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## THE EQUIVALENCE PROBLEM AND RIGIDITY FOR HYPERSURFACES EMBEDDED INTO HYPERQUADRICS

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ABSTRACT. We consider the class of Levi nondegenerate hypersurfaces  $M$  in  $\mathbb{C}^{n+1}$  that admit a local (CR transversal) embedding, near a point  $p \in M$ , into a standard nondegenerate hyperquadric in  $\mathbb{C}^{N+1}$  with codimension  $k := N - n$  small compared to the CR dimension  $n$  of  $M$ . We show that, for hypersurfaces in this class, there is a normal form (which is closely related to the embedding) such that any local equivalence between two hypersurfaces in normal form must be an automorphism of the associated tangent hyperquadric. We also show that if the signature of  $M$  and that of the standard hyperquadric in  $\mathbb{C}^{N+1}$  are the same, then the embedding is rigid in the sense that any other embedding must be the original embedding composed with an automorphism of the quadric.

### 1. INTRODUCTION, MAIN THEOREMS AND IMMEDIATE APPLICATIONS

Our main objective in this paper is to study the class of real hypersurfaces  $M \subset \mathbb{C}^{n+1}$  which admit holomorphic (or formal) embeddings into the unit sphere (or, more generally, Levi-nondegenerate hyperquadrics) in  $\mathbb{C}^{N+1}$  where the codimension  $k := N - n$  is small compared to  $n$ . Such hypersurfaces play an important role e.g. in deformation theory of singularities where they arise as links of singularities (see e.g. [BM97]). Another source is complex representations of compact groups, where the orbits are always embeddable into spheres due to the existence of invariant scalar products.

One of our main results is a complete normal form for hypersurfaces in this class with a rather explicit solution to the equivalence problem in the following form (Theorem 1.3): *Two hypersurfaces in normal form are locally biholomorphically equivalent if and only if they coincide up to an automorphism of the associated hyperquadric.* Our normal form here is different from the classical one by Chern–Moser [CM74] (which, on the other hand, is valid for the whole class of Levi nondegenerate hypersurfaces), where, in order to verify equivalence of two hypersurfaces, one needs to apply a general automorphism of the associated hyperquadric to one of the hypersurfaces, possibly loosing its normal form, and then perform an algebraically complicated procedure of putting the transformed hypersurface back in normal form. Another advantage of our normal form, comparing

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with the classical one, is that it can be directly produced from an embedding into a hyperquadric and hence does not need any normalization procedure.

Our second main result is a rigidity property for embeddings into hyperquadrics (Theorem 1.6), where we show, under a restriction on the codimension, that any two embeddings into a hyperquadric of a given hypersurface coincide up to an automorphism of the hyperquadric.

Before stating our main results more precisely, we introduce some notation. Let  $\mathbb{H}_\ell^{2n+1}$  denote the standard Levi-nondegenerate hyperquadric with signature  $\ell$  ( $0 \leq \ell \leq n$ ):

$$(1.1) \quad \mathbb{H}_\ell^{2n+1} := \left\{ (z, w) \in \mathbb{C}^n \times \mathbb{C} : \operatorname{Im} w = - \sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^n |z_j|^2 \right\}.$$

Since the hyperquadric  $\mathbb{H}_\ell^{2n+1}$  is clearly (linearly) equivalent to  $\mathbb{H}_{N-\ell}^{2n+1}$ , we may restrict our attention to  $\ell \leq \frac{n}{2}$ . When  $\ell = 0$ ,  $\mathbb{H}_\ell^{2n+1}$  is the Heisenberg hypersurface, also denoted by  $\mathbb{H}^{2n+1}$ , which is locally biholomorphically equivalent to the unit sphere in  $\mathbb{C}^{n+1}$ . For brevity, we shall use the notation  $\langle \cdot, \cdot \rangle_\ell$  for the standard complex-bilinear scalar product form of signature  $\ell$  in  $\mathbb{C}^n$ :

$$(1.2) \quad \langle a, b \rangle_\ell := - \sum_{j=1}^{\ell} a_j b_j + \sum_{j=\ell+1}^n a_j b_j, \quad a, b \in \mathbb{C}^n.$$

The dimension  $n$  will be clear from the context and we shall not further burden the notation by indicating also the dependence of  $\langle \cdot, \cdot \rangle_\ell$  on  $n$ . Recall that a smooth ( $C^\infty$ ) real hypersurface  $M$  in  $\mathbb{C}^{n+1}$  is Levi-nondegenerate of signature  $\ell$  (with  $\ell \leq n/2$ ) at  $p \in M$  if it can be locally approximated, at  $p$ , by a biholomorphic image of  $\mathbb{H}_\ell^{2n+1}$  to third order, i.e. if there are local coordinates  $(z, w) \in \mathbb{C}^n \times \mathbb{C}$  vanishing at  $p$  such that  $M$  is defined, near  $p = (0, 0)$ , by

$$(1.3) \quad \operatorname{Im} w = \langle z, \bar{z} \rangle_\ell + A(z, \bar{z}, \operatorname{Re} w),$$

where  $A(z, \bar{z}, u)$  is a smooth function which vanishes to third order at 0. (In fact, one can assume, after possibly another local change of coordinates, that the function  $A$  vanishes at least to fourth order at 0, cf. [CM74].) In this paper, we shall consider *formal hypersurfaces* defined by formal power series equations of the form (1.3) and — more generally — of the (not necessary graph) form

$$(1.4) \quad \operatorname{Im} w = \langle z, \bar{z} \rangle_\ell + A(z, \bar{z}, w, \bar{w}),$$

where  $A(z, \bar{z}, w, \bar{w})$  is a real-valued formal power series vanishing at least to fourth order. Our motivation for considering equations of this form lies in the fact that they arise naturally in the study of Levi-nondegenerate hypersurfaces  $M \subset \mathbb{C}^{n+1}$  admitting embeddings into hyperquadrics; see Proposition 1.2 and section 5 for a detailed discussion. In this paper, we shall only concern ourselves with the formal study of real hypersurfaces and,

hence, we shall identify smooth functions  $A(z, w, \bar{z}, \bar{w})$  with their formal Taylor series in  $(z, w, \bar{z}, \bar{w})$  at 0.

Given a formal power series  $A(Z, \bar{Z})$ ,  $Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}$ , we can associate to it two linear subspaces  $V_A \subset \mathbb{C}[[Z]]$ ,  $U_A \subset \mathbb{C}[[\bar{Z}]]$ , where  $\mathbb{C}[[X]]$  denotes the ring of formal power series in  $X$  with complex coefficients, as follows:

$$V_A := \text{span}_{\mathbb{C}} \left\{ \frac{\partial^\alpha A}{\partial \bar{Z}^\alpha}(Z, 0) \right\}, \quad U_A := \text{span}_{\mathbb{C}} \left\{ \frac{\partial^\alpha A}{\partial Z^\alpha}(0, \bar{Z}) \right\},$$

where “ $\text{span}_{\mathbb{C}}$ ” stands for the linear span over  $\mathbb{C}$  and  $\alpha$  runs over the set  $\mathbb{N}^{n+1}$  of all multi-indices of nonnegative integers. We shall only consider formal series  $A(Z, \bar{Z})$  which are *real-valued* (i.e. which formally satisfy  $A(Z, \bar{Z}) = \overline{A(Z, \bar{Z})}$ ). For such power series one has  $V_A = \overline{U_A}$ . We define the *rank of A*, denoted  $R(A)$ , to be the dimension of  $V_A$ , possibly infinite. We shall see (Proposition 1.2) that hypersurfaces admitting (CR transversal, see below) embeddings into hyperquadrics can be represented by equations (1.4) with  $A$  being of finite rank. If  $R(A) =: r < \infty$ , then a linear algebra argument shows (cf. [W78, D82, W99]) that there are a nonnegative number  $S(A) =: s \leq R(A)$ , called here the *signature of A*, and formal power series  $\phi_j \in \mathbb{C}[[Z]]$ ,  $j = 1, \dots, r$ , linearly independent (over  $\mathbb{C}$ ), such that

$$(1.5) \quad A(Z, \bar{Z}) = - \sum_{j=1}^s |\phi_j(Z)|^2 + \sum_{j=s+1}^r |\phi_j(Z)|^2$$

(where as usual, in the special cases  $s = 0$  and  $s = r$ , the corresponding void sums in (1.5) are understood to be zero). See also [D01] for conditions on  $A$  yielding  $s = 0$ . Moreover, it holds that the span of the  $\phi_j$  is equal to  $V_A$  (indeed, in the linear algebra argument alluded to above, the  $\phi_j$  are chosen as a “diagonal” basis for  $V_A$ ) and the collection of vectors  $\left( \frac{\partial^\alpha \phi}{\partial \bar{Z}^\alpha}(0) \right)_\alpha$ , where  $\phi = (\phi_1, \dots, \phi_r)$  and  $\alpha$  is as above, spans  $\mathbb{C}^r$ . We should point out here, however, that for an arbitrary collection  $(\phi_j)_{j=1}^r$  in  $\mathbb{C}[[Z]]$ , a number  $s \leq r$  and the corresponding series  $A(Z, \bar{Z})$  defined by (1.5), it only holds, in general, that  $V_A$  is contained in the span of the  $\phi_j$ . In the case where  $V_A$  is strictly contained in the latter span, however, there is another representation of  $A$  in the form (1.5) with another collection of  $r' < r$  formal power series.

**Definition 1.1.** For each nonnegative integer  $k$ , define the class  $\mathcal{H}_k$  to consist of those real-valued formal power series  $A(z, \bar{z}, w, \bar{w})$  in  $(z, w) \in \mathbb{C}^n \times \mathbb{C}$  satisfying either of the following equivalent conditions:

- (i)  $R(A) \leq k$  and no formal power series in  $V_A$  has a constant or linear term;
- (ii)  $A(z, \bar{z}, w, \bar{w}) = \sum_{j=1}^k \phi_j(z, w) \overline{\psi_j(z, w)}$  for some formal power series  $\phi_j, \psi_j \in \mathbb{C}[[z, w]]$  having no constant or linear terms.

Before stating our main results, we give an auxiliary result relating  $\mathcal{H}_k$  to the class of hypersurfaces admitting formal (CR) embeddings into the hyperquadric  $\mathbb{H}_r^{2N+1}$ . Observe

first that, for  $\ell' \geq n+1$  (and  $\ell' \leq N/2$ , according to our convention),  $\mathbb{C}^{n+1}$  can be linearly embedded into  $\mathbb{H}_{\ell'}^{2N+1}$  via the map

$$Z \mapsto (Z, 0, Z, 0) \in \mathbb{C}^{n+1} \times \mathbb{C}^{\ell'-n-1} \times \mathbb{C}^{n+1} \times \mathbb{C}^{N-n-\ell'}.$$

Hence also any hypersurface  $M \subset \mathbb{C}^{n+1}$  can be “trivially” embedded into  $\mathbb{H}_{\ell'}^{2N+1}$ . In order to avoid this kind of embedding, we restrict our attention here to formal CR embeddings  $H: (M, 0) \rightarrow (M', 0)$  (where  $M' \subset \mathbb{C}^{N+1}$  is another hypersurface) such that

$$dH(T_0M) \not\subset T_0^c M' := T_0M' \cap iT_0M'.$$

We shall call such embeddings *CR transversal*. In particular, in the case  $M' = \mathbb{H}_{\ell'}^{2N+1}$  considered in present paper, CR transversality is equivalent to  $\partial H_{N+1}(0) \neq 0$ . It is easy to see that any CR embedding of a hypersurface  $M \subset \mathbb{C}^{n+1}$  into  $\mathbb{H}^{2N+1}$  is automatically CR transversal. More generally, a CR embedding of  $(M, 0)$  into  $(M', 0)$ , where  $M'$  is Levi-nondegenerate with signature  $\ell' < n$  at 0, is CR transversal. Indeed, if it were not, then, as is well known, the Levi form of  $M'$  at 0 must vanish on the complex tangent space of  $M$  at 0 and this cannot happen unless  $\ell' \geq n$ . We have:

**Proposition 1.2.** *Let  $M$  be a formal Levi-nondegenerate hypersurface in  $\mathbb{C}^{n+1}$  of signature  $\ell \leq n/2$  at a point  $p$ . Then,  $M$  admits a formal CR transversal embedding into  $\mathbb{H}_{\ell'}^{2N+1}$ ,  $\ell' \leq N/2$ , if and only if  $\ell' \geq \ell$  and there are formal coordinates  $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ , vanishing at  $p$ , such that  $M$  is defined by an equation of the form (1.4) with  $A \in \mathcal{H}_{N-n}$ .*

*More precisely, if  $A \in \mathcal{H}_k$  for some  $k$ , then the hypersurface  $M$ , given by (1.4), admits a formal CR transversal embedding into  $\mathbb{H}_{\ell'}^{2N+1}$  with  $N = n+k$  and  $\ell' = \min(\ell + \mathbf{S}(A), N - \ell - \mathbf{S}(A))$ . Conversely, if  $M$  admits a formal CR transversal embedding into  $\mathbb{H}_{\ell'}^{2N+1}$ , then there are formal coordinates  $(z, w)$  as above and a formal power series  $A(z, \bar{z}, w, \bar{w})$  such that  $M$  is defined by (1.4) with  $\mathbf{R}(A) \leq N - n$  and  $\mathbf{S}(A) \leq \max(\ell' - \ell, N' - \ell' - \ell)$ . Moreover, if in addition  $\ell' < n - \ell$ , then  $\mathbf{S}(A) \leq \ell' - \ell$  holds.*

We shall use the notation  $\text{Aut}(\mathbb{H}_{\ell}^{2N+1}, 0)$  for the stability group of  $\mathbb{H}_{\ell}^{2N+1}$  at 0, i.e. for the group of all local biholomorphisms  $(\mathbb{C}^{N+1}, 0) \rightarrow (\mathbb{C}^{N+1}, 0)$  preserving  $\mathbb{H}_{\ell}^{2N+1}$ . Recall that every  $T \in \text{Aut}(\mathbb{H}_{\ell}^{2N+1}, 0)$  is a linear fractional transformation of  $\mathbb{C}^{N+1}$  (see e.g. (1.6) below or [BER00] for an explicit formula). Our first main result states that a defining equation of the form (1.4) with  $A \in \mathcal{H}_k$ ,  $k < n/2$ , is unique modulo automorphisms of the associated quadric. More precisely, we have the following.

**Theorem 1.3.** *Let  $M_j$ ,  $j = 1, 2$ , be formal Levi-nondegenerate hypersurfaces of signature  $\ell$  in  $\mathbb{C}^{n+1}$  given by*

$$\text{Im } w = \langle z, \bar{z} \rangle_{\ell} + A_j(z, \bar{z}, w, \bar{w})$$

*respectively, where  $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ . Assume that  $A_j \in \mathcal{H}_{k_j}$  with  $k_1 + k_2 < n$ . Then,  $(M_1, 0)$  and  $(M_2, 0)$  are formally equivalent if and only if there exists  $T \in \text{Aut}(\mathbb{H}_{\ell}^{2n+1}, 0)$  sending  $M_1$  to  $M_2$ . More precisely, any formal equivalence sending  $(M_1, 0)$  to  $(M_2, 0)$  is*

the Taylor series of an automorphism

$$(1.6) \quad T(z, w) := \frac{(\lambda(z + aw)U, \sigma\lambda^2 w)}{1 - 2i\langle z, \bar{a} \rangle_\ell - (r + i\langle a, \bar{a} \rangle_\ell)w} \in \mathbf{Aut}(\mathbb{H}_\ell^{2n+1}, 0)$$

for some  $\lambda > 0$ ,  $r \in \mathbb{R}$ ,  $a \in \mathbb{C}^n$ ,  $\sigma = \pm 1$  and an invertible  $n \times n$  matrix  $U$  with  $\langle zU, \bar{z}\bar{U} \rangle_\ell = \sigma\langle z, \bar{z} \rangle_\ell$  such that

$$(1.7) \quad A_1 \equiv \sigma\lambda^{-2}|q|^2 A_2 \circ (T, \bar{T}),$$

where  $q = q(z, w)$  is the denominator in (1.6).

**Remark 1.4.** It follows immediately from (1.7) that  $\mathbf{R}(A_1) = \mathbf{R}(A_2)$ , and either  $\mathbf{S}(A_1) = \mathbf{S}(A_2)$  or  $\mathbf{S}(A_1) = \mathbf{R}(A_2) - \mathbf{S}(A_2)$ . Thus, the rank  $\mathbf{R}(A)$  and the two-point-set  $\{\mathbf{S}(A), \mathbf{R}(A) - \mathbf{S}(A)\}$  are invariants of  $M$  (and not only of the equation (1.4)) provided  $A \in \mathcal{H}_k$  for some  $k < n/2$ .

By combining Theorem 1.3 with Proposition 1.2, one can produce examples of formal real hypersurfaces  $M$  which do not admit formal CR transversal embeddings into  $\mathbb{H}_\ell^{2N+1}$ . Let us give an explicit example.

**Example 1.5.** Let  $M \subset \mathbb{C}^{n+1}$  be defined by

$$\operatorname{Im} w = \langle z, \bar{z} \rangle_\ell + A(z, \bar{z}, w, \bar{w}),$$

where  $\ell < n/2$  and

$$A(z, \bar{z}, w, \bar{w}) = \operatorname{Re}(w^s \overline{h(z)}),$$

for some  $s \geq 2$  and  $h(z)$  a (nontrivial) homogeneous polynomial of degree  $\geq 2$ . It is not difficult to see that  $\mathbf{R}(A) = 2$  and, since  $A$  takes both positive and negative values,  $\mathbf{S}(A) = 1$ . Hence, in view of Proposition 1.2 and Theorem 1.3,  $(M, 0)$  cannot be formally embedded into  $\mathbb{H}_\ell^{2N+1}$  for any  $N$  with  $N < 2n - 2$ . On the other hand,  $M$  can be embedded into  $\mathbb{H}_{\ell+1}^{2(n+2)+1}$ , again by Proposition 1.2.

Our second main result of this paper is a rigidity result for formal embeddings into hyperquadrics of the same signature. Since the signature  $\ell$  of  $M$  is  $\leq n/2$  (and then, in particular,  $< n$ ) it follows that every formal embedding in this case is automatically CR transversal.

**Theorem 1.6.** *Let  $M$  be a formal Levi-nondegenerate hypersurface in  $\mathbb{C}^{n+1}$  of signature  $\ell \leq n/2$  at a point  $p$ . If  $\ell = n/2$ , then we shall also assume that  $(M, p)$  is not formally equivalent to  $(\mathbb{H}_{n/2}^{2n+1}, 0)$ . Suppose that, for some integers  $k_1 \leq k_2$ , there are two formal embeddings*

$$H_1: (M, p) \rightarrow \left(\mathbb{H}_\ell^{2(n+k_1)+1}, 0\right), \quad H_2: (M, p) \rightarrow \left(\mathbb{H}_\ell^{2(n+k_2)+1}, 0\right).$$

*If  $k_1 + k_2 < n$ , then there exists an automorphism  $T \in \mathbf{Aut}(\mathbb{H}_\ell^{2(n+k_2)+1}, 0)$  such that*

$$H_2 = T \circ L \circ H_1,$$

where  $L: \mathbb{C}^{n+k_1} \times \mathbb{C} \rightarrow \mathbb{C}^{n+k_2} \times \mathbb{C}$  denotes the linear embedding  $(z, w) \mapsto (z, 0, w)$  which sends  $\mathbb{H}_\ell^{2(n+k_1)+1}$  into  $\mathbb{H}_\ell^{2(n+k_2)+1}$ .

If  $(M, p)$  is formally equivalent to  $(\mathbb{H}_{n/2}^{2n+1}, 0)$ , then the conclusion of Theorem 1.6 fails. Indeed, if  $n = 2\ell$ , then the two embeddings  $L, L_-: (\mathbb{H}_\ell^{2n+1}, 0) \rightarrow (\mathbb{H}_\ell^{2N+1}, 0)$ ,  $N > n$ , where  $L$  denotes the linear embedding as above and

$$(1.8) \quad L_-(z, w) := (z_{\ell+1}, \dots, z_n, z_1, \dots, z_\ell, 0, -w),$$

cannot be transformed into each by composing to the left with  $T \in \text{Aut}(\mathbb{H}_\ell^{2N+1}, 0)$  (cf. e.g. the proof of Theorem 1.3). However, the following holds.

**Theorem 1.7.** *Let  $n$  be even and  $H: (\mathbb{H}_{n/2}^{2n+1}, 0) \rightarrow (\mathbb{H}_{n/2}^{2(n+k)+1}, 0)$  a formal embedding. If  $k < n$ , then there exists  $T \in \text{Aut}(\mathbb{H}_{n/2}^{2(n+k)+1}, 0)$  such that  $T \circ H$  is either the linear embedding  $L$  or the embedding  $L_-$  given by (1.8).*

We would like to point out that the conclusion of Theorem 1.6 fails in general when  $k_1 + k_2 \geq n$ . Indeed, in the case  $k_1 = 0$ ,  $k_2 = n$ ,  $\ell = 0$ , the linear embedding of the sphere can not be represented by a composition of any  $T \in \text{Aut}(\mathbb{H}^{2(n+k_2)+1}, 0)$  and the Whitney embedding; cf. [Fa86, Hu99]). It can also fail in the case where the hyperquadric  $\mathbb{H}_\ell^{2N+1}$  is replaced by one with a greater signature. Such an example is given by the following:

**Example 1.8.** Let  $k_1 = 0$ ,  $k_2 = 2$ ,  $\ell = 0$ , and  $M = \mathbb{H}^{2n+1}$ . Then, for any formal power series  $\phi(z, w)$ , the map

$$H(z, w) := (\phi(z, w), \phi(z, w), z, w) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}$$

embeds  $M$  in  $\mathbb{H}_{\ell'}^{2(n+k_2)+1}$  with  $\ell' = 1$ . Clearly, not every such mapping can be represented by a composition of some  $T \in \text{Aut}(\mathbb{H}_{\ell'}^{2(n+k_2)+1}, 0)$  and the linear embedding  $(z, w) \mapsto (0, 0, z, w)$ .

However, the situation in Example 1.8 is essentially the only way in which the conclusion of Theorem 1.6 can fail in case of different signatures. The reader is referred to Theorem 5.3 (and the subsequent remark) below for the precise statement.

We should point out that if  $M \subset \mathbb{C}^{n+1}$  is a smooth Levi-nondegenerate hypersurface and  $H_j: M \rightarrow \mathbb{H}_\ell^{2(n+k_j)+1}$ ,  $j = 1, 2$ , are smooth embeddings (rather than formal), then the sion of Theorem 1.6 (and, similarly, of Theorem 1.7) is, *a priori*, only that the Taylor series of  $T \circ L \circ H_1$  and  $H_2$  are equal at  $p$ . However, it then follows that  $T \circ L \circ H_1$  and  $H_2$  are equal also as smooth CR mappings. In the case  $\ell = 0$ , this follows by applying a finite determination result due to the first author and B. Lamel [EL02, Corollary 4], and in the case  $\ell > 0$ , when the Levi form has eigenvalues of both signs, by using the well-known holomorphic extension of smooth CR mappings to a neighborhood of  $p$ .

The case of smooth CR embeddings in Theorem 1.6, for  $\ell = 0$  and  $k_1 = k_2$ , was also treated, by a different method, in the authors' recent paper [EHZ02] and previously, for

$k_1 = k_2 = 1$ , by Webster [W79]. The case  $k_1 = 0$ , i.e. of mappings between hyperquadrics, is of special interest. This situation, for  $\ell = 0$ , has been studied by Webster [W79], Cima-Suffridge [CS83], Faran [Fa86], Forstnerič [Fo89], and the second author [Hu99]. We should mention that in many of the later papers treating the case  $k_1 = 0$  and  $\ell = 0$  (mappings between balls), the focus has been on the study of CR mappings with *low initial regularity* (see also the survey [Hu01]). We shall not address this issue in the present paper.

## 2. ADMISSIBLE CLASSES AND UNIQUE SOLVABILITY OF THE CHERN-MOSER OPERATOR

In the celebrated paper [CM74], S. S. Chern and J. K. Moser constructed a formal (or convergent in the case of a real-analytic hypersurface) normal form for Levi-nondegenerate hypersurfaces. In doing so, they introduced a linear operator  $\mathcal{L}$ , which we shall refer to as the *Chern-Moser operator*, sending  $(n+1)$ -tuples  $(f, g) = (f_1, \dots, f_n, g)$  of  $\mathbb{C}^n \times \mathbb{C}$ -valued complex formal power series in  $(z, w) \in \mathbb{C}^n \times \mathbb{C}$  to real formal power series in  $(z, \bar{z}, u) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R}$  defined by

$$\mathcal{L}(f, g) := \operatorname{Im} \left( g(z, w) - 2i \langle \bar{z}, f(z, w) \rangle_\ell \right) \Big|_{w=u+i\langle z, \bar{z} \rangle_\ell},$$

where  $\langle \cdot, \cdot \rangle_\ell$  is defined by (1.2). We shall think of the signature  $\ell$  as being fixed and suppress the dependence of  $\mathcal{L}$  on  $\ell$  in the notation. The fundamental importance of the Chern-Moser operator is that a formal normal form for Levi-nondegenerate hypersurfaces can be produced (as will be explained below) through the study of the unique solvability of  $\mathcal{L}$  over certain spaces of formal power series.

Let us fix  $n \geq 1$  and choose coordinates  $(z, w) \in \mathbb{C}^n \times \mathbb{C}$  with  $z = (z_1, \dots, z_n)$ . We shall also write  $w = u + iv \in \mathbb{R} \oplus i\mathbb{R}$ . Let  $\mathcal{O}$  denote the space of real-valued formal power series in  $(z, \bar{z}, w, \bar{w})$  vanishing of order at least 2 at 0. In particular, any formal power series in  $(z, \bar{z}, u)$  can be viewed as an element in  $\mathcal{O}$  via the substitution  $u = (w + \bar{w})/2$ . Also, let  $\mathcal{F}$  denote the space of  $(n+1)$ -tuples  $(f, g)$  of complex formal power series in  $(z, w)$  satisfying the following normalization condition (which characterizes the unit element of the automorphism group of  $\mathbb{H}_\ell^{2n+1}$  among those of the form  $(z, w) \mapsto (z + f(z, w), w + g(z, w))$ ):

$$(2.1) \quad \frac{\partial(f, g)}{\partial(z, w)}(0) = 0, \quad \operatorname{Re} \left( \frac{\partial^2 g}{\partial w^2} \right)(0) = 0.$$

It is not difficult to see that  $\mathcal{L}$  sends  $\mathcal{F}$  into  $\mathcal{O}$ .

By an *admissible class* we shall mean a subset  $\mathcal{S} \subset \mathcal{O}$  such that the equation

$$\mathcal{L}(f, g)(z, \bar{z}, u) = (A_1(z, \bar{z}, w, \bar{w}) - A_2(z, \bar{z}, w, \bar{w})) \Big|_{w=u+i\langle z, \bar{z} \rangle_\ell}, \quad (f, g) \in \mathcal{F}, \quad A_1, A_2 \in \mathcal{S},$$

has only the trivial solution  $(f, g, A_1, A_2) \equiv 0$ . Admissible classes can be useful in the study of equivalence problems: Let  $\mathcal{S}$  be an admissible class and  $(M_1, 0)$  and  $(M_2, 0)$

be germs of (formal) Levi-nondegenerate real hypersurfaces at 0 defined respectively by equations of the form

$$v = \langle z, z \rangle_\ell + A_j(z, \bar{z}, w, \bar{w}), \quad A_j \in \mathcal{S}, \quad j = 1, 2.$$

Then it follows from the proof of Theorem 4.3 below that, if there is a (formal) holomorphic mapping of the form  $H = \text{id} + (f, g)$  with  $(f, g) \in \mathcal{F}$  sending  $M_1$  to  $M_2$ , it must be the identity map (and hence  $M_1 = M_2$ ). We should also point out that if  $H$  is any (formal) biholomorphic mapping (not necessarily normalized as above) sending  $M_1$  to  $M_2$ , then there exists a unique automorphism  $T \in \text{Aut}(\mathbb{H}_\ell^{2n+1}, 0)$  such that  $H$  factors as  $H = T \circ \tilde{H}$ , where  $\tilde{H}$  is normalized as above (cf. e.g. Lemma 5.1 for a slightly more general statement). Thus, if there are two mappings  $H$  and  $G$  as above, both sending  $M_1$  to  $M_2$ , and which factor through the same automorphism  $T$ , i.e.  $H = T \circ \tilde{H}$  and  $G = T \circ \tilde{G}$  with  $\tilde{H}$  and  $\tilde{G}$  normalized as above, then  $H$  and  $G$  must be equal (since  $H^{-1} \circ G$  will satisfy (2.1) and send  $M_1$  to itself). Hence, in this case a defining equation of the form (1.4) with  $A \in \mathcal{S}$  is unique up to an action by the finite-dimensional stability group  $\text{Aut}(\mathbb{H}_\ell^{2n+1}, 0)$ . However, in general, the action of  $\text{Aut}(\mathbb{H}_\ell^{2n+1}, 0)$  can be complicated, since an additional renormalization may be required to put  $M$  in the form (1.3) with  $A \in \mathcal{S}$  after applying an automorphism  $T \in \text{Aut}(\mathbb{H}_\ell^{2n+1}, 0)$ .

If an admissible class  $\mathcal{S}$  has the property that, for any  $A \in \mathcal{O}$ , the equation

$$\mathcal{L}(f, g) = A \pmod{\mathcal{S}}$$

is always solvable with  $(f, g)$  satisfying (2.1), then  $\mathcal{S}$  is called a *normal space*. It follows from the definition of an admissible class that such a solution  $(f, g)$  must be unique. In the case when  $\mathcal{S}$  is a normal space, any Levi-nondegenerate hypersurface of signature  $\ell$  can be transformed into a hypersurface defined by (1.3) with  $A \in \mathcal{S}$ . In [CM74] the authors construct an explicit normal space, which we call the *Chern-Moser normal space* and denote by  $\mathcal{N}$ . Moreover, they prove there that, for a real-analytic  $M$ , its normal form and the associated transformation map are given by convergent power series.

In this paper, we shall introduce a new admissible class, which is different from the Chern-Moser normal space, and study the unique solvability property of the Chern-Moser operator for this class. Throughout this paper, for a formal power series  $A(z, \bar{z}, w, \bar{w})$ , we shall use the expansion

$$(2.2) \quad A(z, \bar{z}, w, \bar{w}) = \sum_{\mu, \nu, \gamma, \delta} A_{\mu\nu\gamma\delta}(z, \bar{z}) w^\gamma \bar{w}^\delta, \quad (z, w) \in \mathbb{C}^n \times \mathbb{C},$$

where  $A_{\mu\nu\gamma\delta}(z, \bar{z})$  is a bihomogeneous polynomial in  $(z, \bar{z})$  of bidegree  $(\mu, \nu)$  for every  $(\mu, \nu, \gamma, \delta)$ .

**Definition 2.1.** For every positive integer  $k$ , define  $\mathcal{S}_k \subset \mathcal{O}$  to be the subset of all real-valued formal power series  $A(z, \bar{z}, w, \bar{w})$  with  $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ , satisfying either of the following equivalent conditions:

- (i)  $A_{\mu\nu\gamma\delta} = 0$  if  $\nu + \delta \leq 1$  and  $R(A_{\mu\nu\gamma\delta}) \leq k$  if  $\delta \leq 1$ ;
- (ii) for each fixed  $(\mu, \nu, \gamma, \delta)$  with  $\delta \leq 1$ , we can write

$$A_{\mu\nu\gamma\delta}(z, \bar{z})w^\gamma\bar{w}^\delta = \sum_{j=1}^k \phi_j(z, w)\overline{\psi_j(z, w)}$$

for some holomorphic polynomials  $\phi_j, \psi_j$  with no constant or linear terms.

Note that, since  $A$  is real-valued, condition (i) implies also  $A_{\mu\nu\gamma\delta} = 0$  if  $\mu + \gamma \leq 1$  and  $R(A_{\mu\nu\gamma\delta}) \leq k$  if  $\gamma \leq 1$ . Our main result regarding the set  $\mathcal{S}_k$  is the following uniqueness property.

**Theorem 2.2.** *Given any  $A \in \mathcal{S}_{n-1}$ , the equation*

$$(2.3) \quad \mathcal{L}(f, g)(z, \bar{z}, u) = A(z, \bar{z}, w, \bar{w})|_{w=u+i\langle z, \bar{z} \rangle_\ell},$$

for  $(f, g)$  satisfying (2.1) has only the trivial solution  $(f, g) \equiv 0$ .

The conclusion of Theorem 2.2 reduces to a fundamental result of Chern-Moser [CM74], when the right-hand side in (2.3) is in their normal space  $\mathcal{N}$ . Here, we recall that  $\mathcal{N}$  consists of all formal power series  $\sum_{\alpha, \beta} A_{\alpha\bar{\beta}}(u)z^\alpha\bar{z}^\beta$  with  $|\alpha|, |\beta| \geq 2$ , and  $A_{2\bar{2}}, A_{2\bar{3}}, A_{3\bar{3}}$  satisfying certain trace conditions as described in [CM74, pp 233]. It is clear that the right-hand side of (2.3), with  $A \in \mathcal{S}_{n-1}$ , is in general not in the Chern-Moser normal space. For instance,  $A(z, \bar{z}, w, \bar{w}) := |w|^4$  is easily seen to be in  $\mathcal{S}_1$  whereas  $A|_{w=u+i\langle z, \bar{z} \rangle_\ell}$  is not in the Chern-Moser normal space  $\mathcal{N}$ .

### 3. PROOF OF THEOREM 2.2

The proof of Theorem 2.2 is based on the following lemma from [Hu99].

**Lemma 3.1.** *Let  $\phi_j, \psi_j, j = 1, \dots, n-1$ , be germs at 0 of holomorphic functions in  $\mathbb{C}^n$  and  $H(z, \bar{z})$  a germ at 0 of a real-analytic function satisfying*

$$(3.1) \quad H(z, \bar{\xi})\langle z, \bar{\xi} \rangle_\ell = \sum_{j=1}^{n-1} \phi_j(z)\overline{\psi_j(\xi)}.$$

Then

$$H(z, \bar{\xi}) = \sum_{j=1}^{n-1} \phi_j(z)\overline{\psi_j(\xi)} = 0.$$

Lemma 3.1 is the content of Lemma 3.2 in [Hu99] when  $\ell = 0$ . Also, in the statement of [Hu99, Lemma 3.2], it is assumed that  $\phi_j(0) = \psi_j(0) = 0$ . However, the argument there can be used for the proof of Lemma 3.1 without any change. We should also point out that in Lemma 3.1 it is enough to assume the identity

$$H(z, \bar{z})\langle z, \bar{z} \rangle_\ell = \sum_{j=1}^{n-1} \phi_j(z)\overline{\psi_j(z)},$$

since the identity (3.1) then follows by a standard complexification argument. Using Lemma 3.1 and a simple induction argument, which we leave to the reader, we obtain the following generalization of Lemma 3.1.

**Lemma 3.2.** *Let  $\phi_{jp}, \psi_{jp}, j = 1, \dots, n-1, p = 0, \dots, q$ , be germs at 0 of holomorphic functions in  $\mathbb{C}^n$  and  $H(z, \bar{z})$  a germ at 0 of a real-analytic function satisfying*

$$H(z, \bar{\xi}) \langle z, \bar{\xi} \rangle_\ell^{q+1} = \sum_{p=0}^q \left( \sum_{j=1}^{n-1} \phi_{jp}(z) \overline{\psi_{jp}(\xi)} \right) \langle z, \bar{\xi} \rangle_\ell^p.$$

Then

$$H(z, \bar{\xi}) = \sum_{j=1}^{n-1} \phi_{jp}(z) \overline{\psi_{jp}(\xi)} = 0, \quad p = 0, \dots, q.$$

*Proof of Theorem 2.2.* Recall that the Segre variety  $Q_{(\bar{\xi}, \bar{\eta})}$  of  $\mathbb{H}_\ell^{2n+1}$  with respect to  $(\bar{\xi}, \bar{\eta}) \in \mathbb{C}^n \times \mathbb{C}$  is the complex hyperplane defined by

$$Q_{(\bar{\xi}, \bar{\eta})} := \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : w - \eta = 2i \langle z, \xi \rangle_\ell\}.$$

If we set

$$(3.2) \quad L_j = \frac{\partial}{\partial z_j} + 2i\delta_j \xi_j \frac{\partial}{\partial w},$$

with  $\delta_j = -1$  for  $j \leq \ell$  and  $\delta_j = 1$  for  $j > \ell$ , then  $(L_j)_{1 \leq j \leq n}$  forms a basis of the space of  $(1, 0)$ -vector fields along  $Q_{(\bar{\xi}, \bar{\eta})}$ . By noticing that the equation (2.3) is the same as

$$g(z, w) - \overline{g(z, w)} - 2i \langle \bar{z}, f(z, w) \rangle_\ell - 2i \langle z, \overline{f(z, w)} \rangle_\ell = 2iA(z, \bar{z}, w, \bar{w}), \quad (z, w) \in \mathbb{H}_\ell^{2n+1},$$

and then complexifying in the standard way, we obtain

$$(3.3) \quad g(z, w) - \bar{g}(\xi, \eta) - 2i \langle \xi, f(z, w) \rangle_\ell - 2i \langle z, \bar{f}(\xi, \eta) \rangle_\ell = 2iA(z, \xi, w, \eta)$$

which holds for  $(z, \xi, w, \eta)$  satisfying  $w - \eta = 2i \langle z, \xi \rangle_\ell$  (or, if we think of  $(\xi, \eta)$  as being fixed, for  $(z, w) \in Q_{(\bar{\xi}, \bar{\eta})}$ ). We shall also use the identity obtained by applying  $L_j$  to (3.3):

$$(3.4) \quad L_j(g(z, w)) - 2i \langle \xi, L_j(f(z, w)) \rangle_\ell - 2i \bar{f}_j(\xi, \eta) = 2iL_j A(z, \xi, w, \eta), \quad w = \eta + 2i \langle z, \xi \rangle_\ell.$$

In view of (2.1) we have the expansions

$$(3.5) \quad f(z, w) = \sum_{\mu+\nu \geq 2} f_{\mu\nu}(z)w^\nu, \quad g(z, w) = \sum_{\mu+\nu \geq 2} g_{\mu\nu}(z)w^\nu,$$

where  $f_{\mu\nu}(z)$  and  $g_{\mu\nu}(z)$  are homogeneous polynomials of degree  $\mu$ . We also use the expansion (2.2). We allow the indices to be arbitrary integers by using the convention that all coefficients not appearing in (2.2) and (3.5) are zero. We now let  $w = 0, \eta = \eta(z, \xi) = -2i \langle z, \xi \rangle_\ell$  in (3.3) and (3.4), and equate bihomogeneous terms in  $(z, \xi)$  of a fixed bidegree  $(\alpha, \beta)$ . Comparing terms with  $\beta = 1$  and  $\beta = 0$  in (3.3) we obtain

$$(3.6) \quad f(z, 0) = 0, \quad g(z, 0) = 0.$$

For terms of a fixed bidegree  $(\alpha, \beta)$  with  $\beta \geq 2$ , we have

$$(3.7) \quad -\bar{g}_{\beta-\alpha,\alpha}(\xi)\eta^\alpha - 2i\langle z, \bar{f}_{\beta-\alpha+1,\alpha-1}(\xi)\eta^{\alpha-1} \rangle_\ell = 2i \sum_{p=0}^{\alpha-2} A_{\alpha-p,\beta-p,0,p}(z, \xi)\eta^p,$$

where  $\eta = -2i\langle z, \xi \rangle_\ell$ . Since  $R(A_{\mu\gamma\delta}) < n$  for all  $(\mu, \gamma, \delta)$ , Lemma 3.2 implies  $A_{\mu\gamma\delta} = 0$  for  $\eta = -2i\langle z, \xi \rangle_\ell$  and hence

$$(3.8) \quad \bar{g}_{\beta-\alpha,\alpha}(\xi)\eta + 2i\langle z, \bar{f}_{\beta-\alpha+1,\alpha-1}(\xi) \rangle_\ell = 0, \quad \eta = -2i\langle z, \xi \rangle_\ell, \quad \beta \geq 2.$$

In particular, in view of (3.2) and (3.6), we have

$$(3.9) \quad (L_j(f, g))(z, 0) = 2i\delta_j\xi_j \sum_{\mu} (f_{\mu 1}, g_{\mu 1})(z, 0), \quad (L_j A)(z, \xi, 0, \eta) = 2i\delta_j\xi_j \sum_{\mu,\gamma,\delta} A_{\mu\gamma\delta}(z, \xi)\eta^\delta.$$

We now apply the same procedure (i.e. collect terms of a fixed bidegree  $(\alpha, \beta)$  in  $(z, \xi)$ ) to (3.4) using (3.9). For  $\beta = 0$  we then obtain no terms and for  $\beta = 1$  and 2 respectively the identities

$$(3.10) \quad g_{\alpha 1}(z) = 0, \quad \langle \xi, 2i\delta_j\xi_j f_{\alpha 1}(z) \rangle_\ell + \bar{f}_{j;2-\alpha,\alpha}(\xi)\eta^\alpha = 0, \quad \eta = -2i\langle z, \xi \rangle_\ell,$$

where we use the notation  $f_{\mu\nu} = f_{\mu,\nu} = (f_{1;\mu,\nu}, \dots, f_{n;\mu,\nu})$ . Finally, for  $\beta \geq 3$ , the same comparison yields

$$-2i\bar{f}_{j;\beta-\alpha,\alpha}(\xi)\eta^\alpha = 2i\delta_j\xi_j \sum_{p=0}^{\alpha-1} A_{\alpha-p,\beta-p-1,1,p}(z, \xi)\eta^p.$$

Using the assumption  $R(A_{\mu\gamma\delta}) < n$  and applying Lemma 3.2 as above, we conclude that  $f_{\mu\nu}(z) = 0$  for  $\mu + \nu \geq 3$ . Substituting this into (3.8) we conclude that  $g_{\mu\nu}(z) = 0$  for  $\mu + \nu \geq 3$ . Also, substituting  $f_{21}(z) = 0$  into (3.10) yields  $f_{02} = 0$ . Since  $f_{20} = 0$  in view of (3.6), the identity (3.8) for  $(\alpha, \beta) = (1, 2)$  implies  $g_{11} = 0$ .

It remains to show that  $(f_{11}(z), g_{02}) = 0$ , where we drop the argument  $z$  for  $g_{02}$  since the latter is a constant. Rewriting (3.8) for  $\alpha = \beta = 2$  and the second identity in (3.10) for  $\alpha = 1$ , we obtain respectively

$$(3.11) \quad \bar{g}_{02}\langle z, \xi \rangle_\ell = \langle z, \bar{f}_{11}(\xi) \rangle_\ell, \quad \delta_j\xi_j\langle \xi, f_{11}(z) \rangle_\ell = \bar{f}_{j;11}(\xi)\langle z, \xi \rangle_\ell.$$

Setting  $\xi = \bar{z}$  and using the normalization (2.1), (3.11) implies

$$(3.12) \quad \operatorname{Re} \langle z, \overline{f_{11}(z)} \rangle_\ell = 0, \quad \delta_j\bar{z}_j\langle \bar{z}, f_{11}(z) \rangle_\ell = \overline{f_{j;11}(z)}\langle z, \bar{z} \rangle_\ell.$$

Recall that  $f_{j;11}(z)$  is a linear function in  $z$  that we write as  $f_{j;11}(z) = \sum_k f_j^k z_k$ . Then the second identity in (3.12) can be rewritten as

$$\delta_j\bar{z}_j \sum_{s,k} \delta_s\bar{z}_s f_s^k z_k = \sum_{l,m} \delta_m \overline{f_j^l} \bar{z}_l z_m \bar{z}_m,$$

from which we conclude that  $f_{j;11}(z) = c_j z_j$  for some  $c_j \in \mathbb{R}$ ,  $j = 1, \dots, n$ . Substituting this into the first identity in (3.12) we see that  $\operatorname{Re} c_j = 0$  for all  $j$  and hence  $f_{11}(z) = 0$ . Now the first identity in (3.11) yields  $g_{02} = 0$ . The proof is complete.  $\square$

We mention that in Definition 2.1, it is important to assume that  $\phi_j, \psi_j$  have no linear terms in  $(z, w)$ . Otherwise, the conclusion in Theorem 2.2 fails as can be easily seen by the example  $f(z, w) := (0, \dots, 0, -\chi(z))$ ,  $g := 0$  and  $A := z_n \overline{\chi(z)} + \chi(z) \bar{z}_n$  with  $\chi(z)$  being any holomorphic function in  $z$  vanishing at 0. Also the class  $\mathcal{S}_{n-1}$  in Theorem 2.2 cannot be replaced by any  $\mathcal{S}_k$  with  $k \geq n$  as the following example shows.

**Example 3.3.** Set  $f := (0, \dots, 0, \frac{i}{2} z_n w)$ ,  $g := 0$  and  $A := |z_n|^2 \langle z, \bar{z} \rangle_\ell$ . Then  $(f, g) \neq 0$  satisfies (2.1) and solves the equation (2.3).

#### 4. APPLICATION TO THE EQUIVALENCE PROBLEM

In the situation of Theorem 2.2 we have shown that  $(f, g)$  must vanish. In view of (2.3), it follows that the restriction of  $A$  to  $\mathbb{H}_\ell^{2n+1}$  also vanishes. However, it is easy to see that the full power series  $A(z, \bar{z}, w, \bar{w})$  need not necessarily vanish. In this section we shall refine the class  $\mathcal{S}_{n-1}$  to a smaller one  $\tilde{\mathcal{S}}_{n-1} \subset \mathcal{S}_{n-1} \subset \mathcal{O}$  with the property that any  $A \in \tilde{\mathcal{S}}_{n-1}$  which vanishes on  $\mathbb{H}_\ell^{2n+1}$  vanishes identically.

**Definition 4.1.** For every positive integer  $k$ , define  $\tilde{\mathcal{S}}_k \subset \mathcal{O}$  to be the subset of all real-valued formal power series  $A(z, \bar{z}, w, \bar{w})$  with  $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ , satisfying either of the following equivalent conditions:

- (i)  $A_{\mu\nu\gamma\delta} = 0$  if  $\nu + \delta \leq 1$  and  $\mathbb{R}(A_{\mu\nu\gamma\delta}) \leq k$  otherwise;
- (ii) for each fixed  $(\mu, \nu, \gamma, \delta)$ , we can write

$$A_{\mu\nu\gamma\delta}(z, \bar{z}) w^\gamma \bar{w}^\delta = \sum_{j=1}^k \phi_j(z, w) \overline{\psi_j(z, w)}$$

for some holomorphic polynomials  $\phi_j, \psi_j$  with no constant or linear terms.

The property of  $\tilde{\mathcal{S}}_{n-1}$  mentioned above is a consequence of the following statement.

**Lemma 4.2.** *Let  $A(z, \bar{z}, w, \bar{w})$  with  $(z, w) \in \mathbb{C}^n \times \mathbb{C}$  be a formal power series in the class  $\tilde{\mathcal{S}}_{n-1}$ . Assume that  $A(z, \bar{z}, w, \bar{w})|_{w=u+i\langle z, \bar{z} \rangle_\ell} \equiv 0$  as a formal power series in  $(z, \bar{z}, u)$ . Then  $A(z, \bar{z}, w, \bar{w}) \equiv 0$  as a formal power series in  $(z, \bar{z}, w, \bar{w})$ .*

*Proof.* We use the expansion (2.2) for  $A$ . The same complexification argument as in the proof of Theorem 2.2 yields

$$A(z, \xi, w, \eta) = 0 \text{ for } w = \eta + 2i\langle z, \xi \rangle_\ell$$

which can be rewritten as

$$\sum_{\mu, \nu, \gamma, \delta} A_{\mu\nu\gamma\delta}(z, \xi) (\eta + 2i\langle z, \xi \rangle_\ell)^\gamma \eta^\delta \equiv 0.$$

Assume, in order to reach a contradiction, that  $A(z, \bar{z}, w, \bar{w}) \not\equiv 0$ . Then, there is a smallest nonnegative integer  $\delta_0$  such that  $A_{\mu\nu\gamma\delta_0}(z, \xi) \not\equiv 0$  for some  $(\mu, \nu, \gamma)$ . By factoring out  $\eta^{\delta_0}$  (of course, if  $\delta_0 = 0$ , then we do not need to factor anything) and setting  $\eta = 0$ , we obtain

$$\sum_{\mu, \nu, \gamma} A_{\mu\nu\gamma\delta_0}(z, \xi) (2i\langle z, \xi \rangle_\ell)^\gamma \equiv 0.$$

Isolating terms of a fixed bidegree  $(\alpha, \beta)$  in  $(z, \xi)$ , we deduce

$$(4.1) \quad \sum_p A_{\alpha-p, \beta-p, p, \delta_0}(z, \xi) (2i\langle z, \bar{\xi} \rangle_\ell)^p \equiv 0.$$

By the definition of the class  $\tilde{\mathcal{S}}_{n-1}$ , we have  $R(A_{\mu\nu\gamma\delta_0}) < n$  for every  $(\mu, \nu, \gamma)$ . Hence Lemma 3.2, applied to the identities (4.1) for all  $(\alpha, \beta)$ , yields  $A_{\mu\nu\gamma\delta_0} \equiv 0$  for all  $(\mu, \nu, \gamma)$  in contradiction with the choice of  $\delta_0$ . This completes the proof of the lemma.  $\square$

Note that for  $\alpha_1, \alpha_2 \in \mathbb{R}$ ,  $A_1 \in \mathcal{S}_{k_1}$ ,  $A_2 \in \mathcal{S}_{k_2}$ , it holds that  $\alpha_1 A_1 + \alpha_2 A_2 \in \mathcal{S}_{k_1+k_2}$ . Hence Theorem 2.2 together with Lemma 4.2 imply that  $\mathcal{S}_k$  is an admissible class in the sense of section 2 for  $k < n/2$ .

Given a formal power series  $A(z, \bar{z}, w, \bar{w})$ , we associate to it a formal power series  $A^0(z, \bar{z}, u)$ , with  $u \in \mathbb{R}$ , defined by

$$A^0(z, \bar{z}, u) := A(z, \bar{z}, w, \bar{w})|_{w=u+i\langle z, \bar{z} \rangle_\ell};$$

in other words,  $A^0$  can be viewed as the trace of  $A$  along the hyperquadric  $\mathbb{H}_\ell^{2n+1}$ . Thus, given a formal power series  $A \in \mathcal{S}_k$ , we can associate to it two formal hypersurfaces  $M^0$  and  $M$  in  $\mathbb{C}^{n+1}$  as follows:

$$(4.2) \quad M^0 := \{\operatorname{Im} w = \langle z, \bar{z} \rangle_\ell + A^0(z, \bar{z}, u)\}, \quad M := \{\operatorname{Im} w = \langle z, \bar{z} \rangle_\ell + A(z, \bar{z}, w, \bar{w})\}.$$

(Observe that the equation defining  $M$  is not in graph form.) For these classes of hypersurfaces, Theorem 2.2 and Lemma 4.2 can be used for the study of the equivalence problem as follows.

**Theorem 4.3.** *Let  $A_1 \in \tilde{\mathcal{S}}_{k_1}$ ,  $A_2 \in \tilde{\mathcal{S}}_{k_2}$ , and define the formal hypersurfaces  $M_j^0$  and  $M_j$  in  $\mathbb{C}^{n+1}$  for  $j = 1, 2$ , by (4.2) (with  $A_j$  in the place of  $A$ ). Let*

$$H(z, w) = (z + f(z, w), w + g(z, w))$$

*be a formal biholomorphic map with  $(f, g)$  satisfying the normalization condition (2.1) and which sends  $M_1$  into  $M_2$  (or  $M_1^0$  into  $M_2^0$ ). Then, if  $k_1 + k_2 < n$ , it must hold that  $H(z, w) \equiv (z, w)$  and  $A_1 \equiv A_2$ .*

*Proof.* We shall prove the theorem when  $H$  maps  $M_1$  to  $M_2$ ; the proof of the other case is similar and left to the reader. By the implicit function theorem,  $M_1$  can be represented by the equation

$$\operatorname{Im} w = \langle z, \bar{z} \rangle_\ell + \tilde{A}_1(z, \bar{z}, u),$$

where  $\tilde{A}_1$  is uniquely obtained by solving for  $v$  in the equation

$$v = \langle z, \bar{z} \rangle_\ell + A_1(z, \bar{z}, u + iv, u - iv).$$

Since the formal map  $H = \text{id} + (f, g) =: (F, G)$  sends  $M_1$  to  $M_2$ , we have the following identity

$$(4.3) \quad \text{Im } G - \langle F, \bar{F} \rangle_\ell = A_2(F, \bar{F}, G, \bar{G}),$$

when we set

$$w = u + i\langle z, \bar{z} \rangle_\ell + iA_1(z, \bar{z}, w, \bar{w}) = u + i\langle z, \bar{z} \rangle_\ell + i\tilde{A}_1(z, \bar{z}, u).$$

As in [CM74], we assign the variable  $z$  the weight 1 and  $w$  (as well as  $u$  and  $v$ ) the weight 2. Recall that a holomorphic polynomial  $h(z, w)$  is then said to be weighted homogeneous of weighted degree  $\sigma$  if  $h(tz, t^2w) = t^\sigma h(z, w)$  for any complex number  $t$ . A real polynomial  $h(z, \bar{z}, u)$  (resp.  $h(z, \bar{z}, w, \bar{w})$ ) is said to be weighted homogeneous of weighted degree  $\sigma$ , if for any real number  $t$ ,  $h(tz, t\bar{z}, t^2u) = t^\sigma h(z, \bar{z}, u)$  (resp.  $h(tz, t\bar{z}, t^2w, t^2\bar{w}) = t^\sigma h(z, \bar{z}, w, \bar{w})$ ). In all cases, we write  $\text{deg}_{wt}(h) = \sigma$ . (By convention, 0 is a weighted homogeneous holomorphic polynomial of any degree.) We shall denote by  $h^{(\sigma)}$  the term of weighted degree  $\sigma$  in the expansion of  $h$  into weighted homogeneous terms.

It is sufficient to prove the following claim:  $(f^{(\tau-1)}, g^{(\tau)}) \equiv 0$  and  $A_1^{(\tau)} \equiv A_2^{(\tau)}$  for  $\tau \geq 1$ . For  $\tau = 1$ , this follows directly from the assumptions. We shall prove the claim for a general  $\tau \geq 1$  by induction. Assume that it holds for  $\tau < \sigma$ . Then, using the fact that, on  $M_1$ ,

$$w = u + i\langle z, \bar{z} \rangle + iA_1(z, \bar{z}, u + i\langle z, \bar{z} \rangle + i\tilde{A}_1(z, \bar{z}, u), u - i\langle z, \bar{z} \rangle - i\tilde{A}_1(z, \bar{z}, u)),$$

and collecting terms of weighted degree  $\sigma \geq 2$  in (4.3), we obtain

$$(4.4) \quad \begin{aligned} \mathcal{L}(f^{(\sigma-1)}, g^{(\sigma)})(z, \bar{z}, u) &= (A_2(z, \bar{z}, u + i\langle z, \bar{z} \rangle + i\tilde{A}_1(z, \bar{z}, u), u - i\langle z, \bar{z} \rangle - i\tilde{A}_1(z, \bar{z}, u))^{(\sigma)} \\ &\quad - (A_1(z, \bar{z}, u + i\langle z, \bar{z} \rangle + i\tilde{A}_1(z, \bar{z}, u), u - i\langle z, \bar{z} \rangle - i\tilde{A}_1(z, \bar{z}, u))^{(\sigma)} \\ &= (A_2^{(\sigma)}(z, \bar{z}, w, \bar{w}) - A_1^{(\sigma)}(z, \bar{z}, w, \bar{w}))|_{w=u+i\langle z, \bar{z} \rangle_\ell}. \end{aligned}$$

Since  $A_2 - A_1 \in \mathcal{S}_{n-1}$  by the assumption, we conclude by Theorem 2.2 that  $(f^{(\sigma-1)}, g^{(\sigma)}) \equiv 0$ . Furthermore, since also  $A_2 - A_1 \in \tilde{\mathcal{S}}_{n-1}$ , Lemma 4.2 implies that  $A_1^{(\sigma)} \equiv A_2^{(\sigma)}$ , which completes the induction. The proof of the theorem is complete.  $\square$

*Proof of Theorem 1.3.* Let  $M_1$  and  $M_2$  be as in Theorem 1.3 and let  $H = (F, G)$  be a formal invertible mapping  $(\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$  sending  $M_1$  to  $M_2$ ; that is, (4.3) holds when  $(z, w) \in M_1$ . By collecting terms of weighted degree 1 and 2, as in the proof of Theorem 4.3, we see that there are  $\lambda > 0$ ,  $r \in \mathbb{R}$ ,  $a \in \mathbb{C}^n$ ,  $\sigma = \pm 1$  and an invertible  $n \times n$

matrix  $U$  such that

$$\frac{\partial(F, G)}{\partial(z, w)}(0) := \begin{pmatrix} F_z(0) & G_z(0) \\ F_w(0) & G_w(0) \end{pmatrix} = \begin{pmatrix} \lambda U & 0 \\ \lambda a U & \sigma \lambda^2 \end{pmatrix}, \quad \operatorname{Re} \left( \frac{\partial^2 G}{\partial w^2} \right)(0) = 2\sigma \lambda^2 r,$$

where the linear transformation  $z \rightarrow zU$  preserves the Hermitian product  $\langle \cdot, \cdot \rangle_\ell$  modulo  $\sigma$ , i.e.  $\langle zU, \bar{z}\bar{U} \rangle_\ell = \sigma \langle z, \bar{z} \rangle_\ell$ . Observe that  $\sigma$  must be  $+1$  unless  $\ell = n/2$ . Now, let us define  $T \in \operatorname{Aut}(\mathbb{H}_\ell^{2n+1}, 0)$  by (1.6) (See e.g. [CM74] for the fact that  $T \in \operatorname{Aut}(\mathbb{H}_\ell^{2n+1}, 0)$ .) It is easy to verify that  $\tilde{H} := T^{-1} \circ H$  satisfies the normalization condition (2.1). Also, a straightforward calculation, which is left to the reader, shows that  $\tilde{H}$  maps  $M_1$  (via  $M_2$ ) into a real (formal) hypersurface in  $\mathbb{C}^{n+1}$  which can be defined by

$$\operatorname{Im} w = \langle z, \bar{z} \rangle_\ell + \tilde{A}_2(z, \bar{z}, w, \bar{w})$$

with

$$(4.5) \quad \tilde{A}_2(z, \bar{z}, w, \bar{w}) \equiv \sigma \lambda^{-2} |q(z, w)|^2 A_2 \circ (T(z, w), \overline{T(z, w)}),$$

where  $q(z, w)$  denotes the denominator in (1.6). Observe that if  $A_2$  belongs to  $\mathcal{H}_k$ , then, in view of Definition 1.1, so does  $\tilde{A}_2$ . Thus, by our assumptions, we have  $A_1 \in \mathcal{H}_{k_1}$  and also  $\tilde{A}_2 \in \mathcal{H}_{k_2}$ . Since  $\mathcal{H}_k \subset \tilde{\mathcal{S}}_k$  for all  $k$  as is easy to see, it follows from Theorem 4.3 that  $\tilde{H} \equiv \operatorname{id}$  and  $A_1 \equiv \tilde{A}_2$ . The conclusion of Theorem 1.3 now follows from (4.5) and the proof is complete.  $\square$

## 5. EMBEDDINGS OF REAL HYPERSURFACES IN HYPERQUADRICS

Let  $M$  be as in Proposition 1.2 and  $H: (\mathbb{C}^{n+1}, p) \rightarrow (\mathbb{C}^{N+1}, 0)$  be a formal holomorphic embedding sending  $M$  into  $\mathbb{H}_\ell^{2N+1}$  (with  $N \geq n$  and  $\ell \leq N/2$ ). We shall assume that  $H$  is CR transversal (i.e.  $\partial H_{N+1}(p) \neq 0$ , see section 1; also, recall that this is automatic when  $\ell < n$ ). Let us write  $H = (F, G) \in \mathbb{C}^N \times \mathbb{C}$ . The fact that  $H$  sends  $(M, 0)$  into  $(\mathbb{H}_\ell^{2N+1}, 0)$  means that

$$(5.1) \quad \operatorname{Im} G(z, w) = \langle F(z, w), \overline{F(z, w)} \rangle_{\ell'} \text{ for } (z, w) \in M.$$

By collecting terms of weighted degree 1 and 2 as in the proofs of Theorem 1.3 and 4.3 and using the CR transversality assumption, we see that

$$(5.2) \quad \frac{\partial(F, G)}{\partial(z, w)}(0) = \begin{pmatrix} \lambda V & 0 \\ \lambda a & \sigma \lambda^2 \end{pmatrix},$$

for some  $\sigma = \pm 1$ ,  $\lambda > 0$ ,  $a \in \mathbb{C}^N$ , and an  $n \times N$  matrix  $V$  of rank  $n$  satisfying

$$(5.3) \quad \langle zV, \bar{z}\bar{V} \rangle_{\ell'} = \sigma \langle z, \bar{z} \rangle_\ell.$$

Observe that, in view of (5.3), if  $\sigma = -1$  then  $\ell' \geq n - \ell$ . If  $\ell' < n - \ell$ , then we must have  $\sigma = 1$ . It will be convenient to renumber the coordinates  $z' = (z'_1, \dots, z'_N) \in \mathbb{C}^N$ ,

according to the sign of  $\sigma$ , so that

$$(5.4) \quad \langle z', \bar{z}' \rangle_{\ell'} = -\sigma \sum_{j=1}^{\ell} |z'_j|^2 + \sigma \sum_{j=\ell+1}^n |z'_j|^2 - \sum_{j=n+1}^{n+s} |z'_j|^2 + \sum_{j=n+s+1}^N |z'_j|^2,$$

where  $s = \ell' - \ell$  if  $\sigma = 1$  and  $s = \ell' - (n - \ell)$  if  $\sigma = -1$ . We choose coordinates  $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ , vanishing at  $p$ , such that  $M$  is defined by an equation of the form (1.4), where  $A$  vanishes to fourth order at 0. We shall need the following lemma.

**Lemma 5.1.** *Let  $M \subset \mathbb{C}^{n+1}$  be defined by an equation of the form (1.4) and  $H = (F, G): (\mathbb{C}^{n+1}, p) \rightarrow (\mathbb{C}^{N+1}, 0)$  a formal holomorphic, CR transversal embedding sending  $(M, 0)$  into  $\mathbb{H}_{\ell'}^{2N+1}$ , with  $N \geq n$ . Let  $\sigma = \pm 1$  so that*

$$(5.5) \quad \sigma \frac{\partial G}{\partial w}(0) > 0$$

and renumber the coordinates  $z' \in \mathbb{C}^N$  such that  $\langle z', \bar{z}' \rangle_{\ell'}$  is given by (5.4). Then, there exists  $T \in \text{Aut}(\mathbb{H}_{\ell'}^{2N+1}, 0)$  such that  $T^{-1} \circ H$  is of the form

$$(5.6) \quad (z, w) \mapsto (z + f(z, w), \phi(z, w), \sigma w + g(z, w)) \in \mathbb{C}^n \times \mathbb{C}^{N-n} \times \mathbb{C}$$

with  $d\phi(0) = 0$  and  $(f, g)$  satisfying the normalization conditions (2.1).

**Remark 5.2.** Recall that if  $\ell' < n - \ell$  then, as observed above, we must have  $\sigma = +1$ .

*Proof.* As mentioned above,  $H = (F, G)$  satisfies (5.2), where  $\lambda > 0$ ,  $a \in \mathbb{C}^N$ , and  $V$  satisfies (5.3). Let  $r \in \mathbb{R}$  be such that

$$\text{Re} \left( \frac{\partial^2 G}{\partial w^2} \right) (0) = 2\lambda^2 r.$$

By standard linear algebra, we can extend  $V$  to an  $N \times N$  matrix  $U$  such that its first  $n$  rows are the those of  $V$  and such that the linear transformation  $z' \mapsto z'U$  preserves the Hermitian form  $\langle \cdot, \cdot \rangle_{\ell'}$ , i.e.  $\langle z'U, \bar{z}'\bar{U} \rangle_{\ell'} = \langle z', \bar{z}' \rangle_{\ell'}$ . (Recall that  $\langle \cdot, \cdot \rangle_{\ell'}$  is given by (5.4).) Now define  $T: (\mathbb{C}^{N+1}, 0) \rightarrow (\mathbb{C}^{N+1}, 0)$ , as in the proof of Theorem 1.3 above, to be the element of  $\text{Aut}(\mathbb{H}_{\ell'}^{2N+1}, 0)$  given by

$$(5.7) \quad T(z', w') := \frac{(\lambda(z' + bw')U, \sigma\lambda^2 w')}{1 - 2i\langle z', \bar{b} \rangle_{\ell'} - (r + i\langle b, \bar{b} \rangle_{\ell'})w'},$$

where  $b = \sigma a U^{-1}$ . A straightforward calculation, which is left to the reader, shows that  $T$  satisfies the conclusion of the lemma.  $\square$

*Proof of Proposition 1.2.* In view of Lemma 5.1, we may assume, after composing  $H$  with some  $T \in \text{Aut}(\mathbb{H}_{\ell'}^{2N+1}, 0)$ , that  $H$  is of the form (5.6) with  $d\phi(p) = 0$  and  $(f, g)$  satisfying (2.1). Then the mapping  $H^0(z, w) := (z, \sigma w) + (f(z, w), g(z, w))$  is a local biholomorphism  $(\mathbb{C}^{n+1}, p) \rightarrow (\mathbb{C}^{n+1}, 0)$  and therefore we may consider a formal change of coordinates given by  $(\hat{z}, \hat{w}) := H^0(z, w)$  or, equivalently,  $(\hat{z}, \hat{w}) := (z + f, w + \sigma g)$ ; by a slight abuse

of notation, we shall drop the  $\hat{\phantom{x}}$  over the new coordinates  $(\hat{z}, \hat{w})$  and denote also these coordinates by  $(z, w)$ . The fact that  $H$  is an embedding sending  $M$  into  $\mathbb{H}_{\ell'}^{2N+1}$  means, in view of (5.1), that, in the new coordinates,  $M$  can be defined by the equation (1.4), where

$$(5.8) \quad A(z, \bar{z}, w, \bar{w}) := -\sigma \sum_{j=1}^s |\tilde{\phi}_j(z, w)|^2 + \sigma \sum_{j=s+1}^{N-n} |\tilde{\phi}_j(z, w)|^2, \quad \tilde{\phi}_j := \phi_j \circ (H^0)^{-1},$$

where  $s = \ell' - \ell$  if  $\sigma = 1$  and  $s = \ell' - (n - \ell)$  if  $\sigma = -1$ . (Cf. also [Fo86].) By Definition 1.1, the fact that  $A$  is given by (5.8) means that  $R(A) \leq N - n$  (but not necessarily that  $R(A) = N - n$ ) and hence  $A \in \mathcal{H}_{N-n}$ . Also, observe that the linear subspace  $V_A$  defined in the introduction is an  $R(A)$ -dimensional subspace of the span, over  $\mathbb{C}$ , of the formal series  $\tilde{\phi}_1, \dots, \tilde{\phi}_{N-n}$ . By choosing a basis  $\psi_1, \dots, \psi_{R(A)}$  for  $V_A$  such that  $A$  is given by the analogue of (1.5) with  $\psi_j$  instead of  $\tilde{\phi}_j$  (cf. section 1), one can conclude using standard linear algebra that  $S(A) \leq \ell' - \ell$  when  $\sigma = +1$  and  $S(A) \leq N - \ell' - \ell$  when  $\sigma = -1$ . Since  $\sigma = -1$  is only possible if  $\ell \geq n - \ell$ , this proves one implication in Proposition 1.2.

For the other implication, we must show that given coordinates  $(z, w) \in \mathbb{C}^n \times \mathbb{C}$  vanishing at  $p$  such that  $M$  is defined by (1.4) with  $R(A) = k$ , then  $M$  admits a formal, CR transversal embedding into  $\mathbb{H}_{\ell'}^{2N+1}$  with  $N = n + k$  and  $\ell' = \min(\ell + S(A), N - \ell - S(A))$ . Note that, since  $\mathbb{H}_{\ell'}^{2N+1}$  is equivalent to  $\mathbb{H}_{N-\ell'}^{2N+1}$ , it is enough to show that there is an embedding into  $\mathbb{H}_{\ell'}^{2N+1}$  with  $\ell' = \ell + S(A)$ .

As pointed out in the introduction, if  $R(A) = k$  and  $S(A) = s$ , then there are formal power series  $(\phi_j(z, w))_{1 \leq j \leq k}$  such that

$$A(z, \bar{z}, w, \bar{w}) = -\sum_{j=1}^s |\phi_j(z, w)|^2 + \sum_{j=s+1}^k |\phi_j(z, w)|^2.$$

The map  $H(z, w) := (z, \phi(z, w), w)$ , where  $\phi := (\phi_1, \dots, \phi_k)$ , defines a formal, CR transversal embedding of  $M$  into  $\mathbb{H}_{\ell'}^{2N+1}$  with  $N = n + k$  and  $\ell' = \ell + s$ . This completes the proof of Proposition 1.2.  $\square$

*Proofs of Theorems 1.6 and 1.7.* Let  $M, H_1, H_2, k_1$  and  $k_2$  be as in Theorem 1.6 and set  $N_1 := n + k_1, N_2 := n + k_2$ . Observe that, as was mentioned in the introduction, the embeddings  $H^1, H^2$  must be CR transversal, since  $\ell' = \ell < n$ . Choose a local coordinate system  $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ , vanishing at  $p \in M$ , such that  $M$  is given by an equation of the form (1.4). Let  $\sigma_q = \pm 1$  so that, in the notation of Lemma 5.1, (5.5) holds for  $H_q$ . In view of the remark following the lemma, we must have  $\sigma_q = +1$  unless  $\ell = \ell' = n/2$ . By Lemma 5.1, there are  $T_q \in \text{Aut}(\mathbb{H}_{\ell}^{2N_q+1}, 0)$ ,  $q = 1, 2$ , with the following property: If we set  $\tilde{H}_q := T_q^{-1} \circ H_q$ , then

$$\tilde{H}_q = (F_q, \phi_q, G_q) \in \mathbb{C}^n \times \mathbb{C}^{N_q-n} \times \mathbb{C},$$

where  $\phi_q$  have no constant or linear terms, and  $(F_q(z, w), G_q(z, w) = (z + f_q(z, w), \sigma_q w + g_q(z, w))$ , where  $(f_q, g_q)$  both satisfy the normalization conditions (2.1).

Let us first consider the case  $\ell = \ell' < n/2$ , so that  $\sigma_q = +1$ . By making the local changes of coordinates  $(z_q, w_q) = (F_q(z, w), G_q(z, w)) = (z + f_q(z, w), \sigma_q w + g_q(z, w))$ , for  $q = 1, 2$ , we observe, in view of the discussion in the proof of Proposition 1.2 above, that  $M$  is defined, in the coordinates  $(z_q, w_q)$ , by an equation of the form (1.4) with  $A = A_q$  given by

$$A_q(z, \bar{z}, w, \bar{w}) := \sum_{j=1}^{k_q} |\tilde{\phi}_{q,j}(z, w)|^2, \quad \tilde{\phi}_q := \phi_q \circ (F_q, G_q)^{-1},$$

where, for convenience, we have dropped the subscript  $q$  on the variables. Since  $k_1 + k_2 < n$  by the assumptions, Theorem 4.3 (or Theorem 1.3 for that matter) implies that  $(F_1, G_1) \equiv (F_2, G_2)$  (since  $(F_1, G_1) \circ (F_2, G_2)^{-1} := \text{id} + (\tilde{f}, \tilde{g})$  where  $(\tilde{f}, \tilde{g})$  also satisfies the normalization conditions (2.1)) and

$$(5.9) \quad \sum_{j=1}^{k_1} |\tilde{\phi}_{1,j}(z, w)|^2 \equiv \sum_{j=1}^{k_2} |\tilde{\phi}_{2,j}(z, w)|^2.$$

Let us write  $\hat{\phi}_1 := (\tilde{\phi}_1, 0) \in \mathbb{C}^{k_1} \times \mathbb{C}^{k_2 - k_1}$ , so that  $\hat{\phi}_1$  has  $k_2$  components (recall that  $k_2 \geq k_1$ ). Then it follows from (5.9) and [D93, Proposition 3, pp 102] that there exists a constant unitary transformation  $U$  of  $\mathbb{C}^{k_2}$  such that  $\hat{\phi}_1 \equiv U \tilde{\phi}_2$ . Hence, by composing  $\tilde{H}_2$  with a (unitary linear) automorphism  $T \in \text{Aut}(\mathbb{H}_\ell^{2N_2+1}, 0)$ , we obtain  $T \circ \tilde{H}_2 \equiv L \circ \tilde{H}_1$ , where  $L$  denotes the linear embedding as the statement of Theorem 1.6 (so that  $L \circ \tilde{H}_1 = (F_1, \hat{\phi}_1, G_1)$ .) The conclusion of Theorem 1.6, with  $\ell' = \ell < n/2$ , follows from the construction of  $\tilde{H}_q$ ,  $q = 1, 2$ , and the easily verified fact that any embedding  $L \circ T_1$ , with  $T_1 \in \text{Aut}(\mathbb{H}_\ell^{2N_1+1}, 0)$ , is of the form  $T_2 \circ L$  for some  $T_2 \in \text{Aut}(\mathbb{H}_\ell^{2N_2+1}, 0)$ .

In the case  $\ell' = \ell = n/2$ , we cannot exclude the case  $\sigma_q = -1$ . We consider the local coordinates  $(z_q, w_q) = (F_q(z, w), \sigma_q G_q(z, w))$  and proceed as above. The exact same arguments show that  $(F_1, \sigma_1 G_1) \equiv (F_2, \sigma_2 G_2)$  and that

$$(5.10) \quad \sigma_1 \sum_{j=1}^{k_1} |\tilde{\phi}_{1,j}(z, w)|^2 \equiv \sigma_2 \sum_{j=1}^{k_2} |\tilde{\phi}_{2,j}(z, w)|^2,$$

where  $\tilde{\phi}_q = \phi \circ (F_q, \sigma_q G_q)^{-1}$ . Now, there are two cases to consider, namely  $\sigma_1 \sigma_2 = 1$  and  $\sigma_1 \sigma_2 = -1$ . In the former case, the conclusion that  $H_2 = T \circ L \circ H_1$ , where  $T$  and  $L$  are as in the statement of the theorem, follows as above.

In the latter case  $\sigma_1 \sigma_2 = -1$ , we conclude from (5.10) that  $\phi_{q,j} \equiv 0$  for  $q = 1, 2$  and  $j = 1, \dots, N - n$ . Hence,  $M$ , in the coordinates say  $(z_1, w_1)$ , is equal to the quadric  $\mathbb{H}_{n/2}^{2n+1}$  and, hence, this case never occurs in the situation of Theorem 1.6. This completes the proof of Theorem 1.6.

To prove Theorem 1.7, we let  $H_2 = H$  and define  $\tilde{H}_1$  as above. Then, we define  $H_1: (\mathbb{H}_{n/2}^{2n+1}, 0) \rightarrow (\mathbb{H}_{n/2}^{2n+1}, 0)$ , i.e. with  $k_1 = 0$ , according to the sign of  $\sigma_2$  as follows:

$H_1(z, w) = (z, w)$ , if  $\sigma_2 = 1$ ; and equals to  $(z, -w)$  if  $\sigma_2 = -1$ .

Now,  $\sigma_1\sigma_2 = 1$  and by proceeding as above, we conclude that  $H_2 = T \circ L \circ H_1$ , where  $T$  and  $L$  are as in Theorem 1.7. Recall that when  $\sigma_2 = -1$  we have renumbered the coordinates  $z' \in \mathbb{C}^{N_2}$  so that (5.4) holds. If we undo this reordering, then  $L \circ H_2$  is equal to  $L_-$  as defined by (1.8). This completes the proof of Theorem 1.7.  $\square$

Let us briefly consider the case where  $M$  is embedded into  $\mathbb{H}_{\ell'}^{2N+1}$  with  $\ell' > \ell$ . As demonstrated by Example 1.8, we cannot hope for the full conclusion of Theorem 1.6 in this case. In what follows, we assume that the signature  $\ell$  of  $M$  is fixed and that there is a CR transversal embedding of  $M$  into  $\mathbb{H}_{\ell'}^{2(n+k_1)+1}$ . For simplicity, we shall only consider the case where  $\ell' < n - \ell$ , so that  $\sigma$ , defined by (5.5), must equal  $+1$ . (When  $\ell' \geq n - \ell$  there are different cases to consider, as in the proofs of Theorems 1.6 and 1.7 above.) We shall assume that the Hermitian product  $\langle \cdot, \cdot \rangle_{\ell'}$  on  $\mathbb{C}^N$ , with  $N := n + k$ , is given by (5.4). A similar argument to the one in the proof of Theorem 1.6 above, yields the following result; the details of modifying the argument are left to the reader.

**Theorem 5.3.** *Let  $M$  be a formal Levi nondegenerate hypersurface in  $\mathbb{C}^{n+1}$  of signature  $\ell$  at a point  $p$ . Suppose that there are two formal, CR transversal embeddings  $H_q: (M, p) \rightarrow (\mathbb{H}_{\ell_q}^{2(n+k_q)+1}, 0)$ ,  $q = 1, 2$ , (with  $k_2 \geq k_1 \geq 0$  and  $\ell \leq \ell_q < n - \ell$ ). If  $k_1 + k_2 < n$ , then there exist automorphisms  $T_q \in \text{Aut}(\mathbb{H}_{\ell_q}^{2(n+k_q)+1}, 0)$  and formal holomorphic coordinates  $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ , vanishing at  $p \in M$ , such that*

$$(5.11) \quad (T_q \circ H_q)(z, w) = (z, \phi_q(z, w), w) \in \mathbb{C}^n \times \mathbb{C}^{k_q} \times \mathbb{C}, \quad \langle \phi_1, \phi_1 \rangle_{\ell_1 - \ell} \equiv \langle \phi_2, \phi_2 \rangle_{\ell_2 - \ell},$$

where the  $\phi_q$ 's have no constant or linear terms.

**Remark 5.4.** In the setting of Theorem 5.3, let us suppose that  $k_2 - k_1 \geq 2(\ell_2 - \ell_1)$  (the other case can be treated analogously). We write  $\phi_q = (\phi_q^1, \phi_q^2) \in \mathbb{C}^{\ell_q - \ell} \times \mathbb{C}^{k_q - \ell_q + \ell}$ . It then follows from Theorem 5.3, as in the proof of Theorem 1.6 above, that there exists a unitary transformation  $U$  of  $\mathbb{C}^{k_2 + \ell_1 - \ell_2}$  such that  $L(\phi_2^1, \phi_2^2) = U(\phi_1^1, \phi_1^2)$ , where  $L$  is the linear embedding of  $\mathbb{C}^{k_1 + \ell_2 - \ell_1}$  into  $\mathbb{C}^{k_2 + \ell_1 - \ell_2}$  via  $z \mapsto (z, 0)$ . For instance, in the situation of Example 1.8 with  $n \geq 2$ , where  $k_1 = 0$ ,  $\ell_1 = \ell = 0$ ,  $k_2 = 2$ ,  $\ell_2 = \ell + 1 = 1$  (so that  $\ell_2 < n - \ell = n$  and  $k_2 + \ell_1 - \ell_2 = k_1 + \ell_2 - \ell_1 = 1$ ), we conclude that there is a unimodular complex number  $u$  such that  $\phi_2^1 \equiv u\phi_1^2$ .

We conclude this paper with two more applications of our main theorems. First, Theorem 1.6, combined with the remarks on finite determination of smooth mappings following Example 1.8 in the introduction, yields the following corollary:

**Corollary 5.5.** *Let  $M_1, M_2$  be compact connected smooth  $(2n+1)$ -dimensional CR manifolds of hypersurface type admitting smooth CR embeddings into the unit sphere in  $\mathbb{C}^{n+k+1}$  with  $k < n/2$ . Then  $M_1$  and  $M_2$  are globally CR equivalent if and only if their germs at*

some points  $p_1 \in M_1$  and  $p_2 \in M_2$  are locally CR equivalent. More precisely, if  $f$  is a germ at  $p_1 \in M_1$  of a local CR diffeomorphism sending an open neighborhood of  $M_1$  into  $M_2$ , then  $f$  extends as a CR diffeomorphism of  $M_1$  onto  $M_2$ .

As an application of the above corollary, we obtain remarkable families of pairwise locally inequivalent compact homogeneous hypersurfaces:

**Example 5.6.** The following CR manifolds naturally appear as  $SO(m)$ -orbits contained in the boundaries of Lie balls. For  $m \geq 3$  and  $0 < \lambda < 1$ , set

$$M_\lambda := \{|z_1|^2 + \cdots + |z_m|^2 = 1 + \lambda^2, z_1^2 + \cdots + z_m^2 = 2\lambda\} \subset \mathbb{C}^m.$$

Then  $M_\lambda$  are CR submanifolds of  $\mathbb{C}^m$  of hypersurface type and it was shown by W. Kaup and the third author [KZ02, Proposition 13.4] that they are pairwise globally CR inequivalent (even pairwise not CR homeomorphic). In view of Corollary 5.5 we conclude that, if  $m \geq 5$ , these CR manifolds are even locally CR inequivalent.

Theorem 1.6 can also be viewed as interpolating between the extreme cases  $k_1 = k_2 \leq n/2$  and  $k_1 = 0, k_2 < n$  (both of which had been previously studied in the strictly pseudoconvex case; see the remarks above). We illustrate this with an example.

**Example 5.7.** Let  $M \subset \mathbb{C}^{n+1}$  be a nonspherical ellipsoid, i.e. defined in real coordinates  $Z_j = x_j + iy_j, j = 1, \dots, n+1$ , by

$$\sum_{j=1}^{n+1} \left( \frac{x_j}{a_j} \right)^2 + \left( \frac{y_j}{b_j} \right)^2 = 1,$$

with  $a_j \neq b_j$  for, at least, some  $j$ . After a simple scaling (if necessary), the ellipsoid  $M$  can be described by the equation

$$\sum_{j=1}^{n+1} (A_j Z_j^2 + A_j \bar{Z}_j^2 + |Z_j|^2) = 1,$$

where  $0 \leq A_1 \leq \dots \leq A_{n+1} < 1$ ; the fact that  $M$  is nonspherical means that at least  $A_{n+1} > 0$ . It was observed in [W78, W99] that the polynomial mapping (of degree two)  $G: \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+2}$  given by

$$G(Z) = \left( Z, i \left( 1 - 2 \sum_{j=1}^{n+1} A_j Z_j^2 \right) \right)$$

sends  $M$  into the Heisenberg hypersurface  $\mathbb{H}^{2(n+1)+1} \subset \mathbb{C}^{n+2}$ . Let  $p \in M$  and  $S$  be any automorphism of  $\mathbb{H}^{2(n+1)+1}$  which sends  $G(p)$  to 0. A direct application of Theorem 1.6 (with  $k_1 = 1$  and  $k_2 = k$ ) then shows that any formal embedding  $H: (M, p) \rightarrow (\mathbb{H}^{2(n+k)+1}, 0)$ , with  $k < n - 1$ , must be of the form  $H = T \circ L \circ S \circ G$ , where  $L$  is the linear embedding  $(\mathbb{H}^{2(n+1)+1}, 0) \rightarrow (\mathbb{H}^{2(n+k)+1}, 0)$  and  $T \in \text{Aut}(\mathbb{H}_\rho^{2(n+k)+1}, 0)$ . In particular any such formal embedding is a rational mapping (of degree two).

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