Several Results for Holomorphic Mappings from $\mathbf{B}^n$ into $\mathbf{B}^N$

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Dedicated to Professor F. Treves

§1. Introduction

Let $\mathbf{B}^n$ be the unit ball in $\mathbf{C}^n$. Write $\mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$ for the collection of proper holomorphic maps $F : \mathbf{B}^n \to \mathbf{B}^N$. Write $\mathrm{Prop}_k(\mathbf{B}^n, \mathbf{B}^N) \subset \mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$ for proper holomorphic maps that are C^k-smooth up to the boundary; and denote by $\mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^N)$ the set of rational proper holomorphic maps from $\mathbf{B}^n$ into $\mathbf{B}^N$. Recall that $f, g \in \mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$ are said to be *equivalent* if there are automorphisms $\sigma \in Aut(\mathbf{B}^n)$ and $\tau \in Aut(\mathbf{B}^N)$ such that $f = \tau \circ g \circ \sigma$. When $f \in \mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$ is equivalent to the standard big circle embedding $z \to (z, 0)$, we call f a linear map or a totally geodesic embedding. In all that follows, we always assume that $N \geq n > 1$.

There has been much work done in the past thirty years on the rigidity (linearity) and classification problems for elements in $\mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$. (See [H02] [H01] for references and historical discussions).

In [H02], the author assigned to each $F \in \mathrm{Prop}_2(\mathbf{B}^n, \mathbf{B}^N)$ an invariant integer $\kappa_0(F) \in \{0, 1, 2, ..., n-1\}$ which is called the *geometric rank* of F (see [H02] or § 2 below for the precise definition). Using the language of geometric ranks, [H99, Theorem 4.2] is stated as follows: $F \in \mathrm{Prop}_2(\mathbf{B}^n, \mathbf{B}^N)$ has $\kappa_0(F) = 0$ if and only if F is equivalent to the linear map. Therefore, to understand proper holomorphic mappings between balls, it suffices to study maps with geometric rank $\kappa_0 \geq 1$. Meanwhile, it was also shown in [H02, Lemma 3.2] that one always has $N \geq n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$. Namely, the least dimension of the target space is $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$ to allow the existence of elements in $\mathrm{Prop}_2(\mathbf{B}^n, \mathbf{B}^N)$ with geometric rank $\kappa_0 \geq 0$.

The invariant integer κ_0 was introduced in [H02] to study a semi-rigidity problem for holomorphic maps, which we recall as follows:

Definition 1.0 ([H02]): Let $F \in \mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$. Then F is called k-linear if for any point $p \in \mathbf{B}^n$ there is an affine complex subspace S^a_p containing p and of complex dimension k such that the restriction of F to S^a_p is a linear fractional map.

Theorem 1.1 ([Theorem 2.3, H02]): Suppose that $F \in \mathrm{Prop}_3(\mathbf{B}^n, \mathbf{B}^N)$ has geometric rank $\kappa_0 \leq n-2$. Then F is $(n-\kappa_0)$-linear.

The first part of this paper is to give an application of Theorem 1.1 for the study of rationality for proper holomorphic maps between balls. We will prove the following:

Theorem 1.2: *Suppose that $F \in Prop_2(\mathbf{B}^n, \mathbf{B}^N)$ is k-linear with $k \geq 2$. Then F is a rational map.*

Corollary 1.3 ([Theorem 2.3, H02] and Theorem 1.2): *Suppose that $F \in Prop_3(\mathbf{B}^n, \mathbf{B}^N)$ has geometric rank $\kappa_0(F) < n-1$. Then F is a rational map.*

Corollary 1.4 ([Lemma 3.2, H02] and Corollary 1.3): *Let $F \in Prop_3(\mathbf{B}^n, \mathbf{B}^N)$ with $N \leq \frac{n(n+1)}{2}$. Then F is a rational map.*

For maps with geometric rank $\kappa_0 = n-1$, the partial linearity fails. The structure of the maps certainly gets more complicated. Notice that the minimum target dimension which allows the existence of such maps is $N = \frac{n(n+1)}{2}$. Hence, one might like to split the study of the classification problem for $\mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^N)$ into of the following four sub-problems:

Problem A: Study maps $F \in \mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^N)$ with $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$ and $1 \leq \kappa_0 \leq n-2$.
Problem B: Study maps $F \in \mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^N)$ with $N > n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$ and $1 \leq \kappa_0 \leq n-2$.
Problem C: Study maps $F \in Rat(\mathbf{B}^n, \mathbf{B}^N)$ with $N = \frac{n(n+1)}{2}$ and $\kappa_0(F) = n-1$.
Problem D: Study maps $F \in \mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^N)$ with $N > \frac{n(n+1)}{2}$ and $\kappa_0 = n-1$.

In the content of Problem A, it follows from the paper [HJ01] that any map $F \in Rat(\mathbf{B}^n, \mathbf{B}^{2n-1})$ with $\kappa_0(F) = 1$ is equivalent to the Whitney map. For the case of $\kappa_0 = 2$ and $n = 4$, Ji-Xu [JX03] proved that any map $F \in Rat(\mathbf{B}^4, \mathbf{B}^9)$ with $\kappa_0(F) = 2$ and of degree 2 is equivalent to the generalized Whitney map $W_{4,2}$.

Along the lines of Problem (C), it is proved by Faran [Fa82] that there are exactly three equivalent classes in $Rat(\mathbf{B}^2, \mathbf{B}^3)$ with $\kappa_0 = 1$. The next simplest case is then to consider maps in $Rat(\mathbf{B}^3, \mathbf{B}^6)$ with $\kappa_0 = 2$. Indeed, the second part of the present paper is to discuss the classification problem in this setting. We will prove the following:

Theorem 1.5: *Let $F \in Rat(\mathbf{B}^3, \mathbf{B}^6)$ with $\kappa_0(F) = 2$ and $deg(F) = 2$. Then $F = (f_1, f_2, \phi_{11}, \phi_{12}, \phi_{22}, g)$ is equivalent either to the generalized Whitney map $\left(z_1 w, z_2 w, z_1^2, \sqrt{2} z_1 z_2, z_2^2, w\right)$ or to the map $\left(\sqrt{2} z_1 w, \sqrt{2} z_2 w, z_1^2, \sqrt{2} z_1 z_2, z_2^2, w^2\right)$.*

One might like to compare the case of $Rat(\mathbf{B}^2, \mathbf{B}^3)$ with Theorem 1.5. By [Fa82], there are two equivalent classes in $Rat(\mathbf{B}^2, \mathbf{B}^3)$ of degree 2 with $\kappa_0 = 1$. The first map is the Whitney map $F(z, w) = (z^2, zw, w)$ and the second one is the map $F(z, w) = (z^2, \sqrt{2} zw, w^2)$. Therefore Theorem 1.5 may be regarded as a generalization of Faran's result.

The idea for the proof of Theorem 1.5 is motivated by the work in [HJ01]. It is based on the fact that the condition of $deg(F) = 2$ also induces some additional equations similar to those derived in [HJ01] and thus can be handled by similar methods developed in [HJ01].

§ 2. Preliminaries

•Maps between balls: The ball $\mathbf{B}^n \subset \mathbf{C}^n$ is equivalent to the Siegel upper-half space $\mathbf{H}_n := \{(z,w) \in \mathbf{C}^{n-1} \times \mathbf{C} : \ \mathrm{Im}(w) > |z|^2\}$ by the Cayley transformation $\rho_n : \mathbf{H}_n \to \mathbf{B}^n$, $\rho_n(z,w) = \left(\frac{2z}{1-iw}, \frac{1+iw}{1-iw}\right)$. We can similarly define the space $\mathrm{Rat}(\mathbf{H}_n, \mathbf{H}_N)$, $\mathrm{Prop}_k(\mathbf{H}_n, \mathbf{H}_N)$ and $\mathrm{Prop}(\mathbf{H}_n, \mathbf{H}_N)$.

We will identify a map $F \in \mathrm{Prop}_k(\mathbf{B}^n, \mathbf{B}^N)$ or $\mathrm{Rat}(\mathbf{B}^n, \mathbf{B}^N)$ with the one in the space $\mathrm{Prop}_k(\mathbf{H}_n, \mathbf{H}_N)$ or $\mathrm{Rat}(\mathbf{H}_n, \mathbf{H}_N)$ by $\rho_N^{-1} \circ F \circ \rho_n$, respectively.

We write $L_j = 2i\overline{z_j}\frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}$ for $j = 1, \cdots, n-1$ and $T = \frac{\partial}{\partial u}$ where $w = u + iv$. Then $\{L_1, \cdots, L_{n-1}\}$ forms a global basis for the complex tangent bundle $\mathrm{T}^{(1,0)}\partial\mathbf{H}_n$ of $\partial\mathbf{H}_n$, and T is a tangent vector field of $\partial\mathbf{H}_n$ transversal to $\mathrm{T}^{(1,0)}\partial\mathbf{H}_n \cup \mathrm{T}^{(0,1)}\partial\mathbf{H}_n$. Parameterize $\partial\mathbf{H}_n$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \to (z, u + i|z|^2)$. In what follows, we will assign the weight of z and u to be 1 and 2, respectively. For a non negative integer m, a function $h(z, \overline{z}, u)$ defined over a small ball U of 0 in $\partial\mathbf{H}_n$ is said to be of quantity $o_{wt}(m)$ if $\frac{h(tz, t\overline{z}, t^2u)}{|t|^m} \to 0$ uniformly for (z,u) on any compact subset of U as $t(\in \mathbf{R}) \to 0$. (In this case, we write $h = o_{wt}(m)$. By convention, we write $h = o_{wt}(0)$ if $h \to 0$ as $(z, \overline{z}, u) \to 0$).

• Geometric rank of F: Let $F = (f, \phi, g) = (\widetilde{f}, g) = (f_1, \cdots, f_{n-1}, \phi_1, \cdots, \phi_{N-n}, g)$ be a C^2-smooth CR map from an open piece M of $\partial\mathbf{H}_n$ into $\partial\mathbf{H}_N$. For each $p \in M$, we have an associated CR map F_p from a small neighborhood U of $0 \in \partial\mathbf{H}_n$ to $\partial\mathbf{H}_N$ with $F_p(0) = 0$, defined by

$$F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p), \tag{2.1}$$

where for each $p = (z_0, w_0) \in M$, we write $\sigma_p^0 \in \mathrm{Aut}(\mathbf{H}_n)$ for the map sending (z,w) to $(z + z_0, w + w_0 + 2i\langle z, \overline{z_0}\rangle)$ and we define $\tau_p^F \in \mathrm{Aut}(\mathbf{H}_N)$ by $\tau_p^F(z^*, w^*) = (z^* - \widetilde{f}(z_0, w_0), w^* - \overline{g(z_0, w_0)} - 2i\langle z^*, \overline{\widetilde{f}(z_0, w_0)}\rangle)$.

Lemma 2.1 ([H99, §2, Lemma 5.3]): Let F be a C^2-smooth CR map from a connected open subset M containing 0 in $\partial\mathbf{H}_n$ into $\partial\mathbf{H}_N$, $2 \le n \le N$. For each $p \in \partial\mathbf{H_n}$, there are $\sigma \in \mathrm{Aut}_0(\partial\mathbf{H}_n)$ and $\tau \in \mathrm{Aut}_0(\partial\mathbf{H}_N)$ such that the map $F_p^{**} = \tau \circ F_p \circ \sigma$ satisfies the following normalization:

$$f_p^{**} = z + \frac{i}{2}a_p^{**(1)}(z)w + o_{wt}(3),\ \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2),\ g_p^{**} = w + o_{wt}(4), \tag{2.2}$$

with

$$\langle \overline{z}, a_p^{**(1)}(z)\rangle |z|^2 = |\phi_p^{**(2)}(z)|^2. \tag{2.3}$$

From (2.3), we see that $a_p^{**(1)}(z) = z\mathcal{A}(p)$ and that $\mathcal{A}(p)$ is a $(n-1)\times(n-1)$ semi-positive Hermitian matrix. The rank of $\mathcal{A}(p) = -2i(P_j^l)_{1\le j,l\le (n-1)}$, which is denoted by $Rk_F(p)$, is called the geometric rank of F at p. Notice that $P_j^l = \frac{\partial^2 (f_p)_l^{**}}{\partial z_j \partial w}|_0$. By [(2.3.1), H02], $Rk_F(p)$ is a well-defined integer ([H02]), depending only on F and p. (See [Definition 2.1, H02]). We define the *geometric rank* of F to be $\kappa_0(F) = max_{p\in\partial\mathbf{H_n}} Rk_F(p)$. Notice that we always have $0 \le \kappa_0 \le n-1$. By [H02, Corollary 5.2], when $F \in \mathrm{Prop}_3(\mathbf{H}_n, \mathbf{H}_N)$, the set $\{p \in \partial\mathbf{H}_n, Rk_F(p) = \kappa_0\}$ is an open dense subset of $\mathbf{H}_n$. We define the geometric rank of $F \in \mathrm{Prop}_2(\mathbf{B}^n, \mathbf{B}^N)$ to be the one for the map $\rho_N^{-1}\circ F\circ\rho_n \in \mathrm{Prop}_2(\mathbf{H}_n, \mathbf{H}_N)$. For such a map, we can similarly define the geometric rank $Rk_F(p)$ of F at $p \in \partial\mathbf{B}^n$.

• **A normalization lemma**: The following normalization will be used later for the proof of theorem 1.5.

Lemma 2.2 ([Lemma 3.2, H02]): Let F be a C^2-smooth CR map from an open piece $M \subset \partial\mathbf{H}_n$ into $\partial\mathbf{H}_N$ with $Rk_F(p) = \kappa_0$. Let $P(n,\kappa_0) = \frac{\kappa_0(2n-\kappa_0-1)}{2}$. Then $N \ge n + P(n,\kappa_0)$ and there are $\sigma \in \mathrm{Aut}_0(\partial\mathbf{H}_n)$ and $\tau \in \mathrm{Aut}_0(\partial\mathbf{H}_N)$ such that $\tau\circ F_p\circ\sigma := (f,\phi,g)$, denoted by F_p^{***}, satisfies the following normalization condition:

$$(4.2)\qquad \begin{aligned} f_j &= z_j + \frac{i\mu_j}{2} z_j w + o_{wt}(3), \quad \frac{\partial^2 f_j}{\partial w^2}(0) = 0,\ j = 1\cdots,\kappa_0,\ \mu_j > 0,\\ f_j &= z_j + o_{wt}(3), \quad j = \kappa_0+1,\cdots,n-1\\ g &= w + o_{wt}(4),\\ \phi_{jl} &= \mu_{jl} z_j z_l + o_{wt}(2),\ \text{where}\ \ (j,l)\in\mathcal{S}\ \text{with}\ \ \mu_{jl} > 0\ \text{for}\ \ (j,l)\in\mathcal{S}_0\\ &\qquad \text{and}\ \mu_{jl} = 0\ \text{otherwise}. \end{aligned}$$

Moreover, $\mu_j \ge \mu_1 = 1$, $\mu_{jl} = \sqrt{\mu_j + \mu_l}$ for $j,l \le \kappa_0$ $j \ne l$; and $\mu_{jl} = \sqrt{\mu_j}$ if $j \le \kappa_0$ and $l > \kappa_0$ or if $j = l \le \kappa_0$. Here we label the components of ϕ by double indices $(j,l) \in \mathcal{S}$ with

$$\begin{aligned} \mathcal{S}_0 &= \{(j,l) : 1 \le j \le \kappa_0, 1 \le l \le (n-1), j \le l\},\\ \mathcal{S} &:= \{(j,l) :\ (j,l)\in\mathcal{S}_0,\ \text{or}\ \ j = \kappa_0+1,\\ &\qquad\qquad l \in \{\kappa_0+1,\cdots,\kappa_0+N-n-\frac{(2n-\kappa_0-1)\kappa_0}{2}\}\}. \end{aligned}$$

• **Degree of a rational map**: For any rational holomorphic map $H = \frac{(P_1,\ldots,P_m)}{Q}$ on $\mathbf{C}^n$, where P_j, Q are holomorphic polynomials and $(P_1,\ldots,P_m,Q) = 1$. We define

$$(2.5)\qquad deg(H) = max\{deg(P_j),\ 1 \le j \le m,\ deg(Q)\}.$$

The following will also be used in our later discussions:

Lemma 2.3 ([HJ01, Lemma 5.3 and 5.4]): Let $F \in Rat(\mathbf{H_n}, \mathbf{H_N})$ and F_p^{***} be as descibed in Lemma 2.2. If $deg(F_p^{***}(z,0)) \leq l$ for any p in an open neighborhood of 0 in $\partial\mathbf{H_n}$, then $deg(F) \leq l$.

§3. Proper maps with partial linearity

In this section, we give the proof of Theorem 1.3 and Corollaries 1.4, 1.5, based on the work in [H02]. We mention that all these results and arguments are of purely local nature. However, we only focus on the global setting for simplicity of notation.

Let $F \in \mathrm{Prop}(\mathbf{B}^n, \mathbf{B}^N)$ with $N \geq n > 1$. For any integer $1 \leq k \leq n$, write $G_{n,k}(\mathbf{C})$ for the complex Grassmannian manifold consisting of complex k-planes in $\mathbf{C}^n$. Define

$$\begin{aligned} \mathcal{V}_F :=& \{(Z, S_Z) \in \mathbf{B}^n \times G_{n,k}(\mathbf{C}), \\ & F \text{ is linear fractional when restricted to } S_Z + Z\}. \end{aligned} \tag{3.1}$$

Then, as in [Lemma 5.1, H02], $\mathcal{V}_F$ is a complex analytic variety with $\pi : \mathcal{V}_F \to \mathbf{B}^n$ as its proper holomorphic projection. In particular, this implies that if there is a subset $E \subset \mathbf{B}^n$ of Hausdorff dimension greater than $2n-2$ such that for any $Z \in E$, there is an affine complex subspace of dimension k through Z along which F is linear, then F is k-linear over $\mathbf{B}^n$. We define the quantity $\kappa^0(F)$ such that F is $\kappa^0(F)$-linear, but not $(\kappa(F)^0+1)$-linear. Then Theorem 2.3 of [H02] states that when F is C^3-smooth up to the boundary and when $\kappa_0(F) < n-1$, it holds

$$\kappa^0(F) = n - \kappa_0(F). \tag{3.2}$$

We next recall the following Lemma from [H02]:

Lemma 3.1([Lemma 5.3, H02]): Let M be a connected open subset of $\partial\mathbf{H}_n$. Let F be a C^2 CR map from M into $\partial\mathbf{H}_N$ with $N \geq n > 1$ and with constant geometric rank $\kappa_0 < n-1$. Assume that F extends holomorphically to a subdomain Ω of $\mathbf{H}_n$, which has M as part of its smooth boundary. Assume that F is $(n-\kappa_0)$-linear over Ω. Let $p_0 \in M$. Then for $Z(\in \Omega \setminus E) \approx p_0$ with E a certain complex analytic variety of positive codimension, there is a unique complex subspace S_Z of dimension $(n-\kappa_0)$ such that F, when restricted to $S_Z + Z$, is linear fractional. Moreover S_Z, as elements in $G_{n,n-\kappa_0}(\mathbf{C})$, depends holomorphically on $Z(\approx p_0) \in \Omega \setminus E$ and extends holomorphically across E.

Proposition 3.2: Let $F \in \mathrm{Prop}_2(\mathbf{B}^n, \mathbf{B}^N)$ with $N \geq n > 1$. Assume that $\kappa_0(F) < n-1$. Let $\mathcal{V}_F$ be as defined in (3.1) with $k = n - \kappa_0(F) = \kappa^0(F)$. Then it has a unique irreducible component of dimension n, denoted by $\mathcal{V}_F^0$, such that the following holds: (i) There is a complex analytic variety E_F of positive codimension in $\mathbf{B}^n$ such that π is surjective from $\mathcal{V}_F^0$ to $\mathbf{B}^n$; (ii) π is one-to-one from $\mathcal{V}_F \setminus \pi^{-1}(E_F)$ to $\mathbf{B}^n \setminus E_F$; (iii) $\mathcal{V}_F \setminus \pi^{-1}(E_F) = \mathcal{V}_F^0 \setminus \pi^{-1}(E_F)$.

Proof of Proposition 3.2: By Lemma 3.1, there is an open subset U of the ball such that π is biholomorphic from $\mathcal{V}_F \cap \pi^{-1}(U)$ to U. Now write $\mathcal{V}_F^0$ for the irreducible component of $\mathcal{V}_F$ which contains $\pi^{-1}(U)$ as an open piece. Then $\mathcal{V}_F^0$ has complex dimension n. Moreover, for each irreducible component V of $\mathcal{V}_F$, either $V \equiv \mathcal{V}_F^0$ or $\pi(V)$ must be a proper complex analytic variety of $\mathbf{B}^n$. Let E_F be the union of such $\pi(V)$'s with V different from $\mathcal{V}_F^0$. Then we see the conclusion of the statements in the Proposition. ∎

We next prove the following result:

Proposition 3.3: Let F be a holomorphic map from $\Omega \subset \mathbf{B}^n$ into $\Omega' \subset \mathbf{B}^N$ with $n \geq 2$, that is k-linear ($k > 1$) over Ω. Assume that Ω (Ω') has a connected open piece $M \subset \partial \mathbf{B}^n$ ($M' \subset \partial \mathbf{B}^N$, respectively) as part of its smooth boundary such that $\lim_{Z \in \Omega \to M} F(Z) \subset M'$. Also, assume that there is an open subset U of Ω, sufficiently close to M, such that for each $Z \in U$, there is a unique affine subspace S_Z^a of dimension k passing through Z such that $F|_{S_Z^a \cap \Omega}$ is linear fractional. Then F is rational.

Proof of Proposition 3.3: Without loss of generality, we assume that $Z = 0$, $S_0^a = \{z_{k+1} = \cdots = z_n = 0\}$ and $F|_{S_0^a} = \mathrm{Id}$. Also, to simplify the notation, we assume that $\Omega = \mathbf{B}^n$, $\Omega' = \mathbf{B}^N$. And π is biholomorphic from a neighborhood of $\pi^{-1}(0)$ in $\mathcal{V}_F$ to a neighborhood of 0 in $\mathbf{B}^n$.

Notice that $S_0^a = S_0 = \mathrm{span}\{e_1, \cdots, e_{n-k}\}$, where e_j is the n-tuple whose component at the l^{th} position is δ_j^l. We will use the standard local coordinates for the Grassmannian $G_{n,k}(\mathbf{C})$ near S_0. Namely, for any S near S_0, we associate it uniquely with the coordinates (ξ_{jl}) where j runs from 1 to k and l runs from $k+1$ to n such that $S = \mathrm{span}\{e_1(S), \cdots, e_k(S)\}$. Here $e_j(S) = (0, \cdots, 1, \cdots, 0, \xi_{j(k+1)} \cdots, \xi_{jn})$. Now, for each $Z \approx 0$, S_Z^a associated with F can be parameterized by $k(n-k)$ holomorphic functions $\xi_{jl}(Z)$, where $j = 1, \cdots, k$ and $l = k+1, \cdots, n$, in the manner such that $S_Z^a = Z + span_j\{e_j(S_Z^a - Z)\}$. Then the assumption above shows that $\xi_{jl}(Z)$ as functions in Z are holomorphic near 0 for each (j, l).

Consider the holomorphic map Ψ which sends $(t, \tau) := (t_1, \cdots, t_k, \tau_1, \cdots, \tau_{n-k})$ to

$$(t_1, \cdots, t_k, \sum_{j=1}^{k} \xi_{j(k+1)}(0, \tau) t_j + \tau_1, \cdots, \sum_{j=1}^{k} \xi_{jn}(0, \tau) t_j + \tau_{n-k}).$$

Then $\Psi(t, \tau) = (t, \tau) + (0, O(\tau)|t|)$ is holomorphic from a neighborhood U_ϵ of $\{\sum_{j=1}^k |t_j|^2 < 1+\epsilon\} \times \{|\tau| < \epsilon\}$ for a certain $\epsilon << 1$. Moreover, Ψ is the identical map when restricted to $S_0 \cap \overline{\mathbf{B}^n}$ and has non zero Jacobian there. Moreover, Ψ sends (t, τ) into $S_{(0,\tau)}^a$. Hence $F \circ \Psi$ is linear fractional in t for each fixed $\tau \approx 0$ by the assumption. Therefore, we have

$$F \circ \Psi(t, \tau) = \frac{F(\tau) + \sum_{j=1}^k A_j(\tau) t_j}{1 + \sum_{j=1}^k b_j(\tau) t_j}. \tag{3.3}$$

Now, we claim that $A_j(\tau), b_j(\tau), F(\tau)$ are holomorphic for $\tau \approx 0$. (See [Lemma 5.1, H01].) For this purpose, write

$$F \circ \Psi(t,\tau) = \sum_{\alpha} C_\alpha(\tau) t^\alpha.$$

Then C_α depends holomorphically on τ for $\tau \approx 0$. Multiplying $(1+\sum_j b_j(\tau)t_j)$ of both sides of (3.3) and then considering the Taylor expansion in t at the origin, we see that

$$C_\alpha + \sum_{j=1}^{k} b_j C_{\alpha - e'_j} \equiv 0 \text{ for } |\alpha| \geq 1, \tag{3.4}$$

$C_0(\tau) = F(\tau)$, $C_{e'_j}(\tau) = D_{t_j}(F\circ\Psi)(t,\tau)|_{t=0}$ and $A_j(\tau) = (F\circ\Psi)(\tau)b_j(0,\tau) + C_{e'_j}(\tau)$. Here e'_j is the vector in $\mathbf{C}^k$ defined as for e_j.

By the Alexander theorem, since $F|_{S_Z^a}$ must be a linear embedding, we see that $\{C_{e'_j}\}_{j=1}^{n-\kappa_0}$ are linearly independent vectors. Hence, we can holomorphically solve $b_j(\tau)'s$ in (3.4) in terms of $C_\alpha(\tau)$ with $|\alpha| = 1, 2$. Hence $A_j(\tau), b_j(\tau), F(\tau)$ are holomorphic for $\tau \approx 0$.

Notice that $b_j(0) = 0$ by our normalization that $F|_{S_0^a} = \text{id}$. It is clear that $F \circ \Psi$ extends holomorphically to a neighborhood $U'_{\epsilon'}(\subset\subset U_\epsilon)$ of $\{\sum_{j=1}^{k} |t_j|^2 < 1+\epsilon'\} \times \{|\tau| < \epsilon'\}$ for a certain $\epsilon' < \epsilon$.

Now, as mentioned before, one can find a point $Z_0 \in U'_{\epsilon'}$ such that $\Phi(Z_0)$ is on the unit sphere and Ψ is locally biholomorphic near Z_0. It thus follows that near $\Phi(Z_0)$, $F = (F\circ\Phi)\circ\Phi^{-1}$ extends holomorphically to a neighborhood of $\Psi(Z_0)$. By [Fo1], we conclude the rationality of F. This completes the proof of Proposition 3.3. ■

Proof of Theorem 1.3, Corollaries 1.4-1.5: Theorem 1.3 now follows from Propositions 3.2-3.3. Corollary 1.4 follows from Theorem 1.3 and [Theorem 2.3, H02]. Corollary 1.5 follows from Theorem 1.3, Corollary 1.4 and [Lemma 3.2, H02] with an argument identical to that in [Corollary 2.1, H01]. ■

As mentioned at the beginning of the section, one similarly has the local version of all these results. For instance, one has the following

Corollary 3.4: *Let M be a connected open subset of $\partial\mathbf{H}_n$. Let F be a non-constant C^3-smooth CR map from M into $\partial\mathbf{H}_N$ with $N \geq n > 1$ and with constant geometric rank $\kappa_0 < n-1$. Then F is rational. In particular, any C^3 CR map from an open piece of $\partial\mathbf{H}_n$ into $\partial\mathbf{H}_N$ with $N = \frac{n(n+1)}{2}$ is rational.*

§ 4. A rough degree estimate

Starting from this section, we let $n = 3$, $N = \frac{n(n+1)}{2} = 6$. We will consider rational proper maps from $\mathbf{B}^3$ into $\mathbf{B}^6$. We first prove in this section the following degree estimate.

Theorem 4.1: *Let $F \in Rat(\mathbf{B}^3, \mathbf{B}^6)$ with $\kappa_0(F) = 2$. Then $deg(F) \leq 4$.*

By the discussions in §2, for the proof of Theorem 4.1, we can assume that $F \in Rat(\mathbf{H}_3, \mathbf{H}_6)$ with $\kappa_0 = 2$. By Lemma 2.2, for any $p \in U \subset \partial\mathbf{H}_3$, there are $\sigma_0 \in \mathrm{Aut}_0(\mathbf{H}_3)$ and $\tau_0 \in \mathrm{Aut}_0(\mathbf{H}_6)$ such that $F_p^{***} = \tau_0 \circ \tau_p^F \circ F \circ \sigma_p^0 \circ \sigma_0 := (f_1, f_2, \phi_{11}, \phi_{12}, \phi_{22}, g) = (f, \phi, g) = (\widetilde{f}, g)$ satisfies the following conditions:

$$
\begin{aligned}
&f_1 = z_1 + \frac{i}{2} z_1 w + o_{wt}(3), \ \frac{\partial^2 f_1}{\partial w^2}(0) = 0, \\
&f_2 = z_2 + \frac{i\mu_2}{2} z_2 w + o_{wt}(3), \ \frac{\partial^2 f_2}{\partial w^2}(0) = 0, \ \mu_2 \geq 1 \\
&\phi_{11} = z_1^2 + o_{wt}(2), \quad \phi_{12} = \sqrt{1+\mu_2} z_1 z_2 + o_{wt}(2), \\
&\phi_{22} = \sqrt{\mu_2} z_2^2 + o_{wt}(2), \quad g = w + o_{wt}(4).
\end{aligned} \tag{4.1}
$$

By the argument in [HJ01, Lemma 5.2], we have

$$
\overline{\widetilde{f}(\zeta, 0)^t} = \begin{pmatrix} I & 0 \\ -B^{-1}A & B^{-1} \end{pmatrix} \begin{pmatrix} \overline{\zeta}^t \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{\zeta}^t \\ -B^{-1}A\overline{\zeta}^t \end{pmatrix}, \tag{4.2}
$$

where

$$
A = \begin{pmatrix} A_{2\times 2} \\ A_{1\times 2} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_1\mathcal{L}_1(f_1) & \mathcal{L}_1\mathcal{L}_1(f_2) \\ \mathcal{L}_1\mathcal{L}_2(f_1) & \mathcal{L}_1\mathcal{L}_2(f_2) \\ \mathcal{L}_2\mathcal{L}_2(f_1) & \mathcal{L}_2\mathcal{L}_2(f_2) \end{pmatrix} |_{(0,0,0,\zeta_1,\zeta_2,0)} = \begin{pmatrix} -2\overline{\zeta}_1 & 0 \\ -\overline{\zeta}_2 & -\mu_2\overline{\zeta}_1 \\ 0 & -2\mu_2\overline{\zeta}_2 \end{pmatrix}, \tag{4.3}
$$

and

$$
B = \begin{pmatrix} \mathcal{L}_1\mathcal{L}_1(\phi_{11}) & \mathcal{L}_1\mathcal{L}_1(\phi_{12}) & \mathcal{L}_1\mathcal{L}_1(\phi_{22}) \\ \mathcal{L}_1\mathcal{L}_2(\phi_{11}) & \mathcal{L}_1\mathcal{L}_2(\phi_{12}) & \mathcal{L}_1\mathcal{L}_2(\phi_{22}) \\ \mathcal{L}_2\mathcal{L}_2(\phi_{11}) & \mathcal{L}_2\mathcal{L}_2(\phi_{12}) & \mathcal{L}_2\mathcal{L}_2(\phi_{22}) \end{pmatrix} |_{(0,0,0,\zeta_1,\zeta_2,0)}. \tag{4.4}
$$

Here we denote by $\mathcal{L}_j = 2i\overline{\zeta_j}\frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}$ the complexification of L_j. By (4.1), we have

$$
\mathcal{L}_j\mathcal{L}_l(f_k)|_{(0,0,0,\zeta_1,\zeta_2,0)} = (2i\overline{\zeta_j}\frac{\partial^2 f_k}{\partial z_l \partial w} + 2i\overline{\zeta_l}\frac{\partial^2 f_k}{\partial z_j \partial w})|_{(0,0,0,\zeta_1,\zeta_2,0)}
$$

and

$$
\begin{aligned}
&\mathcal{L}_j\mathcal{L}_l(\phi_{kt})|_{(0,0,0,\zeta_1,\zeta_2,0)} \\
=&(\frac{\partial^2 \phi_{kt}}{\partial z_j \partial z_l} + 2i\overline{\zeta_j}\frac{\partial^2 \phi_{kt}}{\partial z_l \partial w} + 2i\overline{\zeta_l}\frac{\partial^2 \phi_{kt}}{\partial z_j \partial w} - 4\overline{\zeta_j\zeta_l}\frac{\partial^2 \phi_{kt}}{\partial w^2})|_{(0,0,0,\zeta_1,\zeta_2,0)}.
\end{aligned}
$$

Hence

(4.5)

$$\mathcal{L}_1\mathcal{L}_1(\phi_{11})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 2 + 4i\overline{\zeta_1}b^{(11)}_{101} - 8\overline{\zeta_1}^2 b^{(11)}_{002},$$
$$\mathcal{L}_1\mathcal{L}_1(\phi_{12})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 4i\overline{\zeta_1}b^{(12)}_{101} - 8\overline{\zeta_1}^2 b^{(12)}_{002},$$
$$\mathcal{L}_1\mathcal{L}_1(\phi_{22})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 4i\overline{\zeta_1}b^{(22)}_{101} - 8\overline{\zeta_1}^2 b^{(22)}_{002},$$
$$\mathcal{L}_1\mathcal{L}_2(\phi_{11})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 2i\overline{\zeta_1}b^{(11)}_{011} + 2i\overline{\zeta_2}b^{(11)}_{101} - 8\overline{\zeta}_1\overline{\zeta_2}b^{(11)}_{002},$$
$$\mathcal{L}_1\mathcal{L}_2(\phi_{12})|_{(0,0,0,\zeta_1,\zeta_2,0)} = \sqrt{1+\mu_2} + 2i\overline{\zeta_1}b^{(12)}_{011} + 2i\overline{\zeta_2}b^{(12)}_{101} - 8\overline{\zeta}_1\overline{\zeta_2}b^{(12)}_{002},$$
$$\mathcal{L}_1\mathcal{L}_2(\phi_{22})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 2i\overline{\zeta_1}b^{(22)}_{011} + 2i\overline{\zeta_2}b^{(22)}_{101} - 8\overline{\zeta}_1\overline{\zeta_2}b^{(22)}_{002},$$
$$\mathcal{L}_2\mathcal{L}_2(\phi_{11})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 4i\overline{\zeta_2}b^{(11)}_{011} - 8\overline{\zeta_2}^2 b^{(11)}_{002},$$
$$\mathcal{L}_2\mathcal{L}_2(\phi_{12})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 4i\overline{\zeta_2}b^{(12)}_{011} - 8\overline{\zeta_2}^2 b^{(12)}_{002},$$
$$\mathcal{L}_2\mathcal{L}_2(\phi_{22})|_{(0,0,0,\zeta_1,\zeta_2,0)} = 2\sqrt{\mu_2} + 4i\overline{\zeta_2}b^{(22)}_{011} - 8\overline{\zeta_2}^2 b^{(22)}_{002}$$

where we write $\phi_{kt} = \sum b^{(kt)}_{jls} z_1^j z_2^l w^s$.

Lemma 4.2: Let B be the matrix in (4.4). Then

$$B^{-1} = \frac{1}{det(B)} \begin{pmatrix} b^{-1}_{11} & -b^{-1}_{12} & b^{-1}_{13} \\ -b^{-1}_{21} & b^{-1}_{22} & -b^{-1}_{23} \\ b^{-1}_{31} & -b^{-1}_{32} & b^{-1}_{33} \end{pmatrix} \tag{4.6}$$

where

$$b^{-1}_{11} = A_{00} + A_{10}\overline{\zeta_1} + A_{01}\overline{\zeta_2} + A_{11}\overline{\zeta_1\zeta_2} + A_{02}\overline{\zeta_2}^2 + A_{12}\overline{\zeta_1\zeta_2}^2 + A_{03}\overline{\zeta_2}^3,$$
$$b^{-1}_{12} = B_{10}\overline{\zeta_1} + B_{20}\overline{\zeta_1}^2 + B_{11}\overline{\zeta_1\zeta_2} + B_{21}\overline{\zeta_1}^2\overline{\zeta_2} + B_{12}\overline{\zeta_1\zeta_2}^2,$$
$$b^{-1}_{13} = C_{10}\overline{\zeta_1} + C_{20}\overline{\zeta_1}^2 + C_{30}\overline{\zeta_1}^3 + C_{21}\overline{\zeta_1}^2\overline{\zeta_2},$$
$$b^{-1}_{21} = D_{10}\overline{\zeta_1} + D_{01}\overline{\zeta_2} + D_{11}\overline{\zeta_1\zeta_2} + D_{02}\overline{\zeta_2}^2 + D_{12}\overline{\zeta_1\zeta_2}^2 + D_{03}\overline{\zeta_2}^3,$$
$$b^{-1}_{22} = E_{00} + E_{10}\overline{\zeta_1} + E_{01}\overline{\zeta_2} + E_{20}\overline{\zeta_1}^2 + E_{11}\overline{\zeta_1\zeta_2} + E_{02}\overline{\zeta_2}^2 + E_{21}\overline{\zeta_1}^2\overline{\zeta_2} + E_{12}\overline{\zeta_1\zeta_2}^2,$$
$$b^{-1}_{23} = F_{10}\overline{\zeta_1} + F_{01}\overline{\zeta_2} + F_{20}\overline{\zeta_1}^2 + F_{11}\overline{\zeta_1\zeta_2} + F_{30}\overline{\zeta_1}^3 + F_{21}\overline{\zeta_1}^2\overline{\zeta_2},$$
$$b^{-1}_{31} = G_{01}\overline{\zeta_2} + G_{02}\overline{\zeta_2}^2 + G_{12}\overline{\zeta_1\zeta_2}^2 + G_{03}\overline{\zeta_2}^3,$$
$$b^{-1}_{32} = H_{01}\overline{\zeta_2} + H_{11}\overline{\zeta_1\zeta_2} + H_{02}\overline{\zeta_2}^2 + H_{21}\overline{\zeta_1}^2\overline{\zeta_2} + H_{12}\overline{\zeta_1\zeta_2}^2,$$
$$b^{-1}_{33} = I_{00} + I_{10}\overline{\zeta_1} + I_{01}\overline{\zeta_2} + I_{20}\overline{\zeta_1}^2 + I_{11}\overline{\zeta_1\zeta_2} + I_{30}\overline{\zeta_1}^3 + I_{21}\overline{\zeta_1}^2\overline{\zeta_2}.$$

with

$$A_{00} = 2\sqrt{\mu_2}\sqrt{1+\mu_2},\ A_{10} = 4i\sqrt{\mu_2}b^{(12)}_{011},\ A_{01} = 4i\sqrt{1+\mu_2}b^{(22)}_{011} + 4i\sqrt{\mu_2}b^{(12)}_{101},$$
$$A_{02} = -8\sqrt{1+\mu_2}b^{(22)}_{002} + 8b^{(12)}_{011}b^{(22)}_{101} - 8b^{(12)}_{101}b^{(22)}_{011},\ A_{11} = -16\sqrt{\mu_2}b^{(12)}_{002},$$
$$A_{03} = -16ib^{(12)}_{101}b^{(22)}_{002} + 16ib^{(12)}_{002}b^{(22)}_{101},\ A_{12} = -16ib^{(12)}_{002}b^{(22)}_{011} + 16ib^{(12)}_{011}b^{(22)}_{002};$$

$$B_{10}=8i\sqrt{\mu_2}b^{(12)}_{101},\ B_{11}=-16b^{(12)}_{101}b^{(22)}_{011}+16b^{(12)}_{011}b^{(22)}_{101},\ B_{20}=-16\sqrt{\mu_2}b^{(12)}_{002},$$
$$B_{12}=-32ib^{(12)}_{101}b^{(22)}_{002}+32ib^{(12)}_{002}b^{(22)}_{101},\ B_{21}=-32ib^{(12)}_{002}b^{(22)}_{011}+32ib^{(12)}_{011}b^{(22)}_{002};$$

$$C_{10}=-4i\sqrt{1+\mu_2}b^{(22)}_{101},\ C_{20}=-8b^{(12)}_{101}b^{(22)}_{011}+8\sqrt{1+\mu_2}b^{(22)}_{002}+8b^{(12)}_{011}b^{(22)}_{101},$$
$$C_{21}=16ib^{(12)}_{002}b^{(22)}_{101}-16ib^{(12)}_{101}b^{(22)}_{002},\ C_{30}=-16ib^{(12)}_{002}b^{(22)}_{011}+16ib^{(12)}_{011}b^{(22)}_{002};$$

$$D_{10}=4i\sqrt{\mu_2}b^{(11)}_{011},\ D_{11}=-16\sqrt{\mu_2}b^{(11)}_{002}\ D_{02}=-8b^{(11)}_{101}b^{(22)}_{011}+8b^{(11)}_{011}b^{(22)}_{101},$$
$$D_{12}=-16ib^{(11)}_{002}b^{(22)}_{011}+16ib^{(11)}_{011}b^{(22)}_{002},\ D_{03}=-16ib^{(11)}_{101}b^{(22)}_{002}+16ib^{(11)}_{002}b^{(22)}_{101};$$

$$E_{00}=4\sqrt{\mu_2},\ E_{10}=8i\sqrt{\mu_2}b^{(11)}_{101},\ E_{01}=8ib^{(22)}_{011},$$
$$E_{02}=-16b^{(22)}_{002},\ E_{20}=-16\sqrt{\mu_2}b^{(11)}_{002},\ E_{11}=-16b^{(11)}_{101}b^{(22)}_{011}+16b^{(11)}_{011}b^{(22)}_{101},$$
$$E_{21}=-32ib^{(11)}_{002}b^{(22)}_{011}+32ib^{(11)}_{011}b^{(22)}_{002},\ E_{12}=-32ib^{(11)}_{101}b^{(22)}_{002}+32ib^{(11)}_{002}b^{(22)}_{101};$$

$$F_{11}=-16b^{(22)}_{002},\ F_{20}=-8b^{(11)}_{101}b^{(22)}_{011}+8b^{(11)}_{011}b^{(22)}_{101},$$
$$F_{30}=-16ib^{(11)}_{002}b^{(22)}_{011}+16ib^{(11)}_{011}b^{(22)}_{002},\ F_{21}=-16ib^{(11)}_{101}b^{(22)}_{002}+16ib^{(11)}_{002}b^{(22)}_{101};$$

$$G_{01}=-4i\sqrt{1+\mu_2}b^{(11)}_{011},\ G_{02}=-8b^{(11)}_{101}b^{(12)}_{011}+8b^{(11)}_{011}b^{(12)}_{101}+8\sqrt{1+\mu_2}b^{(11)}_{002},$$
$$G_{12}=16ib^{(11)}_{011}b^{(12)}_{002}-16ib^{(11)}_{002}b^{(12)}_{011},\ G_{03}=-16ib^{(11)}_{101}b^{(12)}_{002}+16ib^{(11)}_{002}b^{(12)}_{101};$$

$$H_{01}=8ib^{(12)}_{011},\ H_{11}=-16b^{(11)}_{101}b^{(12)}_{011}+16b^{(11)}_{011}b^{(12)}_{101},\ H_{02}=-16b^{(12)}_{002},$$
$$H_{21}=-32ib^{(11)}_{002}b^{(12)}_{011}+32ib^{(11)}_{011}b^{(12)}_{002},\ H_{12}=-32ib^{(11)}_{101}b^{(12)}_{002}+32ib^{(11)}_{002}b^{(12)}_{101};$$

$$I_{00}=2\sqrt{1+\mu_2},\ I_{01}=4ib^{(12)}_{101},\ I_{20}=-8b^{(11)}_{101}b^{(12)}_{011}+8b^{(11)}_{011}b^{(12)}_{101}-8\sqrt{1+\mu_2}b^{(11)}_{002}$$
$$I_{10}=4ib^{(12)}_{011}+4i\sqrt{1+\mu_2}b^{(11)}_{101},\ I_{11}=-16b^{(12)}_{002},$$
$$I_{21}=-16ib^{(12)}_{002}b^{(11)}_{101}+16ib^{(11)}_{002}b^{(12)}_{101},\ I_{30}=-16ib^{(11)}_{002}b^{(12)}_{011}+16ib^{(11)}_{011}b^{(12)}_{002}.$$

Proof of Lemma 4.2: Denote by $B=(b_{ij})_{3\times 3}$. By (4.5)

$$\begin{aligned}
b_{11}^{-1}&=det\begin{pmatrix} b_{22} & b_{23}\\ b_{32} & b_{33}\end{pmatrix}=b_{22}b_{33}-b_{32}b_{23}\\
&=\left(\sqrt{1+\mu_2}+2i\overline{\zeta_1}b^{(12)}_{011}+2i\overline{\zeta_2}b^{(12)}_{101}-8\overline{\zeta}_1\overline{\zeta_2}b^{(12)}_{002}\right)\left(2\sqrt{\mu_2}+4i\overline{\zeta_2}b^{(22)}_{011}-8\overline{\zeta_2}^2b^{(22)}_{002}\right)\\
&\quad-\left(4i\overline{\zeta_2}b^{(12)}_{011}-8\overline{\zeta_2}^2b^{(12)}_{002}\right)\left(2i\overline{\zeta_1}b^{(22)}_{011}+2i\overline{\zeta_2}b^{(22)}_{101}-8\overline{\zeta}_1\overline{\zeta_2}b^{(22)}_{002}\right)\\
&=A_{00}+A_{10}\overline{\zeta_1}+A_{01}\overline{\zeta_2}+A_{11}\overline{\zeta_1\zeta_2}+A_{02}\overline{\zeta_2}^2+A_{12}\overline{\zeta_1\zeta_2}^2+A_{03}\overline{\zeta_2}^3
\end{aligned}$$

where the A_j are as above. Other formulas are obtained by the similar computation. ∎

Proof of Theorem 4.1: From (4.6) (4.3) and (4.2), we have

$$\overline{\phi_{11}(\zeta,0)}=\frac{\Phi_{11}}{det(B)},\ \overline{\phi_{12}(\zeta,0)}=\frac{\Phi_{12}}{det(B)},\ \overline{\phi_{22}(\zeta,0)}=\frac{\Phi_{22}}{det(B)}, \tag{4.7}$$

where

(4.8)

$$\begin{aligned}
&\Phi_{11}(\overline{\zeta_1},\overline{\zeta_2})=2\overline{\zeta_1}^2b_{11}^{-1}-(1+\mu_2)\overline{\zeta_1\zeta_2}b_{12}^{-1}+2\mu_2\overline{\zeta_2}^2b_{13}^{-1}\\
&=2\overline{\zeta_1}^2\left(A_{00}+A_{10}\overline{\zeta_1}+A_{01}\overline{\zeta_2}+A_{11}\overline{\zeta_1\zeta_2}+A_{02}\overline{\zeta_2}^2+A_{12}\overline{\zeta_1\zeta_2}^2+A_{03}\overline{\zeta_2}^3\right)\\
&-(1+\mu_2)\overline{\zeta_1\zeta_2}\left(B_{10}\overline{\zeta_1}+B_{20}\overline{\zeta_1}^2+B_{11}\overline{\zeta_1\zeta_2}+B_{21}\overline{\zeta_1}^2\overline{\zeta_2}+B_{12}\overline{\zeta_1\zeta_2}^2\right)\\
&+2\mu_2\overline{\zeta_2}^2\left(C_{10}\overline{\zeta_1}+C_{20}\overline{\zeta_1}^2+C_{30}\overline{\zeta_1}^3+C_{21}\overline{\zeta_1}^2\overline{\zeta_2}\right),
\end{aligned}$$

$$\begin{aligned}
&\Phi_{12}(\overline{\zeta_1},\overline{\zeta_2})=-2\overline{\zeta_1}^2b_{21}^{-1}+(1+\mu_2)\overline{\zeta_1\zeta_2}b_{22}^{-1}-2\mu_2\overline{\zeta_2}^2b_{23}^{-1}\\
&=-2\overline{\zeta_1}^2\left(D_{10}\overline{\zeta_1}+D_{01}\overline{\zeta_2}+D_{11}\overline{\zeta_1\zeta_2}+D_{02}\overline{\zeta_2}^2+D_{12}\overline{\zeta_1\zeta_2}^2+D_{03}\overline{\zeta_2}^3\right)\\
&+(1+\mu_2)\overline{\zeta_1\zeta_2}\Big(E_{00}+E_{10}\overline{\zeta_1}+E_{01}\overline{\zeta_2}+E_{20}\overline{\zeta_1}^2\\
&\quad+E_{11}\overline{\zeta_1\zeta_2}+E_{02}\overline{\zeta_2}^2+E_{21}\overline{\zeta_1}^2\overline{\zeta_2}+E_{12}\overline{\zeta_1\zeta_2}^2\Big)\\
&-2\mu_2\overline{\zeta_2}^2\left(F_{10}\overline{\zeta_1}+F_{01}\overline{\zeta_2}+F_{20}\overline{\zeta_1}^2+F_{11}\overline{\zeta_1\zeta_2}+F_{30}\overline{\zeta_1}^3+F_{21}\overline{\zeta_1}^2\overline{\zeta_2}\right)
\end{aligned} \tag{4.9}$$

and

(4.10)

$$\begin{aligned}
&\Phi_{22}(\overline{\zeta_1},\overline{\zeta_2})=2\overline{\zeta_1}^2b_{31}^{-1}-(1+\mu_2)\overline{\zeta_1\zeta_2}b_{32}^{-1}+2\mu_2\overline{\zeta_2}^2b_{33}^{-1}\\
&=2\overline{\zeta_1}^2\left(G_{01}\overline{\zeta_2}+G_{02}\overline{\zeta_2}^2+G_{12}\overline{\zeta_1\zeta_2}^2+G_{03}\overline{\zeta_2}^3\right)\\
&-(1+\mu_2)\overline{\zeta_1\zeta_2}\left(H_{01}\overline{\zeta_2}+H_{11}\overline{\zeta_1\zeta_2}+H_{02}\overline{\zeta_2}^2+H_{21}\overline{\zeta_1}^2\overline{\zeta_2}+H_{12}\overline{\zeta_1\zeta_2}^2\right)\\
&+2\mu_2\overline{\zeta_2}^2\left(I_{00}+I_{10}\overline{\zeta_1}+I_{01}\overline{\zeta_2}+I_{20}\overline{\zeta_1}^2+I_{11}\overline{\zeta_1\zeta_2}+I_{30}\overline{\zeta_1}^3+I_{21}\overline{\zeta_1}^2\overline{\zeta_2}\right).
\end{aligned}$$

From (4.2), it follows that $f_j(z,0)=z_j$, $j=1,2$. Also $g(z,0)=0$ always holds. In fact, after complexifying $\text{Im}(g)=|\widetilde{f}|^2$, we have

$$\frac{g(z,w)-\overline{g(\zeta,\eta)}}{2i}=f(z,w)\overline{f(\zeta,\eta)}+\phi(z,w)\overline{\phi(\zeta,\eta)},\quad \forall\frac{w-\bar{\eta}}{2i}=\langle z,\bar{\zeta}\rangle.$$

By putting $z = w = \eta = 0$ in the above equation, we get $\overline{g(\zeta,0)} = 0$ and this means $g(z,0) = 0$. Therefore, in order to prove Theorem 4.1, by Lemma 2.3. it suffices to prove

(4.11)
$$deg(\Phi_{11}(\overline{\zeta})) \leq 4,\ deg(\Phi_{12}(\overline{\zeta})) \leq 4,\ deg(\Phi_{22}(\overline{\zeta})) \leq 4,\ deg(det(B)(\overline{\zeta})) \leq 3.$$

To prove the first inequality in (4.11), by (4.8), one needs to show

$$2A_{12} - (1+\mu_2)B_{21} + 2\mu_2 C_{30} = 0,\ 2A_{03} - (1+\mu_2)B_{12} + 2\mu_2 C_{21} = 0.$$

This can be verified by the formulas in Lemma 4.2. The second and the third inequalities in (4.11) can be similarly obtained.

To prove the last inequality in (4.11), we write the 3×3 matrix B in (4.4) as (β_{jl}). Then
(4.12)
$$det(B) = -\beta_{21} det \begin{pmatrix} \beta_{12} & \beta_{13} \\ \beta_{32} & \beta_{33} \end{pmatrix} + \beta_{22} det \begin{pmatrix} \beta_{11} & \beta_{13} \\ \beta_{31} & \beta_{33} \end{pmatrix} - \beta_{23} det \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{31} & \beta_{32} \end{pmatrix}$$

and from (4.5) we obtain

$$\begin{aligned} det(B) = &-\left(2i\overline{\zeta_1}b_{011}^{(11)} + 2i\overline{\zeta_2}b_{101}^{(11)} - 8\overline{\zeta}_1\overline{\zeta_2}b_{002}^{(11)}\right) \\ &\cdot\Big[(4i\overline{\zeta_1}b_{101}^{(12)} - 8\overline{\zeta_1}^2 b_{002}^{(12)})(2\sqrt{\mu_2} + 4i\overline{\zeta_2}b_{011}^{(22)} - 8\overline{\zeta_2}^2 b_{002}^{(22)}) \\ &- (4i\overline{\zeta_2}b_{011}^{(12)} - 8\overline{\zeta_2}^2 b_{002}^{(12)})(4i\overline{\zeta_1}b_{101}^{(22)} - 8\overline{\zeta_1}^2 b_{002}^{(22)})\Big] \\ &+ \left(\sqrt{1+\mu_2} + 2i\overline{\zeta_1}b_{011}^{(12)} + 2i\overline{\zeta_2}b_{101}^{(12)} - 8\overline{\zeta}_1\overline{\zeta_2}b_{002}^{(12)}\right) \\ &\cdot\Big[(2 + 4i\overline{\zeta_1}b_{101}^{(11)} - 8\overline{\zeta_1}^2 b_{002}^{(11)})(2\sqrt{\mu_2} + 4i\overline{\zeta_2}b_{011}^{(22)} - 8\overline{\zeta_2}^2 b_{002}^{(22)}) \\ &- (4i\overline{\zeta_2}b_{011}^{(11)} - 8\overline{\zeta_2}^2 b_{002}^{(11)})(4i\overline{\zeta_1}b_{101}^{(22)} - 8\overline{\zeta_1}^2 b_{002}^{(22)})\Big] \\ &- \left(2i\overline{\zeta_1}b_{011}^{(22)} + 2i\overline{\zeta_2}b_{101}^{(22)} - 8\overline{\zeta}_1\overline{\zeta_2}b_{002}^{(22)}\right) \\ &\cdot\Big[(2 + 4i\overline{\zeta_1}b_{101}^{(11)} - 8\overline{\zeta_1}^2 b_{002}^{(11)})(4i\overline{\zeta_2}b_{011}^{(12)} - 8\overline{\zeta_2}^2 b_{002}^{(12)}) \\ &- (4i\overline{\zeta_2}b_{011}^{(11)} - 8\overline{\zeta_2}^2 b_{002}^{(11)})(4i\overline{\zeta_1}b_{101}^{(12)} - 8\overline{\zeta_1}^2 b_{002}^{(12)})\Big]. \end{aligned}$$

By direct computation, one can verify that the $\overline{\zeta_1}^j\overline{\zeta_2}^l$ terms of degree 4, 5 and 6 above all vanish so that the last inequality in (4.11) holds. ∎

§ 5. A characterization of maps with degree 2

In this section, we give the following:

Theorem 5.1: Let $F \in Rat(\mathbf{H}_3, \mathbf{H}_6)$ with $\kappa_0 = 2$. Assume the same notation as in the beiginning of §4. Then $deg(F) = 2$ if and only if for each $p \approx 0$, the associated map, defined as in Lemma 2.2, $F_p^{***} = (f, \phi, g)$ satisfies

(5.0)

$$\frac{\partial^2 \phi_{11}}{\partial z_2 \partial w}(0) = \frac{\partial^2 \phi_{22}}{\partial z_1 \partial w}(0) = \frac{\partial^2 \phi_{22}}{\partial w^2}(0) = \frac{\partial^2 \phi_{11}}{\partial w^2}(0) = \frac{\partial^2 \phi_{12}}{\partial w^2}(0) = 0,$$
$$\frac{\partial^2 \phi_{12}}{\partial z_2 \partial w}(0) = \frac{\mu_2}{\sqrt{1+\mu_2}} \frac{\partial^2 \phi_{11}}{\partial z_1 \partial w}(0), \quad \frac{\partial^2 \phi_{22}}{\partial z_2 \partial w}(0) = \sqrt{\mu_2}\sqrt{1+\mu_2}\frac{\partial^2 \phi_{12}}{\partial z_1 \partial w}(0)$$

We first prove the following:

Lemma 5.2: Let $F_p^{***} = (f, \phi, g)$ with $p \approx 0$ and $\kappa_0(F) = 2$ be as above. Then $deg(F) = 2$ if and only if

(5.1)

$$f_1(z_1, z_2, 0) = z_1, \quad f_2(z_1, z_2, 0) = z_2, \quad \phi_{11}(z_1, z_2, 0) = \frac{z_1^2}{1 + az_1 + bz_2},$$
$$\phi_{12}(z_1, z_2, 0) = \frac{\sqrt{1+\mu_2} z_1 z_2}{1 + az_1 + bz_2}, \quad \phi_{22}(z_1, z_2, 0) = \frac{\sqrt{\mu_2} z_2^2}{1 + az_1 + bz_2}, \quad g(z_1, z_2, 0) = 0.$$

Proof of Lemma 5.2: If (5.1) holds for any $p \in U$, by Lemma 2.3, $deg(F) \le 2$. From the expressions in (5.1), $deg(F) = deg(F_p^{***}) \ge 2$. Hence $deg(F) = 2$.

Conversely, assume $deg(F) = 2$. In the proof of Theorem 4.1, we already obtained the formulas for f_j and g. To prove the formulas for ϕ_{jl}, since $deg(F) = deg(F_p^{***}) = 2$, we can write $f_j = \frac{P_j}{Q}$, $\phi_{jl} = \frac{P_{jl}}{Q}$ with $(P_j, P_{jl}, Q) = 1$, $deg(P_{jl}) \le 2$ and $deg(Q) \le 2$. We claim that $deg(Q(z,0)) = 1$. Otherwise, by the definition of the degree of F, $P_j(z,0) = z_j Q(z,0)$ and thus $deg(f_j) \ge 3$. This contradicts the fact that $deg(F_p^{***}) = 2$. By the normalization condition (4.1) and $deg(P_{jl}) = 2$, the formulas follow immediately. ■

In terms of Lemma 5.2 and (4.7), $deg(F) = 2$ if and only if, for each p and the corresponding F_p^{***}, there is a polynomial H^* such that

$$\begin{aligned} &\Phi_{11} = H^* \overline{\zeta_1}^2, \quad \Phi_{12} = H^* \sqrt{1+\mu_2}\ \overline{\zeta_1 \zeta_2}, \\ &\Phi_{22} = H^* \sqrt{\mu_2} \overline{\zeta_2}^2, \quad det(B) = H^*(1 + a\overline{\zeta_1} + b\overline{\zeta_2}). \end{aligned} \tag{5.2}$$

To prove Theorem 5.1, it suffices to prove that (5.2) holds if and only if

(5.3)

$$b_{011}^{(11)} = b_{101}^{(22)} = b_{002}^{(22)} = b_{002}^{(11)} = b_{002}^{(12)} = 0,$$
$$b_{011}^{(12)} = \frac{\mu_2 b_{101}^{(11)}}{\sqrt{1+\mu_2}}, \quad b_{011}^{(22)} = \sqrt{\mu_2}\ \sqrt{1+\mu_2} b_{101}^{(12)},$$
$$H^* = 4\sqrt{\mu_2}\sqrt{1+\mu_2} + 8i\sqrt{\mu_2} b_{011}^{(12)} \overline{\zeta_1} + 8i\sqrt{\mu_2} b_{101}^{(12)} \overline{\zeta_2} + 16(\mu_2 - 1)\sqrt{\mu_2} b_{002}^{12} \overline{\zeta}_1 \overline{\zeta}_2,$$
$$a = 2i b_{101}^{(11)}, \quad b = 2i\sqrt{1+\mu_2} b_{101}^{(12)}.$$

We need only to prove that the above are necessary conditions. From (4.7) and (5.2), we obtain $C_{10} = D_{10} = F_{01} = G_{01} = 0$, which can be used with Lemma 4.1 to conclude that

$$b_{011}^{(11)} = b_{101}^{(22)} = 0. \tag{5.4}$$

We also have

(5.5)
$$\begin{aligned} H^* &= 2A_{00} + 2A_{10}\overline{\zeta_1} + (2A_{01} - (1+\mu_2)B_{10})\overline{\zeta_2} + (2A_{11} - (1+\mu_2)B_{20})\overline{\zeta_1\zeta_2} \\ &+ (2A_{02} - (1+\mu_2)B_{11} + 2\mu_2 C_{20})\overline{\zeta_2}^2 + (2A_{12} - (1+\mu_2)B_{21} + 2\mu_2 C_{30})\overline{\zeta_1\zeta_2}^2 \\ &+ (2A_{03} - (1+\mu_2)B_{12} + 2\mu_2 C_{21})\overline{\zeta_2}^3, \end{aligned}$$

$$\begin{aligned} \sqrt{1+\mu_2}H^* &= (1+\mu_2)E_{00} + (-2D_{01} + (1+\mu_2)E_{10})\overline{\zeta_1} \\ &+ ((1+\mu_2)E_{01} - 2\mu_2 F_{10})\overline{\zeta_2} + (-2D_{11} + (1+\mu_2)E_{20})\overline{\zeta_1}^2 \\ &+ (-2D_{02} + (1+\mu_2)E_{11} - 2\mu_2 F_{20})\overline{\zeta_1\zeta_2} + ((1+\mu_2)E_{02} - 2\mu_2 F_{11})\overline{\zeta_2}^2 \\ &+ (-2D_{12} + (1+\mu_2)E_{21} - 2\mu_2 F_{30})\overline{\zeta_1}^2\overline{\zeta_2} \\ &+ (-2D_{03} + (1+\mu_2)E_{12} - 2\mu_2 F_{21})\overline{\zeta_1\zeta_2}^2, \end{aligned} \tag{5.6}$$

(5.7)
$$\begin{aligned} \sqrt{\mu_2}H^* &= 2\mu_2 I_{00} + (-(1+\mu_2)H_{01} + 2\mu_2 I_{10})\overline{\zeta_1} + 2\mu_2 I_{01}\overline{\zeta_2} \\ &+ (2G_{02} - (1+\mu_2)H_{11} + 2\mu_2 I_{20})\overline{\zeta_1}^2 + (-(1+\mu_2)H_{02} + 2\mu_2 I_{11})\overline{\zeta_1\zeta_2} \\ &+ (2G_{12} - (1+\mu_2)H_{21} + 2\mu_2 I_{30})\overline{\zeta_1}^3 + (2G_{03} - (1+\mu_2)H_{12} + 2\mu_2 I_{21})\overline{\zeta_1}^2\overline{\zeta_2}. \end{aligned}$$

From (5.5) (5.6) and (5.7), we obtain

$$\begin{aligned} &2A_{00} = \sqrt{1+\mu_2}E_{00} = 2\sqrt{\mu_2}I_{00}, \\ &2A_{10} = \frac{-2D_{01} + (1+\mu_2)E_{10}}{\sqrt{1+\mu_2}} = \frac{-(1+\mu_2)H_{01} + 2\mu_2 I_{10}}{\sqrt{\mu_2}}, \\ &2A_{01} - (1+\mu_2)B_{10} = \frac{(1+\mu_2)E_{01} - 2\mu_2 F_{10}}{\sqrt{1+\mu_2}} = 2\sqrt{\mu_2}I_{01}, \\ &0 = -2D_{11} + (1+\mu_2)E_{20} = 2G_{02} - (1+\mu_2)H_{11} + 2\mu_2 I_{20}, \\ &2A_{11} - (1+\mu_2)B_{20} = \frac{-2D_{02} + (1+\mu_2)E_{11} - 2\mu_2 F_{20}}{\sqrt{1+\mu_2}} \\ &= \frac{-(1+\mu_2)H_{02} + 2\mu_2 I_{11}}{\sqrt{\mu_2}}, \\ &2A_{02} - (1+\mu_2)B_{11} + 2\mu_2 C_{20} = (1+\mu_2)E_{02} - 2\mu_2 F_{11} = 0, \\ &0 = 2G_{12} - (1+\mu_2)H_{21} + 2\mu_2 I_{30}, \\ &0 = -2D_{12} + (1+u_2)E_{21} - 2\mu_2 F_{30} = 2G_{03} - (1+\mu_2)H_{12} + 2\mu_2 I_{21}, \\ &2A_{12} - (1+\mu_2)B_{21} + \mu_2 C_{30} = -2D_{03} + (1+\mu_2)E_{12} - 2\mu_2 F_{21} = 0, \\ &2A_{03} - (1+\mu_2)B_{12} + 2\mu_2 C_{21} = 0. \end{aligned}$$

Making use of Lemma 4.2, the above equations give

$$b_{011}^{(12)} = \frac{\mu_2 b_{101}^{(11)}}{\sqrt{1+\mu_2}}, \quad b_{011}^{(22)} = \sqrt{\mu_2}\ \sqrt{1+\mu_2} b_{101}^{(12)}, \tag{5.8}$$

From Lemma 4.2 and (5.8), we get

$$\begin{aligned} H^* &= 2A_{00} + 2A_{10}\overline{\zeta}_1 + [2A_{01} - (1+\mu_2)B_{10}]\overline{\zeta}_2 + [2A_{11} - (1+\mu_2)B_{20}]\overline{\zeta}_1\overline{\zeta}_2 \\ &= 4\sqrt{\mu_2}\sqrt{1+\mu_2} + 8i\sqrt{\mu_2} b_{011}^{12}\overline{\zeta}_1 + [8i\sqrt{1+\mu_2} b_{011}^{22} + 8i\sqrt{\mu_2} b_{101}^{12} \\ &\quad - 8i\sqrt{\mu_2}(1+\mu_2) b_{101}^{12}]\overline{\zeta}_2 + [-32\sqrt{\mu_2} b_{002}^{12} + 16\sqrt{\mu_2}(1+\mu_2) b_{002}^{12}]\overline{\zeta}_1\overline{\zeta}_2 \\ &= 4\sqrt{\mu_2}\sqrt{1+\mu_2} + 8i\sqrt{\mu_2} b_{011}^{12}\overline{\zeta}_1 + 8i\sqrt{\mu_2} b_{101}^{12}\overline{\zeta}_2 + 16(\mu_2 - 1)\sqrt{\mu_2} b_{002}^{12}\overline{\zeta}_1\overline{\zeta}_2. \end{aligned}$$

Hence the last equation in (5.2) can be written as
(5.9)

$$\begin{aligned} &\det B = H^* \cdot (1 + a\overline{\zeta}_1 + b\overline{\zeta}_2) \\ &= 4\sqrt{\mu_2}\sqrt{1+\mu_2} + 8i\sqrt{\mu_2} b_{011}^{(12)}\overline{\zeta}_1 + 8i\sqrt{\mu_2} b_{101}^{(12)}\overline{\zeta}_2 + 16(\mu_2-1)\sqrt{\mu_2} b_{002}^{(12)}\overline{\zeta}_1\overline{\zeta}_2 \\ &+ a\overline{\zeta}_1[4\sqrt{\mu_2}\sqrt{1+\mu_2} + 8i\sqrt{\mu_2} b_{011}^{(12)}\overline{\zeta}_1 + 8i\sqrt{\mu_2} b_{101}^{(12)}\overline{\zeta}_2 + 16(\mu_2-1)\sqrt{\mu_2} b_{002}^{(12)}\overline{\zeta}_1\overline{\zeta}_2] \\ &+ b\overline{\zeta}_2[4\sqrt{\mu_2}\sqrt{1+\mu_2} + 8i\sqrt{\mu_2} b_{011}^{(12)}\overline{\zeta}_1 + 8i\sqrt{\mu_2} b_{101}^{(12)}\overline{\zeta}_2 + 16(\mu_2-1)\sqrt{\mu_2} b_{002}^{(12)}\overline{\zeta}_1\overline{\zeta}_2]. \end{aligned}$$

Comparing the terms containing $\overline{\zeta}_1$ and $\overline{\zeta}_2$, we have

$$a = 2ib_{101}^{(11)}, \qquad b = 2i\sqrt{1+\mu_2} b_{101}^{(12)}. \tag{5.10}$$

Here we used the fact that $b_{011}^{(22)} = \sqrt{\mu_2}\sqrt{1+\mu_2} b_{101}^{(12)}$ in (5.8) to get the second equation.

Let $\zeta_1 = 0$ in (5.9). Comparing the terms with $\overline{\zeta}_2^2$ and making use of (5.8),(5.10), we have $b_{002}^{(22)} = 0$.

Similarly, letting $\zeta_2 = 0$ in (5.9) and comparing terms with $\overline{\zeta}_1^2$, we have $b_{002}^{(11)} = 0$.

Making use of the formulas: $b_{011}^{(11)} = b_{101}^{(22)} = b_{002}^{(11)} = b_{002}^{(22)} = 0$, we have

$$\begin{aligned} det(B) = &\big(-2i\overline{\zeta_2} b_{101}^{(11)}\big) \cdot \left[(4i\overline{\zeta_1} b_{101}^{(12)} - 8\overline{\zeta_1}^2 b_{002}^{(12)})(2\sqrt{\mu_2} + 4i\overline{\zeta_2} b_{011}^{(22)})\right] \\ &- 2i\overline{\zeta_1} b_{011}^{(22)}\left[(2 + 4i\overline{\zeta_1} b_{101}^{(11)})(4i\overline{\zeta_2} b_{011}^{(12)} - 8\overline{\zeta_2}^2 b_{002}^{(12)})\right] \\ &+ \left(\sqrt{1+\mu_2} + 2i\overline{\zeta_1} b_{011}^{(12)} + 2i\overline{\zeta_2} b_{101}^{(12)} - 8\overline{\zeta}_1\overline{\zeta_2} b_{002}^{(12)}\right) \\ &\cdot \left[(2 + 4i\overline{\zeta_1} b_{101}^{(11)})(2\sqrt{\mu_2} + 4i\overline{\zeta_2} b_{011}^{(22)})\right]. \end{aligned}$$

By (5.9) (5.10) and by considering terms with $\overline{\zeta}_1\overline{\zeta}_2$, we have $b_{002}^{(12)} = 0$.

Hence (5.3) is proved and thus the proof of Theorem 5.1 is complete. ∎

From Lemma 5.2 and Theorem 5.1, the following lemma follows immediately:

Lemma 5.3: Let $F \in Rat(\mathbf{H}_3, \mathbf{H}_6)$ with $deg(F) = 2$ and $F_p^{***} = (f, \phi, g)$ be as above. Then

$$f_1(z_1, z_2, 0) = z_1, \quad f_2(z_1, z_2, 0) = z_2,$$
$$\phi_{11}(z_1, z_2, 0) = \frac{z_1^2}{1 - 2i\overline{b_{101}^{(11)}} z_1 - 2i\sqrt{1+\mu_2}\,\overline{b_{101}^{(12)}} z_2},$$
$$\phi_{12}(z_1, z_2, 0) = \frac{\sqrt{1+\mu_2} z_1 z_2}{1 - 2i\overline{b_{101}^{(11)}} z_1 - 2i\sqrt{1+\mu_2}\,\overline{b_{101}^{(12)}} z_2},$$
$$\phi_{22}(z_1, z_2, 0) = \frac{\sqrt{\mu_2} z_2^2}{1 - 2i\overline{b_{101}^{(11)}} z_1 - 2i\sqrt{1+\mu_2}\,\overline{b_{101}^{(12)}} z_2}, \quad g(z, 0) = 0.$$

§6. Proof of Theorem 1.5

Proof of Theorem 1.5: Suppse that $F \in Rat(\mathbf{H}_3, \mathbf{H}_6)$ satisfies (5.1) with $deg(F) = 2$. We write

$$f_j = \frac{\sum_{1 \le l+s+t \le 2} A_{lst}^{(j)} z_1^l z_2^s w^t}{\sum_{1 \le l+s+t \le 2} E_{lst} z_1^l z_2^s w^t + 1}, \quad 1 \le j \le 2,$$
$$\phi_{jl} = \frac{\sum_{1 \le m+s+t \le 2} B_{mst}^{(jl)} z^m z_2^s w^t}{\sum_{1 \le m+s+t \le 2} E_{mst} z_1^m z_2^s w^t + 1}, \quad (jl) \in \{(11), (12), (22)\},$$
$$g = \frac{\sum_{1 \le l+s+t \le 2} C_{lst} z_1^l z_2^s w^t}{\sum_{1 \le l+s+t \le 2} E_{lst} z_1^l z_2^s w^t + 1}.$$

By Lemma 5.3, we obtain

$$E_{200} = E_{020} = E_{110} = A_{020}^{(1)} = A_{010}^{(1)} = 0, \ E_{100} = A_{200}^{(1)}, \ E_{010} = A_{110}^{(1)}, \ A_{100}^{(1)} = 1,$$
$$A_{200}^{(2)} = A_{100}^{(2)} = 0, \ A_{020}^{(2)} = E_{010}, \ A_{110}^{(2)} = E_{100} \ A_{010}^{(2)} = 1,$$
$$B_{100}^{(11)} = B_{010}^{(11)} = B_{020}^{(11)} = B_{110}^{(11)} = 0, \ B_{200}^{(11)} = 1,$$
$$E_{100} = -2i\overline{B_{101}^{(11)}}, \quad E_{010} = -2i\sqrt{1+\mu_2}\,\overline{B_{101}^{(12)}},$$
$$B_{100}^{(12)} = B_{010}^{(12)} = B_{200}^{(12)} = B_{020}^{(12)} = 0, \ B_{110}^{(12)} = \sqrt{1+\mu_2},$$
$$B_{100}^{(22)} = B_{010}^{(22)} = B_{200}^{(22)} = B_{110}^{(22)} = 0, \ B_{020}^{(22)} = \sqrt{\mu_2},$$
$$C_{110} = C_{020} = C_{100} = C_{010} = C_{200} = 0.$$

By (5.1), we have

$$A_{001}^{(1)} = A_{011}^{(1)} = A_{002}^{(1)} = 0, \ A_{101}^{(1)} = \frac{i}{2} + E_{001},$$
$$A_{001}^{(2)} = A_{101}^{(2)} = A_{002}^{(2)} = 0, \ A_{011}^{(2)} = \frac{i}{2}\mu_2 + E_{001},$$
$$B_{001}^{(11)} = B_{001}^{(12)} = B_{001}^{(22)} = 0,$$
$$C_{101} = E_{100}, \ C_{011} = E_{010}, \ C_{002} = E_{001}, \ C_{001} = 1.$$

By Theorem 5.1 and making use of the formulas: $B_{001}^{(11)} = B_{001}^{(12)} = B_{001}^{(22)} = 0$, we further have:

$$
\begin{aligned}
&B_{001}^{(11)} = B_{011}^{(11)} = B_{002}^{(11)} = 0, B_{011}^{(12)} = \frac{\mu_2 B_{101}^{(11)}}{\sqrt{1+\mu_2}},\\
&B_{001}^{(12)} = B_{002}^{(12)} = 0,\ B_{011}^{(22)} = \sqrt{\mu_2}\sqrt{1+\mu_2}B_{101}^{(12)}\\
&B_{001}^{(22)} = B_{101}^{(22)} = B_{002}^{(22)} = 0,
\end{aligned}
$$

From this, we have

$$
\begin{aligned}
f_1 &= \frac{-2i\overline{B_{101}^{(11)}}z_1^2 - 2i\sqrt{1+\mu_2}\overline{B_{101}^{(12)}}z_1z_2 + (\frac{i}{2} + E_{001})z_1w + z_1}{E_{002}w^2 + E_{101}z_1w + E_{011}z_2w - 2i\overline{B_{101}^{(11)}}z_1 - 2i\sqrt{1+\mu_2}\overline{B_{101}^{(12)}}z_2 + E_{001}w + 1},\\
f_2 &= \frac{-2i\sqrt{1+\mu_2}\overline{B_{101}^{(12)}}z_2^2 - 2i\overline{B_{101}^{(11)}}z_1z_2 + (\frac{i}{2}\mu_2 + E_{001})z_2w + z_2}{E_{002}w^2 + E_{101}z_1w + E_{011}z_2w - 2i\overline{B_{101}^{(11)}}z_1 - 2i\sqrt{1+\mu_2}\overline{B_{101}^{(12)}}z_2 + E_{001}w + 1},\\
\phi_{11} &= \frac{z_1^2 + B_{101}^{(11)}z_1w}{E_{002}w^2 + E_{101}z_1w + E_{011}z_2w - 2i\overline{B_{101}^{(11)}}z_1 - 2i\sqrt{1+\mu_2}\overline{B_{101}^{(12)}}z_2 + E_{001}w + 1},\\
\phi_{12} &= \frac{\sqrt{1+\mu_2}z_1z_2 + B_{101}^{(12)}z_1w + B_{011}^{(12)}z_2w}{E_{002}w^2 + E_{101}z_1w + E_{011}z_2w - 2i\overline{B_{101}^{(11)}}z_1 - 2i\sqrt{1+\mu_2}\overline{B_{101}^{(12)}}z_2 + E_{001}w + 1},\\
\phi_{22} &= \frac{\sqrt{\mu_2}z_2^2 + B_{011}^{(22)}z_2w}{E_{002}w^2 + E_{101}z_1w + E_{011}z_2w - 2i\overline{B_{101}^{(11)}}z_1 - 2i\sqrt{1+\mu_2}\overline{B_{101}^{(12)}}z_2 + E_{001}w + 1},\\
g &= \frac{E_{001}w^2 + C_{101}z_1w + C_{011}z_2w + w}{E_{002}w^2 + E_{101}z_1w + E_{011}z_2w - 2i\overline{B_{101}^{(11)}}z_1 - 2i\sqrt{1+\mu_2}\overline{B_{101}^{(12)}}z_2 + E_{001}w + 1}
\end{aligned}
$$

Let $z_2 = 0$ and $\mathrm{Im}(w) = |z_1|^2$. We have

$$
\begin{aligned}
f_1 &= \frac{-2i\overline{B_{101}^{(11)}}z_1^2 + (\frac{i}{2} + E_{001})z_1w + z_1}{E_{002}w^2 + E_{101}z_1w - 2i\overline{B_{101}^{(11)}}z_1 + E_{001}w + 1}, f_2 = 0,\\
\phi_{11} &= \frac{z_1^2 + B_{101}^{(11)}z_1w}{E_{002}w^2 + E_{101}z_1w - 2i\overline{B_{101}^{(11)}}z_1 + E_{001}w + 1},
\end{aligned}
$$

$$
\begin{aligned}
\phi_{12} &= \frac{B_{101}^{(12)}z_1w}{E_{002}w^2 + E_{101}z_1w - 2i\overline{B_{101}^{(11)}}z_1 + E_{001}w + 1}, \quad \phi_{22} = 0,\\
g &= \frac{E_{001}w^2 + C_{101}z_1w + w}{E_{002}w^2 + E_{101}z_1w - 2i\overline{B_{101}^{(11)}}z_1 + E_{001}w + 1}.
\end{aligned}
$$

Denote $w = u + i|z_1|^2$. The equation $Im(g) = |\widetilde{f}|^2$ becomes

$$\Big(E_{001}(u^2 - |z_1|^4 + 2iu|z_1|^2) + C_{101}z_1(u + i|z_1|^2) + (u + i|z_1|^2\Big)$$
$$\cdot\Big(\overline{E_{002}}(u^2 - |z_1|^4 - 2iu|z_1|^2) + \overline{E_{101}}\,\overline{z_1}(u - i|z_1|^2)$$
$$+ 2iB^{(11)}_{101}\overline{z_1} + \overline{E_{001}}(u - i|z|^2) + 1\Big)$$
$$-\Big(\overline{E_{001}}(u^2 - |z_1|^4 - 2iu|z_1|^2) + \overline{C_{101}}\,\overline{z_1}(u - i|z|^2) + (u - i|z|^2)\Big)$$
$$\cdot\Big(E_{002}(u^2 - |z_1|^4 + 2iu|z_1|^2) + E_{101}z_1(u + i|z_1|^2)$$
$$- 2i\overline{B^{(11)}_{101}}z_1 + E_{001}(u + i|z|^2) + 1\Big)$$
$$= 2i| - 2i\overline{B^{(11)}_{101}}z_1^2 + (\frac{i}{2} + E_{001})z_1(u + i|z_1|^2) + z_1|^2$$
$$+ 2i|z_1^2 + B^{(11)}_{101}z_1(u + i|z_1|^2)|^2 + 2i|B^{(12)}_{101}z_1(u + i|z_1|^2)|^2.$$

Considering u^3-terms, we see that E_{002} is real. Considering the u^4-terms, we get

$$E_{002}(E_{001} - \overline{E_{001}}) = 0. \tag{6.1}$$

Considering the uz_1-terms, we get $C_{101} = -2i\overline{B^{(11)}_{101}}$. Then considering the u^2z_1-terms, we get

$$E_{101} = 0. \tag{6.2}$$

Comparing the u^3z_1-terms and making use of the fact that $E_{101} = 0$, it follows that

$$B^{(11)}_{101}E_{002} = 0. \tag{6.3}$$

Similarly, if we put $z_1 = 0$ and let $Im(w) = |z_2|^2$, we obtain

$$C_{011} = -2i\sqrt{1+\mu_2}\,\overline{B^{(12)}_{101}},\ E_{011} = 0,\ B^{(12)}_{101}E_{002} = 0. \tag{6.4}$$

Therefore the remaining proof can be divided into two cases: (i) $E_{002} \neq 0$ and (ii) $E_{002} = 0$.

Case (i): Since $E_{002} \neq 0$, we get $C_{011} = C_{101} = B^{(12)}_{101} = B^{(11)}_{101} = 0$. By (6.1) E_{001} is a real number. By (5.8), the map F is given by

$$f_1 = \frac{(\frac{i}{2} + E_{001})z_1w + z_1}{E_{002}w^2 + E_{001}w + 1},\quad f_2 = \frac{(\frac{i}{2}\mu_2 + E_{001})z_2w + z_2}{E_{002}w^2 + E_{001}w + 1},$$
$$\phi_{11} = \frac{z_1^2}{E_{002}w^2 + E_{001}w + 1},\quad \phi_{12} = \frac{\sqrt{1+\mu_2}z_1z_2}{E_{002}w^2 + E_{001}w + 1},$$
$$\phi_{22} = \frac{\sqrt{\mu_2}z_2^2}{E_{002}w^2 + E_{001}w + 1},\quad g = \frac{E_{001}w^2 + w}{E_{002}w^2 + E_{001}w + 1}.$$

where E_{001} and E_{002} are real numbers. Consider the equation $Im(g) = |z|^2$:

$$\Big[E_{001}(u^2 - |z|^4 + 2iu|z|^2) + u + i|z|^2\Big]$$
$$\cdot \Big[E_{002}(u^2 - |z|^4 - 2iu|z|^2) + E_{001}(u - i|z|^2) + 1\Big]$$
$$- \Big[E_{001}(u^2 - |z|^4 - 2iu|z|^2) + u - i|z|^2\Big]$$
$$\cdot \Big[E_{002}(u^2 - |z|^4 + 2iu|z|^2) + E_{001}(u + i|z|^2) + 1\Big]$$
$$= 2i|z_1|^2|(\frac{i}{2} + E_{001})(u + i|z|^2) + 1|^2 + 2i|z_2|^2|(\frac{i}{2}\mu_2 + E_{001})(u + i|z|^2) + 1|^2$$
$$+ 2i|z_1|^4 + 2i(1 + \mu_2)|z_1 z_2|^2 + 2i\mu_2|z_2|^4.$$

Let $z_1 = 0$, consider the terms about $|z_2|^6$, we have

$$-4E_{002} = \mu_2^2.$$

Similarly, if we let $z_2 = 0$, we have

$$E_{002} = -\frac{1}{4}.$$

Hence

$$\mu_2 = 1.$$

Then the map F is of the form

$$f_1 = \frac{(\frac{i}{2} + E_{001})z_1 w + z_1}{-\frac{1}{4}w^2 + E_{001}w + 1}, \quad f_2 = \frac{(\frac{i}{2} + E_{001})z_2 w + z_2}{-\frac{1}{4}w^2 + E_{001}w + 1},$$
$$\phi_{11} = \frac{z_1^2}{-\frac{1}{4}w^2 + E_{001}w + 1}, \quad \phi_{12} = \frac{\sqrt{2}z_1 z_2}{-\frac{1}{4}w^2 + E_{001}w + 1},$$
$$\phi_{22} = \frac{z_2^2}{-\frac{1}{4}w^2 + E_{001}w + 1}, \quad g = \frac{E_{001}w^2 + w}{-\frac{1}{4}w^2 + E_{001}w + 1}.$$

By making use of the Calyay transformation, this map can be identified as the following map, still denoted as F, in $Rat(\mathbf{B}^3, \mathbf{B}^6)$:

$$f_1 = \frac{4z_1[iE_{001} + (1 - iE_{001})w]}{w^2 - 4iE_{001}w + 3 + 4iE_{001}}, \quad f_2 = \frac{4z_2[iE_{001} + (1 - iE_{001})w]}{w^2 - 4iE_{001}w + 3 + 4iE_{001}},$$
$$\phi_{11} = \frac{2\sqrt{2}z_1^2}{w^2 - 4iE_{001}w + 3 + 4iE_{001}}, \quad \phi_{12} = \frac{4z_1 z_2}{w^2 - 4iE_{001}w + 3 + 4iE_{001}},$$
$$\phi_{22} = \frac{2\sqrt{2}z_2^2}{w^2 - 4iE_{001}w + 3 + 4iE_{001}}, \quad g = \frac{(3 - 4iE_{001})w^2 + 4iE_{001}w + 1}{w^2 - 4iE_{001}w + 3 + 4iE_{001}}.$$

Since $F(0,0,0) = (0,\cdots,0,a)$ with $a = \frac{1}{3+4iE_{001}}$, F is equivalent to a new map in $Rat(\mathbf{B}^3, \mathbf{B}^6)$ defined by $\Psi \circ F$ where $\Psi \in Aut(\mathbf{B}^6)$ is given by $\Psi = \left(\frac{sz_1}{1-\overline{a}w}, \cdots, \frac{sz_5}{1-\overline{a}w}, -\frac{a-w}{1-\overline{a}w},\right)$ with $s = \sqrt{1-|a|^2}$. This new map, still denoted by F, satisfies that $F(0,0) = (0,0,0)$ and is of the form

$$
\begin{aligned}
f_1 &= \frac{2s(9+16E_{001}^2)z_1[iE_{001} + (1-iE_{001})w]}{-2iE_{001}(16E_{001}+4iE_{001}+12)w + (3+4iE_{001})(4+8E_{001})}, \\
f_2 &= \frac{2s(9+16E_{001}^2)z_2[iE_{001} + (1-iE_{001})w]}{-2iE_{001}(16E_{001}+4iE_{001}+12)w + (3+4iE_{001})(4+8E_{001})}, \\
\phi_{11} &= \frac{s(9+16E_{001})\sqrt{2}z_1^2}{-2iE_{001}(16E_{001}+4iE_{001}+12)w + (3+4iE_{001})(4+8E_{001})}, \\
\phi_{12} &= \frac{s(9+16E_{001})2z_1z_2}{-2iE_{001}(16E_{001}+4iE_{001}+12)w + (3+4iE_{001})(4+8E_{001})}, \\
\phi_{22} &= \frac{s(9+16E_{001})\sqrt{2}z_2^2}{-2iE_{001}(16E_{001}+4iE_{001}+12)w + (3+4iE_{001})(4+8E_{001})}, \\
g &= \frac{(3-4iE_{001})(4+8E_{001})w^2 + 2iE_{001}(16E_{001}-4iE_{001}+12)w}{-2iE_{001}(16E_{001}+4iE_{001}+12)w + (3+4iE_{001})(4+8E_{001})}.
\end{aligned} \tag{6.5}
$$

Notice that when $E_{001} = 0$, the above map becomes

$$
(f_1, f_2, \phi_{11}, \phi_{12}, \phi_{22}, g) = (\sqrt{2}z_1w,\ \sqrt{2}z_2w,\ z_1^2,\ \sqrt{2}z_1z_2,\ z_2^2,\ w^2). \tag{6.6}
$$

We want to show that the map (6.5) is equivalent to the map (6.6) even for $E_{001} \neq 0$. To do this, it is enough to choose complex numbers α and β so that for the map F in (6.5), the new map $\phi_\beta \circ F \circ \phi_\alpha$ is exactly (6.6). Where

$$
\phi_\alpha = \left(\frac{s_\alpha z_1}{1-\overline{\alpha}w}, \frac{s_\alpha z_2}{1-\overline{\alpha}w}, \frac{\alpha - w}{1-\overline{\alpha}w},\right), \quad \phi_\beta = \left(\frac{s_\beta z_1}{1-\overline{\beta}w}, \cdots, \frac{s_\beta z_5}{1-\overline{\beta}w}, \frac{\beta - w}{1-\overline{\beta}w}.\right)
$$

with $s_\alpha = \sqrt{1-|\alpha|^2}$ and $s_\beta = \sqrt{1-|\beta|^2}$. Since $\phi_\beta \circ F \circ \phi_\alpha$ maps $(0,0)$ to $(0,0,0)$, β is determined by α. In fact, since

$$
\begin{aligned}
&F(\phi_\alpha(0,0)) \\
=&\left(0,\cdots,0, \frac{(3-4iE_{001})(4+8E_{001})\alpha^2 + 2iE_{001}(16E_{001}-4iE_{001}+12)\alpha}{-2iE_{001}(16E_{001}+4iE_{001}+12)\alpha + (3+4iE_{001})(4+8E_{001})}\right),
\end{aligned}
$$

we have to choose

$$
\beta := \frac{(3-4iE_{001})(4+8E_{001})\alpha^2 + 2iE_{001}(16E_{001}-4iE_{001}+12)\alpha}{-2iE_{001}(16E_{001}+4iE_{001}+12)\alpha + (3+4iE_{001})(4+8E_{001})}. \tag{6.7}
$$

In order to determine α, we notice that the first component function of the map $\phi_\beta \circ F \circ \phi_\alpha$ is of the form

$$\text{(6.8)} \qquad \frac{constant \cdot z_1[iE_{001} + \alpha - i\alpha E_{001} + w(-iE_{001}\overline{\alpha} - 1 + iE_{001})]}{\cdots}.$$

Where "$\cdots$" is a polynomial of w. To make the map (6.5) equal to the map (6.6), we need to have $iE_{001} + \alpha - i\alpha E_{001} = 0$. Hence we can choose $\alpha = -\frac{iE_{001}}{1-iE_{001}}$. Then from (6.7)

$$\beta = \frac{E_{001} - iE_{001}}{1 + E_{001}} \cdot \frac{(3-4iE_{001})(4+8E_{001})(E_{001} - iE_{001}) + 2iE_{001}(16E_{001} - 4iE_{001} + 12)(1+E_{001})}{-2iE_{001}(16E_{001} + 4iE_{001} + 12)(E_{001}^2 - iE_{001}) + (3+4iE_{001})(4+8E_{001})(1+E_{001})}.$$

Then the first component function of $\phi_\beta \circ F \circ \phi_\alpha$ becomes

(6.9)

$$\Big(4s_\beta s_\alpha s(9 + 16E_{001})z_1(-i\overline{E}_{001} - 1 + iE_{001})w\Big) \cdot \Big(- 4iE_{001}(16E_{001} + 4iE_{001} + 12)(\alpha - w)(1 - \overline{\alpha}w) + (3+4iE_{001})2(4+8E_{001})(1-\overline{\alpha}w)^2 - \overline{\beta}(3-4iE_{001})2(4+8E_{001}^2)(\alpha - w)^2 - 4iE_{001}\overline{\beta}(16E_{001} - 4iE_{001} + 12)(a - w)(1-\overline{\alpha}w)\Big)^{-1}.$$

By direct computation, the w^2 and w terms of the determinant of (6.9) are zero. Hence the first component function of the map $\phi_\beta \circ F \circ \phi_\alpha$ is of the form $constant \cdot z_1 w$. Similarly, we can prove that: $f_2 = constant \cdot z_2 w$, $\phi_{11} = constant \cdot z_1^2$, $\phi_{12} = constant \cdot z_1 z_2$, $\phi_{22} = constant \cdot z_2^2$, $g = constant \cdot w^2$

Assume $\phi_\beta \circ F \circ \phi_\alpha = (a_1 z_1 w,\ a_2 z_2 w,\ a_3 z_1^2,\ a_4 z_1 z_2,\ a_5 z_2^2,\ a_6 w^2)$ with a_i constants. Since $\phi_\beta \circ F \circ \phi_\alpha$ is a proper mapping from $\mathbf{B}_3$ to $\mathbf{B}_6$, letting $z_2 = 0$, we have:

$$|a_1|^2|z_1|^2|w|^2 + |a_3|^2|z_1|^4 + |a_6|^2|w|^4 = 1$$

for any (z_1, w) with $|z_1|^2 + |w|^2 = 1$. Replacing $|z_1|^2$ by $1 - |w|^2$ in the above equation, we have

$$\text{(6.10)} \qquad (|a_1|^2 - 2|a_3|^2)|w|^2 + (|a_3|^2 + |a_6|^2 - |a_1|^2)|w|^4 + |a_3|^2 = 1.$$

Let $|w|^2 = 0$. From (6.10), $|a_3| = 1$. Similarly, let $|w|^2 = 1$. We then get $|a_6| = 1$ and hence $|a_1|^2 = 2$. Let $z_1 = 0$. By the same proof, we get $|a_2|^2 = 2$, $|a_5| = 1$. Since $|\phi_\beta \circ F \circ \phi_\alpha(z, w)|^2 = 1$ for (z_1, z_2, w) with $|z_1|^2 + |z_2|^2 + |w|^2 = 1$, we must have $|a_4|^2 = 2$. Therefore $\phi_\beta \circ F \circ \phi_\alpha$ is equivalent to the map in (6.6).

Case (ii) Since $E_{002} = 0$, the map F becomes

$$f_1 = \frac{-2i\overline{B^{(11)}_{101}}z_1^2 - 2i\sqrt{1+\mu_2}\overline{B^{(12)}_{101}}z_1z_2 + (\frac{i}{2}+E_{001})z_1w + z_1}{-2i\overline{B^{(11)}_{101}}z_1 - 2i\sqrt{1+\mu_2}\overline{B^{(12)}_{101}}z_2 + E_{001}w + 1},$$

$$f_2 = \frac{-2i\sqrt{1+\mu_2}\overline{B^{(12)}_{101}}z_2^2 - 2i\overline{B^{(11)}_{101}}z_1z_2 + (\frac{i}{2}\mu_2+E_{001})z_2w + z_2}{-2i\overline{B^{(11)}_{101}}z_1 - 2i\sqrt{1+\mu_2}\overline{B^{(12)}_{101}}z_2 + E_{001}w + 1},$$

$$\phi_{11} = \frac{z_1^2 + B^{(11)}_{101}z_1w}{-2i\overline{B^{(11)}_{101}}z_1 - 2i\sqrt{1+\mu_2}\overline{B^{(12)}_{101}}z_2 + E_{001}w + 1}, \tag{6.11}$$

$$\phi_{12} = \frac{\sqrt{1+\mu_2}z_1z_2 + B^{(12)}_{101}z_1w + \frac{\mu_2}{\sqrt{1+\mu_2}}b^{(11)}_{101}z_2w}{-2i\overline{B^{(11)}_{101}}z_1 - 2i\sqrt{1+\mu_2}\overline{B^{(12)}_{101}}z_2 + E_{001}w + 1},$$

$$\phi_{22} = \frac{\sqrt{\mu_2}z_2^2 + \sqrt{\mu_2}\sqrt{1+\mu_2}B^{(12)}_{101}z_2w}{-2i\overline{B^{(11)}_{101}}z_1 - 2i\sqrt{1+\mu_2}\overline{B^{(12)}_{101}}z_2 + E_{001}w + 1}, \quad g = w.$$

The equation $\mathrm{Im}(g) = |\widetilde{f}|^2$ becomes

(6.12)

$$\begin{aligned}
&|z|^2|-2i\overline{B^{(11)}_{101}}z_1 - 2i\sqrt{1+\mu_2}\overline{B^{(12)}_{101}}z_2 + E_{001}w + 1|^2 \\
=&|z_1|^2|-2i\overline{B^{(11)}_{101}}z_1 - 2i\sqrt{1+\mu_2}\overline{B^{(12)}_{101}}z_2 + (\frac{i}{2}+E_{001})w + 1|^2 \\
&|z_2|^2|-2i\overline{B^{(11)}_{101}}z_1 - 2i\sqrt{1+\mu_2}\overline{B^{(12)}_{101}}z_2 + (\frac{i}{2}\mu_2+E_{001})w + 1|^2 \\
&+|z_1|^2|z_1 + B^{(11)}_{101}w|^2 + |\sqrt{1+\mu_2}z_1z_2 + B^{(12)}_{101}z_1w + \frac{\mu_2}{\sqrt{1+\mu_2}}B^{(11)}_{101}z_2w|^2 \\
&+|z_2|^2|\sqrt{\mu_2}z_2 + \sqrt{\mu_2}\sqrt{1+\mu_2}B^{(12)}_{101}w|^2.
\end{aligned}$$

Let $w = u + i|z|^2$ as before. Since the only $z_1\overline{z_2}u^2$ term is contained in $|\sqrt{1+\mu_2}z_1z_2 + B^{(12)}_{101}z_1w + \frac{\mu_2}{\sqrt{1+\mu_2}}B^{(11)}_{101}z_2w|^2$, it implies

$$B^{(12)}_{101}\overline{B^{(11)}_{101}} = 0. \tag{6.13}$$

Let $z_2 = 0$, (6.12) is

$$\begin{aligned}
&|z_1|^2|-2i\overline{B^{(11)}_{101}}z_1 + E_{001}(u+i|z_1|^2) + 1|^2 \\
=&|z_1|^2|-2i\overline{B^{(11)}_{101}}z_1 + (\frac{i}{2}+E_{001})(u+i|z_1|^2) + 1|^2 \\
&+|z_1|^2|z_1 + B^{(11)}_{101}(u+i|z_1|^2)|^2 + |z_1|^2|B^{(12)}_{101}(u+i|z_1|^2)|^2.
\end{aligned} \tag{6.14}$$

Or

$$
\begin{aligned}
&|-2i\overline{B^{(11)}_{101}}z_1+E_{001}(u+i|z_1|^2)+1|^2\\
=&|-2i\overline{B^{(11)}_{101}}z_1+(\frac{i}{2}+E_{001})(u+i|z_1|^2)+1|^2\\
&+|z_1+B^{(11)}_{101}(u+i|z_1|^2)|^2+|B^{(12)}_{101}(u+i|z_1|^2)|^2.
\end{aligned}
\tag{6.15}
$$

Consider the u^2-terms, we have

$$
0=\frac{1}{4}+|B^{(11)}_{101}|^2+|B^{(12)}_{101}|^2+Im(E_{001}).
\tag{6.16}
$$

Similarly, if we let $z_1=0$, from (6.12), we can get

$$
\begin{aligned}
&|-2i\sqrt{1+\mu_2}\overline{B^{(12)}_{101}}z_2+E_{001}(u+i|z_2|^2)+1|^2\\
=&|-2i\sqrt{1+\mu_2}\overline{B^{(12)}_{101}}z_2+(\frac{i}{2}\mu_2+E_{001})(u+i|z_2|^2)+1|^2\\
&+\mu_2|z_2+\sqrt{1+\mu_2}B^{(11)}_{101}(u+i|z_2|^2)|^2+\frac{\mu_2^2}{1+\mu_2}|B^{(12)}_{101}(u+i|z_2|^2)|^2.
\end{aligned}
\tag{6.17}
$$

Consider the u^2-terms, we have

$$
0=\frac{\mu_2}{4}+Im(E_{001})+\frac{\mu_2}{1+\mu_2}|B^{(11)}_{101}|^2+(1+\mu_2)|B^{(12)}_{101}|^2.
\tag{6.18}
$$

From (6.16) and (6.18), we get

$$
\frac{\mu_2-1}{4}+|B^{(12)}_{101}|^2=\frac{1}{1+\mu_2}|B^{(11)}_{101}|^2.
\tag{6.19}
$$

By (6.13), $B^{(12)}_{101}=0$ or $B^{(11)}_{101}=0$.

If $B^{(12)}_{101}=0$, then from (6.19), we get $\mu_2=1,\ B^{(12)}_{101}=0$. Then by (6.16), $Im(E_{001})=\frac{-1}{4}$. Hence F becomes

$$
f_1=\frac{(\frac{i}{2}+E_{001})z_1w+z_1}{E_{001}w+1},\quad f_2=\frac{(\frac{i}{2}+E_{001})z_2w+z_2}{E_{001}w+1},\quad \phi_{11}=\frac{z_1^2}{E_{001}w+1},
$$
$$
\phi_{12}=\frac{\sqrt{2}z_1z_2}{E_{001}w+1},\quad \phi_{22}=\frac{z_2^2}{E_{001}w+1},\quad g=w.
$$

Then simple computation shows that $\mathrm{Im}(g)=|\widetilde{f}|^2$ holds if and only if $\mathrm{Im}(E_{001})=-\frac{1}{4}$. By the same proof used in [JX 2003, § 4], the map F is equivalent to the generalized Whitney map defined in Theorem 1.5.

If $B_{101}^{(12)} = 0$, (6.11) becomes

$$
(6.20)\quad
\begin{aligned}
f_1 &= \frac{-2i\overline{B_{101}^{(11)}} z_1^2 + (\frac{i}{2} + E_{001}) z_1 w + z_1}{-2i\overline{B_{101}^{(11)}} z_1 + E_{001} w + 1},\\
f_2 &= \frac{-2i\overline{B_{101}^{(11)}} z_1 z_2 + (\frac{i}{2}\mu_2 + E_{001}) z_2 w + z_2}{-2i\overline{B_{101}^{(11)}} z_1 + E_{001} w + 1},\\
\phi_{11} &= \frac{z_1^2 + B_{101}^{(11)} z_1 w}{-2i\overline{B_{101}^{(11)}} z_1 + E_{001} w + 1},\\
\phi_{12} &= \frac{\sqrt{1+\mu_2} z_1 z_2 + \frac{\mu_2}{\sqrt{1+\mu_2}} b_{101}^{(11)} z_2 w}{-2i\overline{B_{101}^{(11)}} z_1 + E_{001} w + 1},\\
\phi_{22} &= \frac{\sqrt{\mu_2} z_2^2}{-2i\overline{B_{101}^{(11)}} z_1 + E_{001} w + 1}, \quad g = w.
\end{aligned}
$$

Compare the above map with the one in Theorem 6.1[JX 2003, § 6]. By using Corollary 6.2[JX 2003, § 6] and let $z_3 = 0$, we see that they are the same map if we omit the 0-components. That means by the same proof in §7[JX 2003], we can get $\mu_2 = 1$. By (6.19), $B_{101}^{(12)} = 0$ and hence $Im(E_{001}) = -\frac{1}{4}$. By the proof used in [JX 2003, § 4] again, the map F is equivalent to the generalized Whitney map defined in Theorem 1.5.■

References

[A77] H. Alexander, Proper holomorphic maps in $\mathbf{C}^n$, Indiana Univ. Math. Journal 26, 137-146 (1977).

[BER99] M. S. Baouendi, P. Ebenfelt and L. Rothschild, Real Submanifolds in Complex Spaces and Their Mappings, Princeton Univ. Mathematics Series 47, Princeton University, New Jersey, 1999.

[CS83] J. Cima and T. J. Suffridge, A reflection principle with applications to proper holomorphic mappings, Math Ann. 265, 489-500 (1983).

[CS90] J.Cima and T. J. Suffridge, Boundary behavior of rational proper maps, Duke Math. J. 60, 135-138 (1990).

[DA93] J. P. D'Angelo, Several Complex Variables and the Geometry of Real Hypersurfaces, CRC Press, Boca Raton, 1993.

[DA88] J. P. D'Angelo, Proper holomorphic mappings between balls of different dimensions, *Mich. Math. J.* 35, 83-90 (1988).

[Fa82] J. Faran, Maps from the two ball to the three ball, Invent. Math. 68, 441-475 (1982).

[Fa86] J. Faran, On the linearity of proper maps between balls in the lower dimensional case, Jour. Diff. Geom. 24, 15-17 (1986).

[Fo89] F. Forstneric, Extending proper holomorphic mappings of positive codimension, Invent. Math., 95, 31-62 (1989).

[Fo92] F. Forstneric, A survey on proper holomorphic mappings, Proceeding of Year in SCVs at Mittag-Leffler Institute, Math. Notes 38 (1992), Princeton University Press, Princeton, N.J.
[H99] X. Huang, On a linearity problem of proper holomorphic mappings between balls in complex spaces of different dimensions, Jour. of Diff. Geom. Vol (51) No. 1, 13-33 (1999).
[H01] X. Huang, On some problems in several complex variables and CR geometry. First International Congress of Chinese Mathematicians (Beijing, 1998), 383–396, AMS/IP Stud. Adv. Math., 20, Amer. Math. Soc., Providence, RI, 2001.
[H02] X. Huang, On a Semi-Rigidity Property for Holomorphic Maps, preprint, 2002. (To appear in Asian Jour. of Math., a special issue in honor of Professor Y-T Siu's 60th birthday).
[HJ01] X. Huang and S. Ji, Mapping $\mathbf{B}^n$ into $\mathbf{B}^{2n-1}$, Invent. Math. 145, 219-250(2001).
[JX03] S. Ji and D. Xu, Rational maps between $\mathbf{B^n}$ and $\mathbf{B^N}$ with geometric rank $\kappa_0 \leq n-2$ and minimal target dimension, preprint, 2003.
[P07] H. Poincaré, Les fonctions analytiques de deux variables et la représentation conforme, Ren. Cire. Mat. Palermo, II. Ser. 23, 185-220 (1907).
[T62] N. Tanaka, On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables, J. Math. Soc. Japan 14, 397-429(1962).
[W79] S. Webster, On mappings an (n+1)-ball in the complex space, Pac. J. Math. 81, 267-272 (1979).

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