)(x) \\ &= |D| \nabla_y G(x,0) \cdot (1-c) M \nabla u_0(0) + o(1)(x) \ , \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \frac{1}{\rho^d}(u_\rho^{(+)} - u_\rho^{(-)})(x) &= (1+r_\rho)^d |D| \nabla_y G(x,0) \cdot (1-c) M \nabla u_0(0) \\ &\quad -(1-r_\rho)^d |D| \nabla_y G(x,0) \cdot (1-c) M \nabla u_0(0) + o(1)(x) \\ &= O(r_\rho)(x) + o(1)(x) = o(1)(x) \ , \end{aligned} \tag{4.9}$$

where $o(1)(\cdot)$ represents a term that tends to zero in any $H^s(\partial\Omega)$ norm, uniformly with respect to $c > 0$ and $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ \ , \ \int_{\partial\Omega} f \ d\sigma = 0 \right\}$, as $\rho \to 0$. In the second identities of (4.8) and (4.9) we have used the uniform boundedness of $(1-c)M = (1-c)M(c)$ (see the discussion following Corollary 3). It follows immediately by a combination of (4.5), (4.7), and (4.9) that

$$
\begin{aligned}
|E_{D_\rho}(u_\rho) - E_{(1-r_\rho)\rho D}(u_\rho^{(-)})| &\leq |E_{(1+r_\rho)\rho D}(u_\rho^{(+)}) - E_{(1-r_\rho)\rho D}(u_\rho^{(-)})| \\
&= \frac{1}{2}|\int_{\partial\Omega} f(u_\rho^{(+)} - u_\rho^{(-)}) \ d\sigma| \\
&= \rho^d o(1) \ \ ,
\end{aligned}
$$

where the term $o(1)$ tends to zero, uniformly with respect to $c > 0$ and $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ \ , \ \int_{\partial\Omega} f \ d\sigma = 0 \right\}$, as $\rho \to 0$. A subsequent application of (4.6) and (4.7) yields

$$
\begin{aligned}
\left| \int_{\partial\Omega} f(u_\rho - u_\rho^{(-)}) \ d\sigma \right| &= 2\left| E_{D_\rho}(u_\rho) - E_{(1-r_\rho)\rho D}(u_\rho^{(-)}) \right| \\
&= \rho^d o(1) \ \ .
\end{aligned}
$$

Due to the linear dependence of u_ρ and $u_\rho^{(-)}$ on f we thus obtain

$$
\left| \int_{\partial\Omega} f(u_\rho - u_\rho^{(-)}) \ d\sigma \right| \leq \rho^d o(1) \|f\|^2_{H^{-1/2}(\partial\Omega)} \qquad \forall f \in H^{-\frac{1}{2}} \cap \{ \int_{\partial\Omega} f \ d\sigma = 0 \} \ \ ,
$$

where the term $o(1)$ tends to zero, uniformly with respect to $c > 0$ and f, as $\rho \to 0$. Since the operator $f \to (u_\rho - u_\rho^{(-)})\Big|_{\partial\Omega}$ is selfadjoint, a standard polarization argument (as in the proof of Theorem 1) now gives

$$
\left| \int_{\partial\Omega} g(u_\rho - u_\rho^{(-)}) \ d\sigma \right| \leq \rho^d o(1) \|f\|_{H^{-1/2}(\partial\Omega)} \|g\|_{H^{-1/2}(\partial\Omega)} \ \ ,
$$

for all $f, g \in H^{-\frac{1}{2}} \cap \{ \int_{\partial\Omega} f \ d\sigma = 0 \}$. Maximization over $g \in \left\{ \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ \ , \ \int_{\partial\Omega} g \ d\sigma = 0 \right\}$ yields

$$
\|u_\rho - u_\rho^{(-)}\|_{H^{\frac{1}{2}}(\partial\Omega)} = \rho^d o(1) \ \ ,
$$

where $o(1)$ tends to zero, uniformly with respect to $c > 0$ and $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ \ , \ \int_{\partial\Omega} f \ d\sigma = 0 \right\}$, as $\rho \to 0$. By a combination with (4.8) we finally arrive at the asymptotic representation

$$
\begin{aligned}
\frac{1}{\rho^d}(u_\rho - u_0)(x) &= L_\rho + W_\rho + o(1)(x) \\
&= |D| \nabla_y G(x, 0) \cdot (1-c) M \nabla u_0(0) + o(1)(x) \ \ , \qquad (4.10)
\end{aligned}
$$

where the term $o(1)$ tends to zero in $H^{1/2}(\partial\Omega)$, uniformly in $c > 0$ and $f \in \left\{ \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ \ , \ \int_{\partial\Omega} f \ d\sigma = 0 \ \right\}$, as $\rho \to 0$. If we had included a term of the form $\int_\Omega Fv \ dx$ in the energy (as in section 2) then we would immediately have shown that the $o(1)$ term also converges uniformly to zero in the norm $H^1(\Omega\setminus\overline{B_\delta})$. Standard elliptic estimates now instantly (as in section 2) shows that the $o(1)$ term actually converges uniformly to zero in any norm $H^s(\Omega \setminus \overline{B_\delta})$ (or $H^s(\partial\Omega)$). In summary we have therefore established the following theorem.

Theorem 3. *Suppose f is in $H^{-1/2}(\partial\Omega)$ with $\int_{\partial\Omega} f \ d\sigma = 0$, and suppose γ_ρ is given by (4.2), with the bounded, open set D_ρ satisfying (4.1). Let u_0 and u_ρ be the solutions to (1.1) and (1.3), normalized by $\int_{\partial\Omega} u_0 \ d\sigma = \int_{\partial\Omega} u_\rho \ d\sigma = 0$. Let $L_\rho(x)$ and $W_\rho(x)$ be given by (3.4) and (3.5). Then for any fixed $\delta > 0$ and $s \in I\!R_+$*

$$\lim_{\rho\to 0} \|\frac{1}{\rho^d}(u_\rho - u_0) - (L_\rho + W_\rho)\|_{H^s(\Omega\setminus\overline{B_\delta})} = 0 \quad ,$$

and as a consequence

$$\lim_{\rho\to 0} \|\frac{1}{\rho^d}(u_\rho - u_0) - (L_\rho + W_\rho)\|_{H^s(\partial\Omega)} = 0 \quad ,$$

for any $s \in I\!R_+$. Moreover, these limiting processes are uniform in $c > 0$ and in $f \in \left\{\|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1 \ \ , \int_{\partial\Omega} f \ d\sigma = 0\right\}$.

5 Some remarks on cloaking

The main idea behind "cloaking by mapping" is that the Neumann to Dirichlet data map (or the Dirichlet to Neumann data map) of a domain is appropriately invariant under mappings that preserve points on the boundary. To be quite precise: if Ψ is a continuous, piecewise smooth mapping that maps Ω injectively onto Ω, and with $\Psi(x) = x$ for all $x \in \partial\Omega$, then the conductivities

$$\gamma > 0 \qquad \text{and } \Psi_*\gamma = \frac{D\Psi\gamma D\Psi^t}{|\det D\Psi|} \circ \Psi^{-1} > 0 \tag{5.1}$$

have the same Neumann to Dirichlet data map. What this means is that to any observer outside Ω, γ and $\Psi_*\gamma$ will look like the same conductivity. This of course does not contradict well known uniqueness results for the isotropic conductivity in terms of a given Dirichlet to Neumann map [2, 11, 13, 16], rather these uniqueness results may be interpreted as saying that the only Ψ, with $\Psi|_{\partial\Omega} = id$, for which γ and $\Psi_*\gamma$ are simultanenously positive and isotropic is $\Psi = id$ *on all of* Ω. In the context of the conductivity problem the "push forward" construction (5.1) was introduced in [12], and originated from a discussion with L. Tartar.

In order to create a region inside Ω that is perfectly cloaked one selects a Ψ that opens up a single point to this finite sized region, as first discussed in

[8, 9] and later in [15]. Roughly speaking any conductivity, put inside the perfectly cloaked region, in the corresponding "pulled back" formulation "lives" at a point, and is thus invisible as far as the solution to the boundary value problem is concerned. A rigorous treatment of this phenomenon involves a discussion of what are appropriate (physical) solutions to elliptic problems with degenerate coefficients, since the conductivity cloak $\Psi_*\gamma$ becomes very degenerate when a point is opened up to a finite sized region. Such a rigorous discussion is found in [8] and [10]. To introduce regions that are approximately cloaked – by means of *non-degenerate* cloaks – a natural procedure is now to "blow up" a very small region to a finite sized region. To answer the question, of exactly how good the approximate cloaking is, one must estimate the effect of a small inhomogeneity of completely arbitrary conductivity on the Neumann to Dirichlet data map. For more details about such estimates of the level of approximate cloaking associated with piecewise smooth mappings we refer to [10]. In that paper we used a monotonicity argument and information about the effect of small inhomogeneities of extreme (isotropic) conductivities, derived in [7]. The analysis could have been simplified if we had at our disposal had the estimate of Corollary 1. This corollary is also particularly well suited to the case, when the small domains are not exactly dilatations of a fixed domain, as happens when we consider approximately cloaked regions that are not balls. The results contained in Theorem 2 and Theorem 3 about the two principal terms of the asymptotic expansion of u_ρ also have potential applications to approximate cloaks. These applications concern estimates of the level of approximate cloaking as well as questions of design. The result of Theorem 2 allows the identification of the exact level of cloaking associated with a particular object (the conductivity of which is the appropriate "push-forward" of $\gamma_{1,\rho}$). Since the asymptotics is uniform in $\gamma_{1,\rho}$, the principal terms could very efficiently be used to find the asymtotically most/least visible object for a given background field $\nabla u_0(0)$ (and the circular approximate cloak construction). Alternatively it could be used to determine the worst/best background field to identify a particular object that someone is attempting to cloak. Theorem 3 is of interest when it comes to approximate cloaks that are not of circular shape. A particular construction of such cloaks (by composition of mappings) is discussed in [10]. In that case the small inhomogeneity (that is mapped to the finite sized approximately cloaked region) is of the form $F(\rho B_1)$ for some smooth map F. For simplicity let us assume $F(0) = 0$. Then we have exactly (by performing a Taylor expansion of F around 0) that $(1-C\rho)\rho DF(0)B_1 \subset F(\rho B_1) \subset (1+C\rho)\rho DF(0)B_1$, where C depends on bounds of the second derivatives of F. We are thus exactly in the situation covered by Theorem 3, the domain D being the ellipse $DF(0)B_1$. For a given background field $\nabla u_0(0)$ (or for a given family of background fields) the very precise information about the magnitude of the effect of the small inhomogeneity, described in Theorem 3, may be used to determine the best/worst ellipse from the point of of view of approximate cloaking of a uniformly conducting object.

6 Appendix: auxiliary results for exterior domains

In this section we present some results concerning the solution of exterior problems that were used extensively in sections 3.1 and 3.2. We first introduce some convenient notation. Let U be a connected, smooth open region of $I\!R^d$ $(d = 2, 3)$ with a bounded complement (this includes $U = I\!R^d$). The space $W^1(U)$ is defined as follows

$$W^1(U) = \big\{u \in L^1_{loc}(U) : \frac{u(x)}{\ln(2 + |x|)\sqrt{1 + |x|^2}} \in L^2(U) \text{ and } \nabla u \in L^2(U)\big\} \quad ,$$

for $d = 2$, and

$$W^1(U) = \big\{u \in L^1_{loc}(U) \ : \ \frac{u(x)}{\sqrt{1 + |x|^2}} \in L^2(U) \text{ and } \nabla u \in L^2(U)\big\} \quad ,$$

for $d = 3$. This definition is taken from [14, page 59]. We recall the following result (see e.g. [14, Section 2.5.4]).

Proposition 1. *$W^1(U)$ is a Hilbert space with the scalar product*

$$\langle u, v\rangle_{W^1} = \int_U \Big(\nabla u \cdot \nabla v + \frac{u(x)v(x)}{\ln^2(2 + |x|)(1 + |x|^2)}\Big)\, dx \quad , \qquad \text{for } d = 2 \quad ,$$

and

$$\langle u, v\rangle_{W^1} = \int_U \Big(\nabla u \cdot \nabla v + \frac{u(x)v(x)}{1 + |x|^2}\Big)\, dx \quad , \qquad \text{for } d = 3 \quad .$$

For $d = 3$ we may omit the zero'th order term, i.e., an equivalent scalar product on $W^1(U)$ is given by

$$\langle u, v\rangle_{W^1} = \int_U \nabla u \cdot \nabla v\, dx \quad , \qquad \text{for } d = 3 \quad .$$

Remark 1. *For $d = 2$ we cannot omit the zero'th order term and still obtain an equivalent scalar product, for the simple reason that constants lie in the set $W^1(U)$. On the other hand if $\mathfrak{N}(\cdot)$ is a continuous linear functional on $W^1(U)$ with the property that:*

$$c \text{ constant, and } \mathfrak{N}(c) = 0 \implies c = 0 \quad ,$$

then the scalar product $\langle u, v\rangle_{W^1} = \int_U \nabla u \cdot \nabla v\, dx$ is equivalent to the scalar product $\langle u, v\rangle_{W^1} = \int_U \Big(\nabla u \cdot \nabla v + \frac{u(x)v(x)}{\ln^2(2+|x|)(1+|x|^2)}\Big)\, dx$ on the closed, co-dimension 1 subspace

$$W^1_{\mathfrak{N}}(U) = W^1(U) \cap \{u \ : \ \mathfrak{N}(u) = 0\} \quad ,$$

see Theorem 2.5.13 of [14].

In the following we shall always assume that D is an smooth, bounded, simply connected domain in $\mathbb{R}^d$ $(d = 2, 3)$. We note that as a consequence $U = \mathbb{R}^d \setminus \overline{D}$ has only one connected component.

Proposition 2. *Suppose $g \in H^{\frac{1}{2}}(\partial D)$. There exists a unique solution $V \in W^1(\mathbb{R}^d \setminus \overline{D})$ to*

$$\begin{cases} -\Delta V = 0 & \text{in } \mathbb{R}^d \setminus \overline{D} \quad , \\ V = g & \text{on } \partial D \quad . \end{cases}$$

This solution satisfies

$$\|V\|_{W^1(\mathbb{R}^d \setminus \overline{D})} \leq C \|g\|_{H^{\frac{1}{2}}(\partial D)} \quad . \tag{6.1}$$

Additionally, in the case $d = 2$,

$$\int_{\partial D} \frac{\partial V}{\partial n} \, d\sigma = 0 \quad . \tag{6.2}$$

For both $d = 2$ and $d = 3$ V has the representation

$$V(x) = \int_{\partial D} \frac{\partial}{\partial n_y} \Phi(x,y) V \, d\sigma_y - \int_{\partial D} \Phi(x,y) \frac{\partial V}{\partial n_y} \, d\sigma_y \; + C_V \quad , \quad x \in \mathbb{R}^d \setminus \overline{D} \quad ,$$

where C_V is a constant,

$$\Phi(x,y) = \begin{cases} -\dfrac{1}{2\pi} \ln|x-y| & \text{if } d = 2 \quad , \\ \dfrac{1}{4\pi |x-y|} & \text{if } d = 3 \quad , \end{cases}$$

and n_y denotes the exterior unit normal vector to D at $y \in \partial D$. The constant C_V is equal to 0 in the case $d = 3$.

Proof. We first consider the case $d = 3$. The arguments leading to existence and uniqueness of V are standard, and so is the estimate (6.1) for V. Since $V \in W^1(\mathbb{R}^d \setminus \overline{D})$ one has

$$\lim_{r\to\infty} \int_{B_{2r}\setminus B_r} \left(|\nabla V|^2 + \frac{|V|^2}{|x|^2} \right) dx = 0 \quad .$$

On the other hand, for each r (sufficiently large) there exists $R \in (r, 2r)$ such that

$$r \int_{\partial B_R} \left(|\nabla V|^2 + \frac{|V|^2}{|R|^2} \right) d\sigma = \int_{B_{2r}\setminus B_r} \left(|\nabla V|^2 + \frac{|V|^2}{|x|^2} \right) dx \quad .$$

Therefore, there exists a sequence R_k such that $\lim_{k\to\infty} R_k = \infty$ and

$$\lim_{k\to\infty} R_k \int_{\partial B_{R_k}} \left(|\nabla V|^2 + \frac{|V|^2}{|R_k|^2} \right) d\sigma = 0 \quad . \tag{6.3}$$

Given any $x \in I\!R^d \setminus \overline{D}$, Green's identity gives

$$\begin{aligned} V(x) \quad &= \int_{\partial D} \Big(\frac{\partial}{\partial n_y} \Phi(x,y) V(y) - \Phi(x,y) \frac{\partial V}{\partial n_y}(y) \Big) \, d\sigma_y \quad , \\ &+ \int_{\partial B_{R_k}} \Big(\Phi(x,y) \frac{\partial V}{\partial n_y}(y) - \frac{\partial}{\partial n_y} \Phi(x,y) V(y) \Big) \, d\sigma_y \quad . \end{aligned} \tag{6.4}$$

Here n_y denotes the normal vector directed into the exterior of B_{R_k} or D at the point $y \in \partial B_{R_k}$ or $y \in \partial D$. The second term in this formula satisfies the estimate

$$\begin{aligned} &\left| \int_{\partial B_{R_k}} \Big(\Phi(x,y) \frac{\partial V}{\partial n_y}(y) - \frac{\partial}{\partial n_y} \Phi(x,y) V(y) \Big) \, d\sigma_y \right| \\ &\qquad \le C_x \int_{\partial B_{R_k}} \Big(\frac{|\nabla V|}{R_k} + \frac{|V|}{R_k^2} \Big) \, d\sigma \quad . \end{aligned}$$

An application of Hölder's inequality yields

$$\begin{aligned} &\left| \int_{\partial B_{R_k}} \Big(\Phi(x,y) \frac{\partial V}{\partial n_y}(y) - \frac{\partial}{\partial n_y} \Phi(x,y) V(y) \Big) \, d\sigma_y \right| \\ &\qquad \le C_x \left(\left(\int_{\partial B_{R_k}} |\nabla V|^2 \, d\sigma \right)^{\frac{1}{2}} + \left(\int_{\partial B_{R_k}} \frac{|V|^2}{R_k^2} \, d\sigma \right)^{\frac{1}{2}} \right) \quad . \end{aligned}$$

Letting k tend to infinity, and using (6.3), we get

$$\lim_{k\to\infty} \left| \int_{\partial B_{R_k}} \Big(\Phi(x,y) \frac{\partial V}{\partial n_y}(y) - \frac{\partial}{\partial n_y} \Phi(x,y) V(y) \Big) \, d\sigma_y \right| = 0 \quad .$$

By inserting this into (6.4) we finally obtain

$$V(x) = \int_{\partial D} \Big(\frac{\partial}{\partial n_y} \Phi(x,y) V(y) - \Phi(x,y) \frac{\partial V}{\partial n_y}(y) \Big) \, d\sigma_y \quad ,$$

as desired.

We now turn to the case $d = 2$. The arguments leading to existence and uniqueness of V are standard, and so is the estimate (6.1) for V. Since $V \in W^1(\mathbb{R}^2 \setminus \overline{D})$ we may analogously to (6.3) in this case prove that there exists a sequence $R_k \to \infty$, as $k \to \infty$, such that

$$\lim_{k\to\infty} R_k \int_{\partial B_{R_k}} \Big(|\nabla V|^2 + \frac{|V|^2}{\ln^2 R_k \, |R_k|^2} \Big) \, d\sigma = 0 \quad ,$$

or equivalently

$$\lim_{k\to\infty} R_k \int_{\partial B_{R_k}} |\nabla V|^2 \, d\sigma = 0 \quad , \quad \text{and} \tag{6.5}$$

$$\lim_{k\to\infty} R_k \int_{\partial B_{R_k}} \frac{|V|^2}{\ln^2 R_k \, |R_k|^2} \, d\sigma = 0 \quad . \tag{6.6}$$

The identity (6.2) follows from (6.5) if we let k tend to ∞ in the estimate

$$\Big| \int_{\partial D} \frac{\partial V}{\partial n}\, d\sigma \Big| = \Big| \int_{\partial B_{R_k}} \frac{\partial V}{\partial n}\, d\sigma \Big| \leq (2\pi R_k)^{\frac{1}{2}} \Big| \int_{\partial B_{R_k}} \Big|\frac{\partial V}{\partial n}\Big|^2 d\sigma \Big|^{\frac{1}{2}} \quad .$$

For $r_1 < r_2$ sufficiently large (that $\overline{D} \subset B_{r_1}$) Green's identity applied to the annulus $B_{r_2} \setminus \overline{B_{r_1}}$ gives

$$\int_{\partial B_{r_1}} \Big(\frac{\partial}{\partial n_y} \Phi(0,y) V(y) - \Phi(0,y) \frac{\partial V}{\partial n_y}(y) \Big)\, d\,\sigma_y$$
$$= \int_{\partial B_{r_2}} \Big(\frac{\partial}{\partial n_y} \Phi(0,y) V(y) - \Phi(0,y) \frac{\partial V}{\partial n_y}(y) \Big)\, d\,\sigma_y \quad .$$

Since $\int_{\partial B_r} \frac{\partial V}{\partial n}\, d\sigma = \int_{\partial D} \frac{\partial V}{\partial n}\, d\sigma = 0$ (as just proven above) and since $\frac{\partial}{\partial n_y}\Phi(0,y)$ and $\Phi(0,y)$ are both constant on ∂B_r, with $\frac{\partial}{\partial n_y}\Phi(0,y) = -\frac{1}{2\pi r}$, it now follows that

$$\frac{1}{2\pi r_1} \int_{\partial B_{r_1}} V(y)\, d\,\sigma_y = \frac{1}{2\pi r_2} \int_{\partial B_{r_2}} V(y) \sigma_y \quad ,$$

for r_1 and r_2 sufficiently large. Thus the expression

$$\begin{aligned} C_V &= \int_{\partial B_r} \Big(\Phi(0,y) \frac{\partial V}{\partial n_y}(y) - \frac{\partial}{\partial n_y} \Phi(0,y) V(y) \Big)\, d\sigma_y \quad , \\ &= -\frac{1}{2\pi} \log r \int_{\partial B_r} \frac{\partial V}{\partial n_y}(y)\, d\sigma_y + \frac{1}{2\pi r} \int_{\partial B_r} V(y)\, d\sigma_y \\ &= \frac{1}{2\pi r} \int_{\partial B_r} V(y)\, d\sigma_y \quad , \end{aligned}$$

does indeed define a constant, independent of r, for r sufficently large. It follows directly from Green's identity (see (6.4)) that

$$\begin{aligned} &V(x) - C_V - \int_{\partial D} \Big(\frac{\partial}{\partial n_y} \Phi(x,y) V(y) - \Phi(x,y) \frac{\partial V}{\partial n_y}(y) \Big)\, d\sigma_y \quad , \\ &= \int_{\partial B_{R_k}} \Big(\left(\Phi(x,y) - \Phi(0,y)\right) \frac{\partial V}{\partial n_y}(y) - \frac{\partial}{\partial n_y} \left(\Phi(x,y) - \Phi(0,y)\right) V(y) \Big)\, d\sigma_y \quad . \end{aligned} \tag{6.7}$$

Since

$$|\Phi(x,y) - \Phi(0,y)| = \frac{1}{2\pi} \left| \log \left| \frac{x}{|y|} - \frac{y}{|y|} \right| \right| \leq C_x \frac{1}{|y|} \quad ,$$

and

$$\left| \frac{\partial}{\partial n_y} \left(\Phi(x,y) - \Phi(0,y)\right) \right| = \frac{1}{2\pi} \left| \frac{|y| - x \cdot y/|y|}{|y-x|^2} - \frac{|y|}{|y|^2} \right| \leq C_x \frac{1}{|y|^2}$$

on ∂B_{R_k}, we easily estimate

$$\begin{aligned}
&\left|\int_{\partial B_{R_k}} \Big((\Phi(x,y)-\Phi(0,y)) \frac{\partial V}{\partial n_y}(y) - \frac{\partial}{\partial n_y}\left(\Phi(x,y)-\Phi(0,y)\right) V(y)\Big)\, d\sigma_y\right| \\
&\quad \le C_x \int_{\partial B_{R_k}} \left(\frac{|\nabla V|}{R_k} + \frac{|V|}{R_k^2} \right)\, d\sigma \\
&\quad \le C_x \left(\frac{1}{\sqrt{R_k}} (\int_{\partial B_{R_k}} |\nabla V|^2 \, d\sigma)^{1/2} + \frac{\ln R_k}{\sqrt{R_k}} (\int_{\partial B_{R_k}} \frac{|V|^2}{\ln^2 R_k\, R_k^2}\, d\sigma)^{1/2} \right) \\
&\quad \to 0 \quad \text{as } k \to \infty \quad . \qquad (6.8)
\end{aligned}$$

Here we have used (6.5) – (6.6) to obtain the final convergence. Insertion of (6.8) into (6.7) shows that

$$V(x) = \int_{\partial D} \Big(\frac{\partial}{\partial n_y} \Phi(x,y) V(y) - \Phi(x,y) \frac{\partial V}{\partial n_y}(y) \Big)\, d\sigma_y + C_V \quad ,$$

exactly as desired. This completes the proof of Proposition 2. □

Remark 2. *Even though we shall not use it in this paper, we note that a simple argument shows that*

$$C_V = \lim_{R\to\infty} \frac{1}{2\pi R} \int_{\partial B_R} V(y)\, d\sigma_y = \lim_{|x|\to\infty} V(x) \quad , \ \textit{for } d = 2 \quad .$$

There is an analogue of Proposition 2 for the exterior Neumann problem. In this paper we shall only use this result in the case $d = 3$. Its proof is entirely similar to the proof of Proposition 2, and is left to the reader.

Proposition 3. *Suppose $d = 3$ and $g \in H^{-\frac{1}{2}}(\partial D)$. There exists a unique solution $V \in W^1(I\!R^3 \setminus \overline{D})$ to*

$$\begin{cases} -\Delta V &= 0 \quad \textit{in } I\!R^3 \setminus \overline{D} \quad , \\ \frac{\partial V}{\partial n} &= g \quad \textit{on } \partial D \quad . \end{cases}$$

This solution satisfies

$$\|V\|_{W^1(I\!R^3\setminus\overline{D})} \le C \|g\|_{H^{-\frac{1}{2}}(\partial D)} \quad .$$

Moreover,

$$V(x) = \int_{\partial D} \frac{\partial}{\partial n_y} \Phi(x,y) V\, d\sigma_y - \int_{\partial D} \Phi(x,y) \frac{\partial V}{\partial n_y}\, d\sigma_y \quad , \quad x \in I\!R^3 \setminus \overline{D} \quad ,$$

where

$$\Phi(x,y) = \frac{1}{4\pi |x-y|} \quad ,$$

and n_y denotes the exterior unit normal vector to D at $y \in \partial D$.

For $v \in C^1(I\!R^d \setminus D) \times C^1(\overline{D})$ we define

$$[v] = v|_{\text{ext}} - v|_{\text{int}} \quad \text{on } \partial D \quad .$$

We note that this difference has a continuous extension to $W^1(I\!R^d \setminus \overline{D}) \times H^1(D)$. We use the notation $(H^1(D))^*$ for the dual of $H^1(D)$, and given $f \in (H^1(D))^*$ the action of f on the function 1 is denoted by $\int_D f \; dx$.

Proposition 4. *Let γ be an element of*

$$L^\infty_+(D) = (L^\infty(D))^{d\times d} \cap \{ \ \gamma(x) \ \ \textit{symmetric, positive definite, } \ \textit{ess inf}\, \gamma > 0 \ \} \ ,$$

and suppose $f \in (H^1(D))^*$, $g \in H^{-\frac{1}{2}}(\partial D)$, $h \in H^{\frac{1}{2}}(\partial D)$. In dimension $d = 2$ suppose additionally $\int_D f \; dx + \int_{\partial D} g \; d\sigma = 0$. There exists a solution $V \in W^1(I\!R^d \setminus \overline{D}) \times H^1(D)$ to

$$\begin{cases} \nabla \cdot (\gamma \nabla V) & = f \quad \textit{in } D \ , \\ \Delta V & = 0 \quad \textit{in } I\!R^d \setminus \overline{D} \ , \\ \frac{\partial V}{\partial n}\Big|_{\text{ext}} - (\gamma \nabla V) \cdot n\Big|_{\text{int}} & = g \quad \textit{on } \partial D \ , \\ [V] & = h \quad \textit{on } \partial D \ . \end{cases}$$

In dimension $d = 3$ this solution is unique. In dimension $d = 2$ the solution is unique modulo an additive constant – we may make it unique by imposing the condition $\int_{\partial D} V \; dx = 0$. Moreover,

$$V(x) = \int_{\partial D} \frac{\partial}{\partial n_y} \Phi(x,y) V\Big|_{\text{ext}} d\sigma_y - \int_{\partial D} \Phi(x,y) \frac{\partial V}{\partial n_y}\Big|_{\text{ext}} d\sigma_y \ + C_V \quad , \tag{6.9}$$

$x \in I\!R^d \setminus \overline{D}$, where C_V is a constant,

$$\Phi(x,y) = \begin{cases} -\dfrac{1}{2\pi} \ln|x-y| & \textit{if } d = 2 \ , \\ \dfrac{1}{4\pi|x-y|} & \textit{if } d = 3 \ , \end{cases}$$

and n_y denotes the exterior unit normal vector to D at $y \in \partial D$. The constant C_V is equal to 0 in the case $d = 3$. The solution V depends linearly and continuously on the data f, g and h. In the particular case when $f = 0$ it satisfies the estimate

$$\|\nabla V\|_{L^2(I\!R^d \setminus \overline{D})} + \|\gamma^{1/2} \nabla V\|_{L^2(D)} \le C \left(\|g\|_{H^{-\frac{1}{2}}(\partial D)} + \|h\|_{H^{\frac{1}{2}}(\partial D)} \right) \quad , \tag{6.10}$$

with a constant C that is independent of γ (and of g and h).

Proof A standard variational argument yields the uniqueness of V (modulo a constant in the case $d = 2$). We proceed to verify the existence. Applying the Lax-Milgram Lemma and Proposition 2 we may select $\tilde{W} = (\tilde{W}_1, \tilde{W}_2) \in W^1(I\!R^d \setminus \overline{D}) \times H^1(D)$ by first requiring that

$$\begin{cases} \nabla \cdot (\gamma \nabla \tilde{W}_2) &= f & \text{in } D \quad , \\ (\gamma \nabla \tilde{W}_2) \cdot n &= \frac{1}{|\partial D|} \int_D f \; dx & \text{on } \partial D \quad , \end{cases}$$

and secondly requiring that

$$\begin{cases} \Delta \tilde{W}_1 &= 0 & \text{in } I\!R^d \setminus \overline{D} \quad , \\ \tilde{W}_1 &= h + \tilde{W}_2 & \text{on } \partial D \quad . \end{cases}$$

We notice that if $f = 0$ then we may select $\tilde{W}_2 = 0$ and, according to Proposition 2, $\tilde{W}_1$ therefore satisfies the estimate

$$\|\tilde{W}_1\|_{W^1(I\!R^d \setminus \overline{D})} \leq C \|h\|_{H^{\frac{1}{2}}(\partial D)} \quad .$$

In combination with the fact that $\tilde{W}_1$ is harmonic in $I\!R^d \setminus \overline{D}$, this yields

$$\|\frac{\partial \tilde{W}_1}{\partial n}\|_{H^{-1/2}(D)} \leq C \|h\|_{H^{\frac{1}{2}}(\partial D)} \qquad \text{when } f = 0 \quad , \tag{6.11}$$

with a constant C that is independent of γ. For $d = 2$ we have, according to Proposition 2, that $\int_{\partial D} \frac{\partial \tilde{W}_1}{\partial n} \; d\sigma = 0$. We now decompose the function V as $V = \tilde{V} + \tilde{W}$ where $\tilde{V} \in W^1(I\!R^d \setminus \overline{D}) \times H^1(D)$ is a solution to

$$\begin{cases} \nabla \cdot (\gamma \nabla \tilde{V}) &= 0 & \text{in } D \quad , \\ \Delta \tilde{V} &= 0 & \text{in } I\!R^d \setminus \overline{D} \quad , \\ \frac{\partial \tilde{V}}{\partial n}\Big|_{\text{ext}} - (\gamma \nabla \tilde{V}) \cdot n\Big|_{\text{int}} &= g - \frac{\partial \tilde{W}_1}{\partial n} + \frac{1}{|\partial D|} \int_D f \; dx & \text{on } \partial D \quad , \\ [V] &= 0 & \text{on } \partial D \quad . \end{cases}$$

The existence of such a $\tilde{V}$ is again a classical consequence of the Lax-Milgram Lemma. It is also easy to see that $\tilde{V}$ satisfies the estimate

$$\|\nabla \tilde{V}\|_{L^2(I\!R^d \setminus \overline{D})} + \|\gamma^{1/2} \nabla \tilde{V}\|_{L^2(D)} \leq C \|g - \frac{\partial \tilde{W}_1}{\partial n} + \frac{1}{|\partial D|} \int f \; dx\|_{H^{-\frac{1}{2}}(\partial D)} \quad ,$$

with C independent of γ. In the particular case when $f = 0$ (and thus $\tilde{W}_2 = 0$) we get by a combination of this with (6.11)

$$\begin{aligned} \|\nabla V\|_{L^2(I\!R^d \setminus \overline{D})} + \|\gamma^{1/2} \nabla V\|_{L^2(D)} &\leq C \left(\|g - \frac{\partial \tilde{W}_1}{\partial n}\|_{H^{-\frac{1}{2}}(\partial D)} + \|h\|_{H^{\frac{1}{2}}(\partial D)} \right) \\ &\leq C \left(\|g\|_{H^{-\frac{1}{2}}(\partial D)} + \|h\|_{H^{\frac{1}{2}}(\partial D)} \right) \quad , \end{aligned}$$

as stated in (6.10). Finally, the representation formula (6.9) is an immediate consequence of the analogous formula from Proposition 2. □

Remark 3. *Just as in Proposition 2 the constant C_V may easily be shown to have the form*

$$C_V = \lim_{R\to\infty} \frac{1}{2\pi R} \int_{\partial B_R} V(y)\, d\sigma_y = \lim_{|x|\to\infty} V(x) \quad ,$$

for $d = 2$.

Acknowledgments

The research of M.S. Vogelius was partially supported by NSF grant DMS–0604999.

References

[1] H. Ammari and H. Kang, Reconstruction of Small Inhomogeneities from Boundary Measurements, Lecture Notes in Mathematics 1846, *Springer-Verlag* (2004)

[2] K. Astala, and L. Päivärinta, Calderón's inverse conductivity problem in the plane, *Ann. of Math.* **163** (2006) pp. 265–299

[3] M. Brühl, M. Hanke, and M.S. Vogelius, A direct impedance tomography algorithm for locating small inhomogeneities, *Numer. Math.* **93** (2003) pp. 635–654

[4] Y. Capdeboscq, and M.S. Vogelius, A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction, *Math. Modelling Num. Anal.* **37** (2003) pp. 159–173

[5] D.J. Cedio-Fengya, S. Moskow, and M.S. Vogelius, Identification of conductivity imperfections of small diameter by boundary measurements. Continuous dependence and computational reconstruction, *Inverse Problems* **14** (1998) pp. 553–595

[6] G.B. Folland, Introduction to Partial Differential Equations, *Princeton University Press* (1976)

[7] A. Friedman and M. Vogelius, Identification of small inhomogeneities of extreme conductivity by boundary measurements: a theorem on continuous dependence, *Arch. Rational Mech. Anal.* **105** (1989) pp. 299–326

[8] A. Greenleaf, M. Lassas, and G. Uhlmann, On nonuniqueness for Calderon's inverse problem, *Mathematical Research Letters* **10** (2003) pp. 685–693

[9] A. Greenleaf, M. Lassas, and G. Uhlmann, Anisotropic conductivities that cannot be detected by EIT, *Physiological Measurement* **24** (2003) pp. 413–419

[10] R.V. Kohn, H. Shen, M.S. Vogelius, and M.I. Weinstein, Cloaking via change of variables in Electrical Impedance Tomography, *Inverse Problems* **24** (2008) 015016 (21pp)

[11] R.V. Kohn and M Vogelius, Determining conductivity by boundary measurements II. Interior results, *Comm. Pure & Applied Math.* **38** (1985) pp. 643–667

[12] R.V. Kohn and M Vogelius, Relaxation of a variational method for impedance computed tomography, *Comm. Pure & Applied Math.* **40** (1987) pp. 745–777

[13] A.I. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, *Ann. of Math.* **143** (1996) pp. 71–96

[14] J.-C. Nedelec, Acoustic and Electromagnetic Equations, *Applied Mathematical Sciences* **144** *Springer Verlag* (2001)

[15] J.B. Pendry, D. Schurig, and D.R. Smith, Controlling electromagnetic fields, *Science* **312** (2006) pp. 1780–1782

[16] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, *Ann. of Math.* **125** (1987) pp. 153–169