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On a Semi-Rigidity Property for Holomorphic Maps

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Dedicated to Professor Yum-Tong Siu on the occasion of his 60th birthday

§1. Introduction

In this paper, we are concerned with a rigidity problem for holomorphic maps between balls in complex spaces of different dimensions. Write $\mathbf{B}^n$ and $\mathbf{B}^N$ for the unit balls in $\mathbf{C}^n$ and $\mathbf{C}^N$, respectively. Let F be a proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^N$. We say that F is κ-linear if for any point $p \in \mathbf{B}^n$, there is an affine complex subspace S_p^κ, that passes through p and is of dimension κ, such that for any affine complex line L contained in S_p^κ, $F(L \cap \mathbf{B}^n)$ is contained in an affine complex line in $\mathbf{C}^N$. Our main result is the following:

Theorem 1.1: *Let F be a proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^N$, which is C^3-smooth up to the boundary. Write $P(n,\kappa) = \frac{\kappa(2n-\kappa-1)}{2}$. If $1 \le \kappa \le n-1$ and $N - n < P(n,\kappa)$, then F is $(n-\kappa+1)$-linear.*

For an affine complex subspace $S \subset \mathbf{C}^n$ and a proper holomorphic map F from $\mathbf{B}^n$ into $\mathbf{B}^N$, F maps each affine complex disk in $S \cap \mathbf{B}^n$ to an affine complex disk in $\mathbf{B}^N$ if and only if the restriction of F to S is a linear fractional map. As a consequence, Theorem 1.1 can be used to give the following:

Corollary 1.2: *Let F be a proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^N$, which is C^3-smooth up to the boundary. Write $P(n,\kappa) = \frac{\kappa(2n-\kappa-1)}{2}$. If $1 \le \kappa \le n-1$ and $N - n < P(n,\kappa)$, then for any point $p \in \mathbf{B}^n$, there are $\sigma_p \in Aut(\mathbf{B}^n)$ and $\tau_p \in Aut(\mathbf{B}^N)$ such that $\tau_p \circ F \circ \sigma_p(z_1, \cdots, z_{n-\kappa+1}, 0, \cdots, 0) = (z_1, \cdots, z_{n-\kappa+1}, 0, \cdots, 0)$.*

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The following example shows that in Theorem 1.1, when $N-n \geq P(n,\kappa)$, one can not expect the $(n-k+1)$-linearity for the map in general.

Example 1.3: Let

$$\begin{aligned}
\psi_1 &= (z_1^2, \sqrt{2}z_1z_2, \cdots, \sqrt{2}z_1z_{k-1}, z_1z_k, \cdots, z_1z_n),\\
\psi_2 &= (z_2^2, \sqrt{2}z_2z_3, \cdots, \sqrt{2}z_2z_{k-1}, z_2z_k, \cdots, z_2z_n),\\
&\cdots\\
\psi_{k-1} &= (z_{k-1}^2, z_{k-1}z_k, \cdots, z_{k-1}z_n),\\
\psi_k &= (z_k, \cdots, z_n).
\end{aligned}$$

Let $W_{n,k} = (\psi_1, \cdots, \psi_k)$. Then $W_{n,k}$ is a proper polynomial map from $\mathbf{B}^n$ into $\mathbf{B}^N$ with $N = n + P(n,k)$. Notice that $W_{n,k}$ is not $(n-k+1)$-linear.

Remark 1.4: When $N < \frac{n(n+1)}{2}$ in Theorem 1.1, our result says that at each point in the ball, F has at least two independent directions along which the map is linear. When $N \geq \frac{n(n+1)}{2}$, F usually has no partial linearity. To see this, we just need to notice that the polynomial map that sends

$$(z_1, \cdots, z_n) \ \text{ to } \ (z_1^2, \sqrt{2}z_1z_2, \cdots, \sqrt{2}z_1z_n, z_2^2, \sqrt{2}z_2z_3, \cdots, \sqrt{2}z_2z_n, \cdots, z_{n-1}z_{n-1}, \sqrt{2}z_{n-1}z_n, z_n^2)$$

is proper from $\mathbf{B}^n$ into $\mathbf{B}^N$ with $N = n + P(n, n-1) = \frac{n(n+1)}{2}$. We mention the interesting similarity between the minimal target dimension N for which the rigidity breaks down in the case we are considering here and the minimal target dimension in the classical Cartan-Janet theorem for which there is no more obstruction to locally isometrically embed an analytic Riemannian manifold of dimension n into $\mathbf{R}^N$ ([Sp]).

The study of holomorphic maps between balls in complex Euclidean spaces was initiated from a paper of Poincaré [Po], and has attracted considerable attention since then. When $N = n > 1$, a result of Alexander [Alx] states that any proper holomorphic self-map of the unit ball $\mathbf{B}^n$ in $\mathbf{C}^n$ with $n > 1$ is an automorphism. Notice that for an affine complex line L, $L \cap \mathbf{B}^n$ is a complex geodesic in terms of the hyperbolic Kobayashi metric of the ball and an automorphism of $\mathbf{B}^n$ maps an affine line to an affine line. The result of Alexander hence tells that a proper holomorphic self-map of $\mathbf{B}^n$ preserves the complex geodesics of $\mathbf{B}^n$ $(n > 1)$. More generally, one says that a map from $\mathbf{B}^n$ into $\mathbf{B}^N$ is a linear map or a totally geodesic embedding if it maps a complex geodesic in $\mathbf{B}^n$ to a complex geodesic in $\mathbf{B}^N$. Webster [We1] [We2] was the first one to look at

the geometric structure of proper holomorphic maps between balls in complex spaces of different dimensions. He showed that a proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^{n+1}$ with $n > 2$, which is three times differentiable up to the boundary, is a totally geodesic embedding. Subsequently, Cima-Suffridge [CS1] reduced the boundary regularity in Webster's theorem to the C^2-regularity. Motivated by a conjecture posed in [CS1], Faran in [Fa1] showed that any proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^N$ with $N < 2n-1$, that is analytic up to the boundary, is also a totally geodesic embedding. Forstneric in [Fo1] [Fo2] proved that any proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^N$ is rational, if the map is C^{N-n+1}-regular up to the boundary, which, in particular, reduces the regularity assumption in Faran's linearity theorem to the C^{N-n+1}-smoothness.

The structure of the maps gets more complicated when $N \geq 2n-1$. (See the book [Da1], [BER], and the survey article [Hu2] for more explanations). Recall that two proper holomorphic maps f, g from $\mathbf{B}^n$ into $\mathbf{B}^N$ are called equivalent if there are $\sigma \in \mathrm{Aut}(\mathbf{B}^n)$ and $\tau \in \mathrm{Aut}(\mathbf{B}^N)$ such that $g = \tau \circ f \circ \sigma$. It is easy to verify that a map is linear if and only if it is equivalent to the standard big circle embedding $L(z) : z \to (z, 0)$. In [Fa2], it was shown that there are four different inequivalent maps from $\mathbf{B}^2$ into $\mathbf{B}^3$, which are C^3-smooth up to the boundary. By the work of D'Angelo [Da2], any two proper holomorphic maps from $\mathbf{B}^n$ into $\mathbf{B}^N$ are homotopically equivalent through a family of inequivalent maps in a suitably larger space, which, in particular, can be used to show that there is a continuous family of inequivalent proper holomorphic quadratic polynomial embeddings from $\mathbf{B}^n$ into $\mathbf{B}^{2n}$. See also [DL] for discussions on the classification of proper monomial maps between balls with certain symmetry. At this point, we mention that the discovery of inner functions can be used to show that there is a proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^{n+1}$, which can not be C^2-smooth at any boundary point. (See [HS], [Low], [For2], [Dor], etc).

In [Hu1] and, subsequently, a joint paper with Ji [HJ1], we considered two natural questions arising from the above mentioned work. In [Hu1], we proved that any proper holomorphic map which is only C^2-regular up to the boundary must be linear if $N < 2n-1$, by applying a different method from the above mentioned work. It is not clear to us if this C^2-regularity is optimal or not for the super-rigidity to hold, the result in [Hu1] gives a first result in which the required regularity is independent of the codimension. Moreover, some of the basic approaches developed in [Hu1] seem to be quite useful for the study of many other related problems (see [HJ1], [EHZ1] [EHZ2]), and will also be used in the present paper. In a joint paper with Ji [HJ1], it was shown that any proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^N$ with $N = 2n-1, n \geq 3$, which is C^2-smooth up to the boundary, is either linear or equivalent to the Whitney map $W : z = (z_1, \cdots, z_n) \to (z_1, \cdots, z_{n-1}, z_n z)$ ([Theorem 1, Theorem 2.3; HJ]). Since the Whitney map is not an immersion,

together with the aforementioned work of Faran [Fo2], this shows that any proper holomorphic embedding from $\mathbf{B}^n$ into $\mathbf{B}^N$ with $N = 2n-1$, which is twice continuously differentiable up to the boundary, must be a linear map.

The present paper continues the work in [Hu1] and [HJ1]. Our main result, Theorem 1.1, provides a description of the partial linearity for proper holomorphic maps between balls for $N < n + P(n, n-1) = \frac{n(n+1)}{2}$. When $N \geq n + P(n, n-1)$, the partial linearity breaks down by Remark 1.4. However, there are still many problems left to be understood. We refer the reader to the book of D'Angelo [Da3] for discussions on this matter and we wish at least to mention here a result due to Catlin and D'Angelo [CD] and D'Angelo [Da3], which states that for any polynomial functions $q(z), p_1(z)$ with $|p_1(z)| < |q(z)|$ on the closure of $\mathbf{B}^n$, there exists a vector valued polynomial $p(z)$ with $N(q, p_1)$- components such that $\frac{(p_1(z), p(z))}{q(z)}$ properly holomorphically maps $\mathbf{B}^n$ into $\mathbf{B}^{N(q,p_1)}$, where $N(q, p_1)$ depends on (q, p_1) and $N(q, p_1) \to \infty$ as $\max_{z \in \overline{\mathbf{B}}^n} |p_1/q(z)| \to 1$.

Finally, we mention that in the past years, there has been much work done on various related rigidity problems for holomorphic maps between complex hyperbolic space forms, bounded symmetric domains, etc. To name a few, we refer the reader to the work [We2], [CaMo], [Mok], [MSY], [Tu], [EHZ1], [EHZ2] and the references therein.

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§2. Preliminaries, a geometric invariant and partial linearity

Our proof of Theorem 1.1 is based on the approach developed in [Hu1] and [HJ1]. In this section, we start by recalling some notation, definition and various formulas established in [Hu1] and [HJ1], which will be used throughout the paper. Then we introduce a geometric invariant for the map and discuss how it is related to the partial linearity.

§**2.1**. In this subsection, we recall some notation and formulas established in [Hu1] [HJ1]. Then we introduce the concept of the geometric rank, which is the invariant closely related to the dimension of the affine complex subspaces along which the map is linear.

First, since the ball $\mathbf{B}^n \subset \mathbf{C}^n$ is equivalent to the Siegel upper-half space, denoted by $\mathbf{S}^n := \{(z,w) \in \mathbf{C}^{n-1} \times \mathbf{C} : \ \mathrm{Im}(w) > |z|^2\}$, and the punctured sphere is equivalent to the Heisenberg hypersurface, we will mainly focus on mappings between Heisenberg hypersurfaces. In the last section of the paper, we will see how this will immediately give results for mappings between balls by applying the Cayley-type transformations.

Let $M_1 \subset \mathbf{H}_n$ and $M_2 \subset \mathbf{H}_N$ be two connected open pieces of the standard Heisenberg hypersurfaces in $\mathbf{C}^n$ and $\mathbf{C}^N$, respectively. Here

$$\mathbf{H}_n := \{(z,w) \in \mathbf{C}^n, \ \mathrm{Im}w = \sum_{j=1}^{n-1} |z_j|^2\}; \ \ \mathbf{H}_N := \{(z^*,w^*) \in \mathbf{C}^N, \ \mathrm{Im}w^* = \sum_{j=1}^{N-1} |z_j^*|^2\}. \tag{2.1}$$

Write $L_j = 2i\overline{z_j}\frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}$ for $j = 1, \cdots, n-1$ and $T = \frac{\partial}{\partial u}$ with $w = u + iv$. Then $\{L_1, \cdots, L_{n-1}\}$ forms a global basis for the complex tangent bundle $\mathrm{T}^{(1,0)}\mathbf{H}_n$ of $\mathbf{H}_n$, and T is a tangent vector field of $\mathbf{H}_n$ transversal to $T^{(1,0)}M \cup T^{(0,1)}M$. Parameterize $\mathbf{H}_n$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \to (z, u + i|z|^2)$. In what follows, we will assign the weight of z and u to be 1 and 2, respectively. For a non negative integer m, a function $h(z, \overline{z}, u)$ defined over a small ball U of 0 in $\mathbf{H}_n$ is said to be of quantity $o_{wt}(m)$, if $\frac{h(tz, t\overline{z}, t^2u)}{|t|^m} \to 0$ uniformly for (z, u) on any compact subset of U as $t(\in \mathbf{R}) \to 0$. (In this case, we write $h = o_{wt}(m)$. By convention, we write $h = o_{wt}(0)$ if $h \to 0$ as $(z, \overline{z}, u) \to 0$). Also, we write $\chi(z, \overline{z}, u) \in \mathcal{P} + o_{wt}(m)$ if $\chi = h_1 + h_2$ with h_1 a polynomial in $(z, \overline{z}, u)$ and $h_2 = o_{wt}(m)$. For a function $h(z, \overline{z}, u)$ defined over U, we use $h^{(k)}(z, \overline{z}, u)$ for the sum of terms of weighted degree k in the weighted expansion of h up to order k. If h is not specified, we use it to denote a weighted homogeneous polynomial of weighted degree k. For a weighted homogeneous holomorphic polynomial of degree k, we use the notation: $(\cdot)^{(k)}(z, w)$, or $(\cdot)^{(k)}(z)$ if it depends only on z.

For two m-tuples $x = (x_1, \cdots, x_m), y = (y_1, \cdots, y_m)$, we write $(x, y) = x \cdot y^t = \sum_{j=1}^m x_j y_j$, and $|x|^2 = \sum_{j=1}^m |x_j|^2$.

In all that follows, we assume that $N \geq n > 1$. Also, $S \subset \mathbf{C}^n$ is called an affine complex subspace of dimension k, if $S - p_0$ with $p_0 \in S$ is a k- dimensional complex linear subspace of $\mathbf{C}^n$.

Let

$$F = (f, \phi, g) = (f_1, \cdots, f_{n-1}, \phi_1, \cdots, \phi_{N-n}, g)$$

be a non-constant C^2-smooth CR map from M_1 into M_2. Then for each $p \in M_1$, we have an associated CR map F_p from a small neighborhood of $0 \in \mathbf{H}_n$ to $\mathbf{H}_N$ with $F_p(0) = 0$, defined by

$$F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p), \tag{2.1.0}$$

where for each $p = (z_0, w_0) \in M_1$, we write $\sigma^0_{(z_0,w_0)} \in \mathrm{Aut}(\mathbf{H}_n)$ for the map sending (z, w) to $(z + z_0, w + w_0 + 2i < z, \overline{z_0} >)$ and we define $\tau^F_{(z_0,w_0)} \in \mathrm{Aut}(\mathbf{H}_N)$ by $\tau^F_{(z_0,w_0)}(z^*, w^*) = (z^* - \widetilde{f}(z_0, w_0), w^* - \overline{g(z_0, w_0)} - 2i < z^*, \overline{\widetilde{f}(z_0, w_0)} >)$, where $\widetilde{f} = (f, \phi)$.

Let

$$\lambda(p) = g'_w(p) - 2i < \widetilde{f}'_w(p), \overline{\widetilde{f}(p)} >= |L_j(\widetilde{f})|^2. \tag{2.1.1}$$

Then under the assumption that F is not a constant map, one has $\lambda(p) > 0$. (See [Hu1]).

Let

$$\begin{aligned} E_l(p) &= (\frac{\partial \widetilde{f}_p}{\partial z_l})|_0 = (\frac{\partial f_{p,1}}{\partial z_l}, \cdots \frac{\partial f_{p,n-1}}{\partial z_l}, \frac{\partial \phi_{p,1}}{\partial z_l}, \cdots \frac{\partial \phi_{p,N-n}}{\partial z_l})|_0 = L_l(\widetilde{f})(p), \\ E_w(p) &= (\frac{\partial \widetilde{f}_p}{\partial w})|_0 = (\frac{\partial f_{p,1}}{\partial w}, \cdots \frac{\partial f_{p,n-1}}{\partial w}, \frac{\partial \phi_{p,1}}{\partial w}, \cdots \frac{\partial \phi_{p,N-n}}{\partial w})|_0 = \widetilde{f'_w}(p) = T(f)(p). \end{aligned} \tag{2.1.2}$$

Then the rank of $\{E_1(p), \cdots, E_{n-1}(p)\}$ is $(n-1)$. (See [Hu1]). Let $C_l(p)$ be so chosen that (see [pp 17, Hu1])

$$A(p) = A_{(z,w)} := \begin{pmatrix} E_1(p)/\sqrt{\lambda(p)} \\ \cdot \\ \cdot \\ \cdot \\ E_{n-1}(p)/\sqrt{\lambda(p)} \\ C_1(p) \\ \cdots \\ C_{N-n}(p) \end{pmatrix} \text{ is a unitary matrix.} \tag{2.1.3}$$

Define $F^*_p = (\widetilde{f_p}^*, g^*_p) = ((f_p)^*_1, \cdots, (f_p)^*_{n-1}, (\phi_p)^*_1, \cdots, (\phi_p)^*_{N-n}, g^*_p)$ by

$$\frac{1}{\sqrt{\lambda(p)}} F_p \cdot \begin{pmatrix} \overline{A^t(p)} & 0 \\ 0 & 1/\sqrt{\lambda(p)} \end{pmatrix}. \tag{2.1.4}$$

Then F^*_p is still a C^2 CR map from a neighborhood of the origin of $\mathbf{H}_n$ into $\mathbf{H}_N$ with $F^*_p(0) = 0$ and

$$f^*_p = z + O(|w| + |(z, w)|^2),\ \phi^*_p = O(|w| + |(z, w)|^2),\ g^*_p = w + O(|(z, w)|^2).$$

As in [Hu1], [HJ1], we further modify F^*_p as follows:

Let

$$\begin{aligned} \mathbf{a}(p) &:= \frac{\partial \widetilde{f_p}^*}{\partial w}(0) = (a_1(p), \cdots, a_{n-1}(p), b_1(p), \cdots, b_{N-n}(p)) \text{ with} \\ a_l(p) &= \frac{1}{\lambda(p)} E_w(p) \cdot \overline{E_l(p)}^t \text{ for } l \le n-1. \end{aligned} \tag{2.1.5}$$

It can be easily seen that

$$|\mathbf{a}(p)|^2 = \frac{1}{\lambda(p)}|E_w(p)|^2 \tag{2.1.5$'$}$$

(see [§2, Hu1])). Also let

$$\begin{aligned}
&d_{lj}(p) := \frac{\partial^2 (f_p)_l^*}{\partial z_j \partial w}|_0 = \frac{1}{\lambda(p)}(\widetilde{f}_p)''_{wz_j}(0)\cdot\overline{E_l(p)^t} = \frac{1}{\lambda(p)}L_j(\widetilde{f}'_w)(p)\cdot\overline{E_l(p)^t};\\
&c_l(p) := \frac{\partial^2 g_p^*}{\partial z_l \partial w}|_0 = \frac{1}{\lambda(p)}(g_p)''_{wz_l}(0) = \frac{1}{\lambda(p)}L_l(g'_w - 2i\widetilde{f}'_w\cdot\overline{\widetilde{f}(p)}^t)|_0 = \frac{2i}{\lambda(p)}\overline{T(\widetilde{f})}\cdot L_l(\widetilde{f})^t;\\
&r(p) := \frac{1}{2}\mathrm{Re}(\frac{\partial^2 g_p^*}{\partial w^2})|_0 = \frac{1}{2\lambda(p)}\mathrm{Re}((g_p)''_{ww}(0)) = \frac{1}{2\lambda(p)}\mathrm{Re}(g''_{ww} - 2i\widetilde{f}''_{ww}\cdot\overline{\widetilde{f}(p)}^t).
\end{aligned} \tag{2.1.6}$$

Define

(2.1.7)

$$G_p := \left(\frac{z^* - \mathbf{a}(p)w^*}{1 + 2i\langle z^*, \overline{\mathbf{a}(p)}\rangle + (r(p) - i\langle \mathbf{a}(p), \overline{\mathbf{a}(p)}\rangle)w^*}, \frac{w^*}{1 + 2i\langle z^*, \overline{\mathbf{a}(p)}\rangle + (r(p) - i\langle \mathbf{a}(p), \overline{\mathbf{a}(p)}\rangle)w^*}\right).$$

Then $G_p \in \mathrm{Aut}_0(\mathbf{H}_N)$. Finally, F_p^{**} is the composition of F_p^* with G_p:

$$F_p^{**} := (f_p^{**}, \phi_p^{**}, g_p^{**}) = (\widetilde{f}_p^{**}, g_p^{**}) := G_p \circ F_p^*. \tag{2.1.8}$$

As in the work of [Hu1] [HJ1], the following lemma is of fundamental importance for the understanding of the map F:

Lemma 2.0 ([§2, Lemma 5.3, Hu1]): Let F be a C^2-smooth CR map from a connected open subset $M \subset \mathbf{H}_n$ into $\mathbf{H}_N$. For each $p \in M$, F_p^{**}, defined as above, satisfies the normalization condition:

$$f_p^{**} = z + \frac{i}{2}a_p^{**(1)}(z)w + o_{wt}(3),\ \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2),\ g_p^{**} = w + o_{wt}(4), \tag{2.2.1}$$

with

$$\langle \overline{z}, a_p^{**(1)}(z)\rangle |z|^2 = |\phi_p^{**(2)}(z)|^2. \tag{2.2.2}$$

In passing, we notice that there is a $\tau_p^{**} \in \mathrm{Aut}_0(\mathbf{H}_N)$ such that

$$F_p^{**} = \tau_p^{**} \circ F_p. \tag{2.2.3}$$

Also, for a fixed $p_0 \in M$, we can make $C_l(p)$ in (2.1.3) depend smoothly on E_j and p for $p \approx p_0$.

From (2.2.2), it is easy to see that

(2.2.4) $a_p^{**(1)}(z) = z\mathcal{A}(p)$ with $\mathcal{A}(p)$ a certain $(n-1)\times(n-1)$ semi-positive Hermitian matrix.

Moreover, it is independent of the choice of $C_j(p)$ by the following formula (see [pp224, (2.3), HJ]):

$$P_j^l := \frac{\partial^2 (f_p)_l^{**}}{\partial z_j \partial w}|_0 = d_{lj}(p) - a_l(p)c_j(p) - \delta_j^l(i|\mathbf{a}(p)|^2 + r(p)),\ 1 \le l, j \le n-1. \tag{2.3.1}$$

Using the aforementioned formulas, one gets (see [pp231, (4.1), HJ]):

$$\begin{aligned} 2\lambda P_l^j &= 2L_l(\widetilde{f_w}') \cdot \overline{L_j(\widetilde{f})}^t - \frac{2}{\lambda}\left(\widetilde{f'_w} \cdot \overline{L_j(\widetilde{f})}^t\right) \cdot L_l\left(g'_w - 2i\widetilde{f'_w} \cdot \overline{\widetilde{f}}^t\right) - \\ &- 2i\delta_l^j|\widetilde{f}'_w|^2 - \delta_l^j \mathrm{Re}\{g''_{ww}\} + 2\delta_l^j \mathrm{Re}\{i\widetilde{f''_{ww}} \cdot \overline{\widetilde{f}}^t\} \end{aligned} \tag{2.3.2}$$

and

$$q_l^j := 2\lambda P_l^j - \delta_l^j(2iT^2(\widetilde{f}) \cdot \overline{\widetilde{f}}^t - T^2(g)) = 2L_l(\widetilde{f_w}') \cdot \overline{L_j(\widetilde{f})}^t - \frac{4i}{\lambda(p)}\left(\widetilde{f'_w} \cdot \overline{L_j(\widetilde{f})}^t\right)\left(L_l(\widetilde{f}) \cdot \overline{\widetilde{f'_w}}^t\right) \tag{2.3.3}$$

Another way to see that $\mathcal{A}(p)$ is independent of the choice of C_j is as follows:

In passing, we mention that there is a typo in the expressions of [(4.1)-(4.3), HJ]: The indices $\{l, j\}$ in the second term after the equality sign should be switched to $\{j, l\}$.

First, we notice that

$$(\phi_p^{**})_j = \frac{(\phi_p^*)_j - b_j(p)g_p^*}{1 + 2i < \widetilde{f_p^*}, \overline{\mathbf{a}(p)} > -(-r(p) + i < \mathbf{a}(p), \overline{\mathbf{a}(p)} >)g_p^*}. \tag{2.3.4}$$

As in [Hu1], write $\phi_p^{**(2)} = \sum_{k,l=1}^{n-1} q_{kl} z_k z_l$ with $q_{kl} = q_{lk} = (q_{kl}^1, \cdots, q_{kl}^{(N-n)})$. Comparing the coefficients of terms with factor $z_l z_k$ in the Taylor expansion of (2.3.4) (or see [(2.6), Hu1]), it follows easily that

$$\begin{aligned} q_{kl}^j(p) &= \frac{1}{2}\frac{\partial^2 (\phi_p^*)_j}{\partial z_l \partial z_k}(0) = \frac{1}{2}\frac{\partial^2 (\phi_p^{**})_j}{\partial z_l \partial z_k}(0) \\ &= \frac{1}{2\sqrt{\lambda(p)}}(\widetilde{f}_p)''_{z_l z_k}(0) \cdot \overline{C_j(p)}^t = \frac{1}{2\sqrt{\lambda(p)}} L_l L_k(\widetilde{f})(p) \cdot \overline{C_j(p)}^t \end{aligned} \tag{2.3.5}$$

Hence, by (2.3.5) and (2.2.1) it follows that

(2.3.6)

$$z\mathcal{A}(p)\overline{z}^t|z|^2 = |\phi_p^{**(2)}(z)|^2 = \sum_{k,l,k',l'=1}^{n-1} < q_{kl}, \overline{q_{k'l'}} > z_k z_l \overline{z_{k'} z_{l'}} = \sum_{k,l,k',l'=1}^{n-1} K_{kl\overline{k'l'}}(p) z_k z_l \overline{z_{k'} z_{l'}}$$

$$\text{with } K_{kl\overline{k'l'}}(p) = \frac{1}{4\lambda(p)} < L_l L_k(\widetilde{f})(p), \overline{L_{l'} L_{k'}(\widetilde{f})(p)} > -$$

$$- \frac{1}{4\lambda^2(p)} \sum_{j=1}^{n-1} \left(L_l L_k(\widetilde{f})(p) \cdot \overline{E_j(p)}^t \right) \left(\overline{L_{l'} L_{k'}(\widetilde{f})(p)} \cdot E_j(p)^t \right).$$

In particular, we conclude that $|\phi_p^{**}(z)|^2$ and thus $\mathcal{A}(p)$ is independent of the choice of C_j. With this property at our disposal, we make the following definition:

Definition 2.1: The rank of $\mathcal{A}(p) = -2i(P_j^l)_{1\leq j,l\leq(n-1)}$, which we denote by $Rk_F(p)$, is called the geometric rank of F at p.

Notice that $\mathcal{A}(p)$ depends continuously on p, and $Rk_F(p)$ is a lower semi-continuous function on p.

§**2.2**: In this subsection, we present an elementary but useful lemma based on the calculations in §2.1. Let $\sigma \in \mathrm{Aut}_0(\mathbf{H}_n)$ be given by

$$\sigma(z,w) := \frac{\left(\lambda(z+aw)\cdot U, \lambda^2 w\right)}{q(z,w)} \tag{2.4.1}$$

with $q(z,w) = 1 - 2i < \overline{a}, z > +(r - i|a|^2)w$, where $\lambda > 0$, $r \in \mathbf{R}$, a is an $(n-1)$-tuple and U is an $(n-1)\times(n-1)$ unitary matrix. Let $\tau^* \in \mathrm{Aut}_0(\mathbf{H}_N)$ be given by

$$\tau^*(z^*,w^*) := \frac{\left(\lambda^*(z^*+a^*w^*)\cdot U^*, \lambda^{*2} w^*\right)}{q^*(z^*,w^*)} \tag{2.4.2}$$

with $q^*(z^*,w^*) = 1-2i < \overline{a^*}, z^* > +(r^* - i|a^*|^2)w^*$, where $\lambda^* > 0$, $r^* \in \mathbf{R}$, a^* is an $(N-1)$-tuple and U^* is an $(N-1)\times(N-1)$ unitary matrix.

Lemma 2.2: (A). Suppose that $F = (f,\phi,g)$ and $F^* = (f^*,\phi^*,g^*)$ are two C^2-smooth CR map from a neighborhood of 0 in $\mathbf{H}_n$ into $\mathbf{H}_N$ $(N \geq n > 1)$, that satisfy the normalization condition (2.2.1). Assume that $F^* = \tau^* \circ F \circ \sigma$ with σ, τ^* given in (2.4.1) and (2.4.2), respectively. Then it holds that

$$\lambda^* = \lambda^{-1},\ a_1^* = -\lambda^{-1} a\cdot U,\ a_2^* = 0, r^* = -\lambda^{-2} r, \tag{2.5.1}$$

$$U^* = \begin{pmatrix} U^{-1} & 0 \\ 0 & U^*_{22} \end{pmatrix}, \tag{2.5.2}$$

where $a^* = (a^*_1, a^*_2)$ with a^*_1 its first $(n-1)$ components, U^*_{22} is an $(N-n)\times(N-n)$ unitary matrix. Conversely, suppose τ^* and σ, given in (2.4.1)-(2.4.2), are related by (2.5.1)-(2.5.2). And suppose that F satisfies the normalization condition (2.2.1). Then F^* must also satisfy the normalization in (2.2.1).

(B). Let F_1 be a non-constant C^2-CR map from $M \subset \mathbf{H}_n$ into $\mathbf{H}_N$. Assume that $F_2 = \tau \circ F_1 \circ \sigma$ with $\sigma \in \mathrm{Aut}(\mathbf{H}_n)$ and $\tau \in \mathrm{Aut}(\mathbf{H}_N)$. Then $Rk_{F_2}(p) = Rk_{F_1}(\sigma(p))$.

(C). Suppose that F satisfies the normalization condition as in (2.2.1) and assume that $Rk_F(0) = \kappa_0 > 0$. Then there are $\sigma \in \mathrm{Aut}_0(\mathbf{H}_n)$ and $\tau \in \mathrm{Aut}_0(\mathbf{H}_N)$ such that $\tau \circ F \circ \sigma := (f, \phi, g)$ satisfies the following normalization condition:

$$\begin{aligned} & f_j = z_j + \frac{i\mu_j}{2} z_j w + o_{wt}(3), \text{ for } \ j \le \kappa_0, \\ & f_j = z_j + o_{wt}(3), \text{ for } \ j > \kappa_0 \\ & g = w + o_{wt}(4), \\ & \phi = \phi^{(2)}(z) + o_{wt}(2), \\ & \frac{\partial^2 f_j}{\partial w^2}(0) = 0, \text{ for } j \le \kappa_0 \end{aligned} \tag{2.5.3}$$

where $\mu_1 = 1$ and $\mu_j \ge 1$ for $j \le \kappa_0$.

(D). Let F be a C^2-CR map from a small neighborhood of 0 in $\mathbf{H}_n$ into $\mathbf{H}_N$ satisfying the normalization in (2.5.3) with $\kappa_0 > 0$. Then for any complex line L transversal to $\mathbf{H}_n$ at 0, if $F|_L$ is linear fractional, then L must be contained in $S_0 := \{(z, w) \in \mathbf{C}^n : z_1 = \cdots = z_{\kappa_0} = 0\}$.

Proof of Lemma 2.2: Let F and F^* be as in Lemma 2.2 (A). Write $F^* = (\widetilde{f^*}, g^*) = (f^*, \phi^*, g^*)$. Then a direct computation shows that $g^* = \lambda^2\lambda^{*2} w + O(|(z, w)|^2)$ and $\widetilde{f^*} = \lambda^*(\lambda(z + aw) \cdot U, 0) \cdot U^* + \lambda^* a^* \lambda^2 U^* w + O(|(z, w)|^2)$. Hence, we get $\lambda^* = \lambda^{-1}$, $(zU, 0)U^* = (z, 0)$, and $(aU, 0)U^* + \lambda a^* U^* = 0$. Write

$$U^* = \begin{pmatrix} U^*_{11} & U^*_{12} \\ U^*_{21} & U^*_{22} \end{pmatrix}.$$

Then it follows that $UU^*_{11} = \mathrm{Id}$, or $U^*_{11} = U^{-1}$. Since U^* is unitary, we see that $U^*_{12} = 0, U^*_{21} = 0$ and U^*_{22} is also unitary. Hence, we have $a + \lambda a^*_1 U^*_{11} = 0$ and $a^*_2 U_{22} = 0$. This gives that $a^*_2 = 0$

and $a_1^* = -\lambda^{-1}aU$. To get the relation between r^* and r, we notice that

$$\begin{aligned} g^*(0,w) &= \frac{\lambda^{*2} g\circ\sigma(0,w)}{q^*(\widetilde{f}\circ\sigma(0,w), g\circ\sigma(0,w))} \\ &= \frac{w}{(1+(r-i|a|^2)w)(1-2i<\overline{a_1^*}, f\circ\sigma(0,w)>+(r^*-i|a^*|^2)g\circ\sigma(0,w))} + o(w^2) \\ &= w(1-(r-i|a|^2)w)(1-2i|a|^2w-(r^*\lambda^2-i|a|^2)w) = w(1-(r+r^*\lambda^2)w)+o(w^2). \end{aligned}$$

Therefore, we have $r^* = -\lambda^{-2}r$.

Write $f = z + \frac{i}{2}z\mathcal{A}w + o_{wt}(3)$ and $f^* = z + \frac{i}{2}z\mathcal{A}^*w + o_{wt}(3)$ as before. Notice that

$$f^* = \frac{\left(\lambda^* f\left(\lambda(z+aw)\cdot U/q(z,w), \lambda^2 w/q\right) + \lambda^* a_1^* g\left(\lambda(z+aw)\cdot U/q(z,w), \lambda^2 w/q\right)\right)U^{-1}}{q^*(f\circ\sigma(z,w), g\circ\sigma(z,w))}.$$

A similar calculation then shows that

$$\text{(2.5.4)}\quad \mathcal{A}^* = \lambda^2 U\mathcal{A}U^{-1} \text{ and thus } Rk_{F^*}(0) = Rk_F(0);\ \frac{\partial^2 f^*}{\partial w^2}(0) = i\lambda^2 aU\mathcal{A}U^{-1} + \lambda^3\frac{\partial^2 f}{\partial w^2}(0)U^{-1}.$$

Conversely, suppose the formulas in (2.5.1)-(2.5.2) hold, and suppose that F satisfies the normalization (2.2.1). Then $F^* = (f^*, \phi^*, g^*)$ satisfies the normalization:

$$f^*(z,w) = z + O(|(z,w)|^2),\ \phi^* = \phi^{(2)}(z) + o_{wt}(2),\ \text{and } g^*(z,w) = w + O(|(z,w)|^2),\ T^2(g^*)(0) = 0.$$

By [Lemma 5.3, Hu1], we conclude that F^* also satisfies the normalization (2.2.1).

To prove the statement in (B), we just notice that in the context of (B), there are $\sigma_0 \in \mathrm{Aut}_0(\mathbf{H}_n)$ and $\tau_0 \in \mathrm{Aut}_0(\mathbf{H}_N)$ such that $F^{**}_{2,p} = \tau_0 \circ F^{**}_{1,\sigma(p)} \circ \sigma_0$. Hence, by Lemma 2.2 (A) (2.5.4), we conclude the statement in (B).

Next, let F be as in (C). By (2.5.4), we can apparently make $\mathcal{A} = \mathrm{diag}(\mu_1, \cdots, \mu_{\kappa_0}, 0, \cdots, 0)$ with $\mu_j \geq \mu_1 = 1$, by choosing λ, U suitably. To get the normalization in the last line of (2.5.3), by (2.5.4) we need only to replace F by $\tau\circ F\circ\sigma$, where σ and τ are as in (2.5.1)-(2.5.2), respectively, with $U = \mathrm{Id}$, $\lambda = 1$, $r = 0$, $U^*_{22} = \mathrm{Id}$, and

$$a = \left(\frac{2\sqrt{-1}a_{02,1}}{\mu_1}, \cdots, \frac{2\sqrt{-1}a_{02,\kappa_0}}{\mu_{\kappa_0}}, 0, \cdots, 0\right) \text{ where } a_{02,j} = \frac{1}{2}\frac{\partial^2 f_j}{\partial w^2}(0).$$

This proves (C).

Now, we turn to the proof of (D). Let L be a line defined by $z = ct = (c_1, \cdots, c_{n-1})t, w = t$ and assume that $F|_L$ is linear fractional. Let

$$\sigma = \frac{(z+cw, w)}{1-2i<z,\overline{c}>-i|c|^2w},\quad \tau = \frac{(z^*-(c,0)w^*, w^*)}{1+2i<z^*,\overline{(c,0)}>-i|c|^2w^*},$$

and $F^* = \tau \circ F \circ \sigma$. Then F^* must map $\{(0,w) : \mathrm{Im}(w) \geq 0\}$ into itself. However $f_j^*(0,t) = \frac{i}{2}\mu_j c_j t^2 + o(t^2)$. We see that $c_j = 0$ for $j \leq \kappa_0$. The proof of Lemma 2.2 is now complete. ∎

Let F be a C^2-smooth CR map from $M \subset \mathbf{H}_n$ into $\mathbf{H}_N$ as before. We say that F is k-linear at $p \in M$ if there are $\sigma \in \mathrm{Aut}_0(\mathbf{H}_n)$ and $\tau \in \mathrm{Aut}_0(\mathbf{H}_N)$ such that $\tau \circ F_p \circ \sigma(z_1, \cdots, z_{k-1}, 0, \cdots, 0, w) = (z_1, \cdots, z_{k-1}, 0, \cdots, 0, w)$. From Lemma 2.2 (C) (D), it follows easily that if $Rk_F(p) \geq \kappa_0$, then F cannot be $(n-\kappa_0+1)$-linear at p. The following theorem, which is regarded as the main technical Theorem of the paper, says that under a certain assumption, F is always $(n-\kappa_0)$-linear at p:

Theorem 2.3: *Let F be a non-constant C^3-CR map from a connected open subset M of $\mathbf{H}_n$ into $\mathbf{H}_N$. Assume that $Rk_F(p) \equiv \kappa_0 < n-1$ is constant for each $p \in M$. Then F is $(n-\kappa_0)$-linear at any point in M. Moreover for each $p \in M$, there is a unique affine complex subspace S_p of dimension $(n-\kappa_0)$ transversal to $\mathbf{H}_n$ such that the restriction of F to S_p is linear. Also, $S_p - p$, as an element in the complex Grassmannian manifold $G_{n,n-\kappa_0}(\mathbf{C})$ of $(n-\kappa_0)$-dimensional linear subspaces in $\mathbf{C}^n$, depends C^1-continuously on p.*

§3. Analysis of the associated differential equations

Let F be a CR map from a small neighborhood M of 0 in $\mathbf{H}_n$ into $\mathbf{H}_N$ $(N \geq n > 1)$ with $F(0) = 0$. We will prove in this section and §4 that it must be $(n-\kappa_0)$-linear at each point in M if it is C^3-smooth and $Rk_F \equiv \kappa_0 < n-1$. By [Theorem 4.2, Hu1], we can assume, in what follows, that $\kappa_0 > 0$. By the Lewy extension theorem ([BER]), F extends holomorphically to a subdomain Ω in $\mathbf{S}_n$ which has M as part of its smooth boundary. In this section, F is only assumed to be C^2-smooth over M.

§**3.1**: In this subsection, we derive a system of differential equations by (2.2.2), through which we take control of F. Then we derive some consequences of the system for the purpose of the later application.

For each $p \in M$, write $\mathcal{E}_p := \{\xi(p) = (\xi_1(p), \cdots, \xi_{n-1}(p)) \in \mathbf{C}^{n-1} : \quad \xi(p) \cdot \mathcal{A}_p \cdot \overline{\xi(p)}^t = 0\}$. Since $\mathcal{A}_p$ is semi-positive, $\mathcal{E}_p = \{\xi(p) : \quad \xi(p) \cdot \mathcal{A}_p = 0\}$. By (2.2.2), it follows that $\xi(p) \in \mathcal{E}_p$ if and only if $\phi_p^{**(2)}(\xi(p)) = 0$. We start with the following generalization of [Lemma 2.1, Lemma 4.3] of [Hu1]:

Lemma 3.1: $\xi(p) = (\xi_1(p), \cdots, \xi_{n-1}(p)) \in \mathcal{E}_p$ if and only if

$$\begin{aligned} &\sum_{j,l=1}^{n-1} \xi_j(p)\xi_l(p) L_j L_l(\widetilde{f})(p) = \sum_{l=1}^{n-1} \Lambda_l(p) L_l(\widetilde{f})(p) \text{ with} \\ &\Lambda_l = \sum_{j=1}^{n-1} \frac{4\sqrt{-1}}{\lambda(p)} \xi_j(p)\xi_l(p) \overline{\widetilde{f'_w}(p)} \cdot L_j(\widetilde{f})^t(p). \end{aligned} \tag{3.1.0}$$

Proof of Lemma 3.1: The proof can be done in an essentially identical way as for the proof of [Lemma 2.1, Hu1], with a slight modification due to the fact that we now only know that $\phi_p^{**(2)}(\xi(p)) = 0$. For convenience of the reader, we include here the following details:

Comparing the coefficients of terms of the form $z_l z_k$ in the Taylor expansion of (2.3.4) and using the given hypothesis, it follows easily that $\sum_{k,l=1}^{n-1} \xi_l(p)\xi_k(p) q_{kl}^j(p) = 0$ for any j. By (2.3.5), one gets that the $(N-1)$-tuple $\sum_{k,l} \xi_k \xi_l (\widetilde{f}_p)''_{z_l z_k}(0) = \sum_{k,l=1}^{n-1} \xi_k \xi_l L_l L_k(\widetilde{f})$ stays in the space spanned by $\{E_1, \cdots, E_{n-1}\}$. Namely, there exist scalar numbers $\{\lambda_j^*(p)\}$ such that

$$\sum_{k,l=1}^{n-1} \xi_k \xi_l L_l L_k(\widetilde{f})(p) = \sum_{j=1}^{n-1} \lambda_j^*(p) E_j(p). \tag{3.1.1}$$

Next, considering the Taylor expansion of $(f_p^{**})_j$ in the following

$$(f_p^{**})_j = \frac{(f_p^*)_j - a_j(p) g_p^*}{1 + 2i < \widetilde{f_p^*}, \overline{\mathbf{a}(p)} > -(-r(p) + i < \mathbf{a}(p), \overline{\mathbf{a}(p)} >) g_p^*} \tag{3.1.2}$$

and using (2.2.1), we obtain $\sum e_{kl}^j(p) z_k z_l - 2i \sum_l \overline{a_l}(p) z_j z_l \equiv 0$, where $e_{kl}^j(p) = \frac{1}{2} \frac{\partial^2 (f_p^*)_j}{\partial z_l \partial z_k}(0)$. Hence, it follows that $e_{kl}^j(p) = \sqrt{-1}(\delta_k^j \overline{a_l}(p) + \delta_l^j \overline{a_k}(p))$. Therefore,

$$\frac{1}{2\lambda} L_l L_k(\widetilde{f})(p) \overline{E_j}^t = \frac{\sqrt{-1}\delta_k^j}{\lambda(p)} \overline{E_w}(p) \cdot E_l(p)^t + \frac{\sqrt{-1}\delta_l^j}{\lambda(p)} \overline{E_w}(p) \cdot E_k(p)^t. \tag{3.1.3}$$

Combing this with (3.1.1) and making use of the orthogonality: $E_l(p) \cdot \overline{E_j(p)}^t = \lambda(p)\delta_l^j$, we get $\lambda_j^* = \frac{2\sqrt{-1}}{\lambda} \sum_{k,l=1}^{n-1} \xi_k(p)\xi_l(p)(\delta_k^j \overline{E_w(p)} \cdot E_l(p)^t + \delta_l^j \overline{E_w(p)} \cdot E_k(p)^t)$. Returning to (3.1.1), we get the proof of Lemma 3.1. ∎

Denote by $\mathcal{S}_0 = \{(j,l) : 1 \le j \le \kappa_0, 1 \le l \le (n-1), j \le l\}$ and write $\mathcal{S} := \{(j,l) : \ (j,l) \in \mathcal{S}_0, \text{ or } \ j = \kappa_0 + 1, l \in \{\kappa_0 + 1, \cdots, N - n - \frac{(2n-\kappa_0-1)\kappa_0}{2}\}\}$.

Lemma 3.2: Let F be a C^2-smooth CR map from $M \subset \mathbf{H}_n$ into $\mathbf{H}_N$ with $F(0) = 0$ and $Rk_F(0) = \kappa_0$. Let $P(n, \kappa_0) = \frac{\kappa_0(2n-\kappa_0-1)}{2}$. Then $N \geq n + P(n, \kappa_0)$ and there are $\sigma \in \mathrm{Aut}_0(\mathbf{H}_n)$ and $\tau \in \mathrm{Aut}_0(\mathbf{H}_N)$ such that $\tau \circ F \circ \sigma := (f, \phi, g)$ satisfies the following normalization condition: (3.2)

$$f_j = z_j + \frac{i\mu_j}{2} z_j w + o_{wt}(3), \quad \frac{\partial^2 f_j}{\partial w^2}(0) = 0, \ j = 1 \cdots, \kappa_0, \ \mu_j > 0,$$

$$f_j = z_j + o_{wt}(3), \quad j = \kappa_0 + 1, \cdots, n-1$$

$$g = w + o_{wt}(4),$$

$$\phi_{jl} = \mu_{jl} z_j z_l + o_{wt}(2), \text{ where } (j,l) \in \mathcal{S} \text{ with } \mu_{jl} > 0 \text{ for } (j,l) \in \mathcal{S}_0 \text{ and } \mu_{jl} = 0 \text{ otherwise.}$$

Moreover, $\mu_j \geq \mu_1 = 1$, $\mu_{jl} = \sqrt{\mu_j + \mu_l}$ for $j, l \leq \kappa_0$ $j \neq l$; and $\mu_{jl} = \sqrt{\mu_j}$ if $j \leq \kappa_0$ and $l > \kappa_0$ or if $j = l \leq \kappa_0$. Here and in what follows, we label the components of ϕ by double indices $(j,l) \in \mathcal{S}$.

Proof of Lemma 3.2: By Lemma 2.2, we can assume that F already takes the form (2.5.3). From the equation (2.2.2), we get $\sum_{j=1}^{\kappa_0} \mu_j |z_j|^2 |z|^2 = \sum_l |\phi_l^{(2)}(z)|^2$. Write $\phi_j^{(2)}(z) = \sum_{k \leq l} a_{kl}^j z_k z_l$. We then have

$$\sum_{j=1}^{\kappa_0} \mu_j |z_j|^2 |z|^2 = \sum_j a_{kl}^j a_{k'l'}^j z_k z_l \overline{z_{k'} z_{l'}}.$$

Write $\alpha_{jl} = (a_{jl}^1, \cdots, a_{jl}^{N-n})$. We have $\sum_{j=1}^{\kappa_0} \mu_j |z_j|^2 |z|^2 = \sum_{k \leq l, k' \leq l'} (\alpha_{kl}, \overline{\alpha_{k'l'}}) z_k z_l \overline{z_{k'} z_{l'}}$. We immediately see that $\alpha_{kl} \perp \alpha_{k'l'}$ for $(k,l) \neq (k',l')$, and $|\alpha_{kl}|^2 = \mu_k + \mu_l$ for $k, l \leq \kappa_0$, $k \neq l$; $|\alpha_{kl}|^2 = \mu_k$ for $k \leq \kappa_0, l > \kappa_0$ or $k = l \leq \kappa_0$. Hence, $\{\alpha_{kl}\}_{(k,l) \in \mathcal{S}_0}$ is a linearly independent system, which implies that $N - n \geq |\mathcal{S}_0| = P(n, \kappa_0)$. Next, extend $\{\frac{\alpha_{jl}}{|\alpha_{jl}|}\}_{(j,l) \in \mathcal{S}_0}$ to an $(N-n) \times (N-n)$ unitary matrix $\widetilde{U}$ and replace ϕ by $\phi \cdot \overline{\widetilde{U}}^t$. Then the rest statements in Lemma 3.2 can be easily seen. ∎

For the rest of this subsection, we fix an integer k with $\kappa_0 + 1 \leq k \leq (n-1)$, and let $I_k(p) = (\xi_1, \cdots, \xi_{n-1}) \in \mathcal{E}_p$ be such that its jth component is δ_j^k for $j > \kappa_0$. Apparently, under the assumption that $Rk_F(p) \equiv \kappa_0$ is constant, $I_k(p)$ depends continuously on p and is uniquely determined by the linear equations:

$$\sum_{j=1}^{\kappa_0} P_j^l \xi_j = -P_k^l, \text{ for } l = 1, \cdots, \kappa_0. \tag{3.3.1}$$

The following lemma will be basic for our later discussion:

Lemma 3.3: Assume that F is normalized as in (3.2), and assume that $Rk_F(p) \equiv \kappa_0$ for $p \approx 0$ with $0 < \kappa_0 < n-1$.

(A): For any $j, l, l' \in \{1, \cdots, n-1\}$, it holds that

$$L_j(I_k),\ \overline{L_j}(I_k),\quad L_j(\Lambda_l),\quad \overline{L_j L_l}(I_k),\ \overline{L_j L_l}(\Lambda_{l'}) \in C^0(M).$$

Moreover, write $I_k(p) = (\xi_1(p), \cdots, \xi_{\kappa_0}(p), 0, \cdots, 1, \cdots, 0)$ as above. Then for $j \le \kappa_0,\ k' > \kappa_0$,

$$\begin{aligned}
\xi_j &= \frac{2i}{\mu_j}\sum_{l=j+1}^{n-1}\frac{\partial^2\phi_{jl}}{\partial z_k\partial w}(0)\overline{\mu_{jl}z_l} + \frac{2i}{\mu_j}\sum_{j'<j}\frac{\partial^2\phi_{j'j}}{\partial z_k\partial w}(0)\overline{\mu_{j'j}z_{j'}} + \\
&+ \frac{4i}{\mu_j}\frac{\partial^2\phi_{jj}}{\partial z_k\partial w}(0)\overline{\mu_{jj}z_j} + \frac{2i}{\mu_j}L_kT(f_j) + o_{wt}(1), \qquad (3.3.2)\\
\overline{L_{k'}}(\xi_j) &= \frac{2i}{\mu_j}\frac{\partial^2\phi_{jk'}}{\partial z_k\partial w}(0)\overline{\mu_{jk'}} + o_{wt}(0).
\end{aligned}$$

(B): With the notation in (A), it holds that

$$\sum_{j,l=1}^{n-1}\overline{L_k}(\xi_j\xi_l)L_jL_l(\widetilde{f}) + 4i\sum_{l=1}^{n-1}\xi_lL_lT(\widetilde{f}) = \sum_{l=1}^{n-1}\overline{L_k}(\Lambda_l)L_l(\widetilde{f}) + 2i\Lambda_kT(\widetilde{f}); \qquad (3.4.1)$$

$$\sum_{j,l=1}^{n-1}\overline{L_k}^2(\xi_j\xi_l)L_jL_l(\widetilde{f}) + 8i\sum_{l=1}^{n-1}\overline{L_k}(\xi_l)L_lT(\widetilde{f}) - 8T^2(\widetilde{f}) = \sum_{l=1}^{n-1}\overline{L_k}^2(\Lambda_l)L_l\widetilde{f} + 4i\overline{L_k}(\Lambda_k)T(\widetilde{f}) \qquad (3.4.1)'$$

(C): For any $(j,l) \in \mathcal{S}$, it holds that $\frac{\partial^2\phi_{jl}}{\partial z_k\partial w}(0) = 0$ and $\frac{\partial^2 f}{\partial w^2}(0) = 0$. Also, for $(j,l) \in \mathcal{S}$ with $l \ne k$, it holds that $\frac{\partial^2\phi_{jl}}{\partial w^2}(0) = 0$. When $l = k$, $\frac{\partial^2\phi_{jk}}{\partial w^2}(0) = \frac{1}{4}\overline{L_k}^2(\xi_j)(0)$.
(D): It holds that

$$\sum_{j,l=1}^{n-1}\overline{L_k}^2(\xi_j\xi_l)L_jL_l(g) + 8i\sum_{l=1}^{n-1}\overline{L_k}(\xi_l)L_lT(g) - 8T^2(g) = \sum_{l=1}^{n-1}\overline{L_k}^2(\Lambda_l)L_lg + 4i\overline{L_k}(\Lambda_k)T(g). \qquad (3.4.2)$$

Proof of Lemma 3.3: (A): From (3.3.1), it follows that for $j \le \kappa_0$

$$(\xi_1, \cdots, \xi_{\kappa_0}) = -(P_k^1, \cdots, P_k^{\kappa_0})P^{-1} \text{ and thus } \xi_j = \frac{2i}{\mu_j}P_k^j + o(\sum_{l=1}^{\kappa_0}|P_k^l|). \qquad (3.4.3)$$

where $P = (P_l^j)_{1\le j,l\le\kappa_0}$ with $P(0) = \mathrm{diag}(\frac{i\mu_1}{2}, \cdots, \frac{i\mu_{\kappa_0}}{2})$. Hence, shrinking M if necessary, we see that for each j_0, there is a holomorphic function H_{j_0} in its argument such that $\xi_{j_0} = H_{j_0}(P_l^j)$. Without loss of generality, we assume that M is sufficiently small. We first prove the following:

Claim 3.4: Let G be a C^ν-smooth CR map from a neighborhood of M in $\mathbf{H}_n$ into $\mathbf{C}^N$. Let $h(G) = H(p, \overline{p}, L^\alpha \overline{L^\beta} T^\gamma(G))_{|\alpha|+|\beta|+|\gamma|\leq\nu}$ with H holomorphic in its argument. Let $D_0 = L^{\alpha_1}\overline{L^{\beta_1}}T^{\gamma_1}$ be a differential operator along M. Suppose that there is a certain holomorphic function H_0 in its argument such that for each polynomial map G^* from $\mathbf{C}^n$ into $\mathbf{C}^N$,

$$D_0(h(G^*)) = H_0(p, \overline{p}, L^\alpha \overline{L^\beta} T^\gamma(G^*))_{|\alpha|+|\beta|+|\gamma|\leq\nu}.$$

Then the distribution $D_0(h(G))$, acting on $C_0^\infty(M)$, coincides with the continuous function $H_0(p, \overline{p}, L^\alpha \overline{L^\beta} T^\gamma(G))_{|\alpha|+|\beta|+|\gamma|\leq\nu}$.

Proof of Claim 3.4: The claim is an immediate application of the Baouendi-Treves approximation theorem [BT]. (See already a version of this in [Lemma 4.1, Hu1]). In details, by [BT], there is a sequence of holomorphic polynomial maps, denoted by $\{G_m\}_{m=1}^\infty$, which converges to G in the C^ν-norm over $\overline{M}$. Hence $h(G_m) \to h(G)$ in the C^0-norm over $\overline{M}$, and $D_0(h(G_m)) \to D_0(h(G))$ in the sense of distribution. By the assumption, $D_0(h(G_m))$ converges also to $H_0(p, \overline{p}, L^\alpha \overline{L^\beta} T^\gamma(G))_{|\alpha|+|\beta|+|\gamma|\leq\nu}$ in the C^0-norm over $\overline{M}$. This completes the proof. ∎

We continue our proof of Lemma 3.3. Recall that

$$2\lambda P_l^j(F) = 2L_l(\widetilde{f}'_w)\overline{L_j(\widetilde{f})^t} - \frac{4i}{\lambda}((\widetilde{f}'_w)\cdot\overline{L_j(\widetilde{f})^t})(L_l(\widetilde{f})\cdot\overline{\widetilde{f}'_w}^t) - \delta_l^j(g''_{ww} - 2i\widetilde{f}''_{ww}\cdot\overline{\widetilde{f}}^t). \tag{3.4.4}$$

If we replace F by a polynomial map G in the above formula, a direct computation then shows that $\overline{L_{l_1}}(P_l^j(G))$ can be written as $G^*(p, \overline{p}, L^\alpha \overline{L^\beta} T^\gamma(G))_{|\alpha|+|\beta|+|\gamma|\leq 2}$ with G^* holomorphic in its argument. Hence, by Claim 3.4, $\overline{L_{l_1}}(\xi_j)$ is continuous. Since $\mathcal{A}$ is Hermitian, $P_j^l = -\overline{P_l^j}$, from which it follows easily that $L_{l_1}(\xi_j) \in C^0(M)$. To see $\overline{L_{l_1}L_{l_2}}(\xi_j) \in C^0(M)$, we notice that by considering the ϕ_{jk}-components of $\widetilde{f}$ in (3.1.0) with $(j,l) \in \mathcal{S}_0$, we have for each $j \leq \kappa_0$ that $\xi_j(F) = H_j^*(L_l\widetilde{f}, L_{l_1}L_{l_2}(\widetilde{f}), \lambda(p), \overline{T(\widetilde{f})}\cdot L_{l_3}(\widetilde{f})^t)$, where H_j^* is a certain holomorphic function in its variables. Notice that

$$\begin{aligned}
&\overline{T(\widetilde{f})}\cdot L_j(\widetilde{f})^t = \frac{1}{2i}L_jT(g) - (L_jT(\widetilde{f}))\cdot\overline{\widetilde{f}},\\
&\overline{L_{l_1}L_{l_2}}\left(\frac{1}{2i}L_jT(\widetilde{f}) - (L_jT(\widetilde{f}))\cdot\overline{\widetilde{f}}\right)\\
&= -2i\delta_j^{l_1}T^2(\widetilde{f})\cdot\overline{L_{l_2}(\widetilde{f})} - 2i\delta_j^{l_2}T^2(\widetilde{f})\cdot\overline{L_{l_1}(\widetilde{f})} - L_jT(\widetilde{f})\cdot\overline{L_{l_1}L_{l_2}(\widetilde{f})}
\end{aligned} \tag{3.4.5}$$

(if F is smooth). We easily conclude that if we replace F by a polynomial map G above, we have $\overline{L_{l_1}L_{l_2}}(\xi_j(G)) = H_j^{**}(p, \overline{p}, L^\alpha \overline{L^\beta} T^\gamma(G))_{|\alpha|+|\beta|+|\gamma|\leq 2}$. Hence, we see that $\overline{L_{l_1}L_{l_2}}(\xi_j)$ is continuous over M. Similarly, making use of (3.4.5) and the second formula in (3.1.0):

$$\Lambda_l = \sum \frac{4i}{\lambda}\xi_j(P_{l'}^{j'})\xi_l(P_{l'}^{j'})\overline{T(\widetilde{f})}\cdot L_j(\widetilde{f})^t,$$

we can show that $\overline{L_j}(\Lambda_l)$, $L_j(\Lambda_l)$ and $\overline{L_{j'}L_j}(\Lambda_l)$ are continuous over M. For the later application, we give here the following explicit formulas:

$$
\begin{aligned}
&\overline{L_k}(\Lambda_l) = II_1 + II_2 + II_3 + II_4, \quad \text{where} \\
&II_1 = \sum_{j=1}^{n-1} \frac{-4i}{\lambda^2}\overline{L_k}(\lambda)(\overline{\widetilde{f}'_w} \cdot L_j\widetilde{f}^t)\xi_j\xi_l, \quad II_2 = \sum_{j=1}^{n-1} \frac{4i}{\lambda}(\overline{L_k(\widetilde{f}'_w)} \cdot L_j\widetilde{f}^t)\xi_j\xi_l, \\
&II_3 = \frac{-8}{\lambda}(\overline{\widetilde{f}'_w} \cdot (\widetilde{f}'_w)^t)\xi_l, \quad II_4 = \sum_{j=1}^{n-1} \frac{4i}{\lambda}(\overline{\widetilde{f}'_w} \cdot L_j\widetilde{f}^t)\overline{L_k}(\xi_j\xi_l).
\end{aligned}
\tag{3.5.1}
$$

$$
\begin{aligned}
\overline{L_k}^2(\Lambda_l) = &\sum_{j=1}^{n-1} 4i\overline{L_k}^2(\frac{1}{\lambda(p)})K_j\xi_j\xi_l + \sum_{j=1}^{n-1} \overline{L_k}(\frac{8i}{\lambda(p)})\overline{L_k}(K_j)\xi_j\xi_l \\
&+ \sum_{j=1}^{n-1} \overline{L_k}(\frac{8i}{\lambda(p)})K_j\overline{L_k}(\xi_j\xi_l) + \sum_{j=1}^{n-1} \frac{4i}{\lambda}\overline{L_k^2}(K_j)\xi_j\xi_l \\
&+ \sum_{j=1}^{n-1} \frac{8i}{\lambda}\overline{L_k}(K_j)\overline{L_k}(\xi_j\xi_l) + \sum_{j=1}^{n-1} \frac{4i}{\lambda}K_j\overline{L_k^2}(\xi_j\xi_l), \quad \text{with} \\
&K_j = \overline{T(\widetilde{f})} \cdot L_j(\widetilde{f})^t, \quad \overline{L_k}(K_j) = \overline{L_kT(\widetilde{f})} \cdot L_j(\widetilde{f})^t + 2i\delta_j^k\overline{T(\widetilde{f})} \cdot T(\widetilde{f})^t, \\
&\overline{L_k}^2(K_j) = -4i\delta_j^k T^2(\widetilde{f}) \cdot \overline{L_k(\widetilde{f})}^t - L_jT(\widetilde{f}) \cdot \overline{L_k^2(\widetilde{f})}^t.
\end{aligned}
\tag{3.5.2}
$$

Next, we have for $j \le \kappa_0$ that

$$
\begin{aligned}
2\lambda P_k^j = 2L_k(\widetilde{f}'_w) \cdot \overline{L_j(\widetilde{f})^t} - \frac{4i}{\lambda}((\widetilde{f}'_w) \cdot \overline{L_j(\widetilde{f})^t}))(L_k(\widetilde{f}) \cdot \overline{\widetilde{f}'_w}^t) = 2\sum_{l=j+1}^{n-1} \frac{\partial^2\phi_{jl}}{\partial z_k\partial w}(0)\overline{\mu_{jl}z_l} + \\
+ 4\frac{\partial^2\phi_{jj}}{\partial z_k\partial w}(0)\overline{\mu_{jj}z_j} + 2\sum_{j'<j} \frac{\partial^2\phi_{j'j}}{\partial z_k\partial w}(0)\overline{\mu_{j'j}z_{j'}} + 2L_kT(f_j) + o_{wt}(1)
\end{aligned}
$$

With (3.4.3), this gives immediately (3.3.2).

To see (B), we just need to apply $\overline{L_k}$ and $\overline{L_k}^2$ to (3.1.0), respectively, with an application of Claim 3.4.

Notice that $\xi_j = o_{wt}(0)$ for $j \neq k$ and $\Lambda_l = o_{wt}(0)$. For a $(j,l) \notin \mathcal{S}_0$, considering the ϕ_{jl}-component of $\widetilde{f}$ in (3.4.1), and then collecting terms of weighted degree 0, we see that $\frac{\partial^2\phi_{jl}}{\partial w\partial z_k}(0) = 0$. For $(j,l) \in \mathcal{S}_0$, considering the ϕ_{jl}-component in (3.4.1) we have $(2-\delta_j^l)\overline{L_k}(\xi_j\xi_l)L_jL_l(\phi_{jl})(0) +$

$4iL_kT(\phi_{jl})(0) = 0$. Assume that $l \neq k$. Since $\overline{L_k}(\xi_j\xi_l)(0) = 0$, we derive that $L_kT(\phi_{jl})(0) = 0$. When $l = k$, by (3.3.2), we have

$$0 = \overline{L_k}(\xi_j)L_jL_k(\phi_{jl})(0) + 4iL_kT(\phi_{jl})(0) = \frac{2i}{\mu_j}\mu_{jk}^2 L_kT(\phi_{jk})(0) + 4iL_kT(\phi_{jk})(0) = 6iL_kT(\phi_{jk})(0).$$

It thus follows that $L_kT(\phi_{jk})(0) = 0$. This proves that $\frac{\partial^2\phi_{jl}}{\partial w\partial z_k}(0) = 0$ for any $(j,l) \in \mathcal{S}$. Also, it implies by (3.3.2) that

$$\overline{L_{k'}}(\xi_j)(0) = 0, \text{ for} k' > \kappa_0 \tag{3.5.3}$$

Next, we proceed to show that $T^2(f)(0) = 0$. By the normalization condition, we need to show that $T^2(f_j)(0) = 0$ for $j > \kappa_0$. By (3.5.2), we have $\overline{L_k}^2(\Lambda_l)(0) = 16\delta_l^k T^2(f_k)(0)$. Applying $(3.4.1)'$, we arrive at $-8T^2(f_j)(0) = 16\delta_j^k T^2(f_k)(0)$, from which it follows easily that $T^2(f_j) = 0$. Moreover, considering the ϕ_{jl} component with $(j,l) \notin \mathcal{S}_0$ in (3.4.1), we get $T^2(\phi_{jl})(0) = 0$. When $(j,l) \in \mathcal{S}_0$, we get $\overline{L_k}^2(\xi_j\xi_l)(0)L_jL_l(\phi_{jl})(0) - 4T^2(\phi_{jl})(0) = 0$. Hence, if $l \neq k$, since $\overline{L_k}^2(\xi_j\xi_l)(0) = 0$, it holds that $T^2(\phi_{jl})(0) = 0$. If $l = k$, we get $T^2(\phi_{jk})(0) = \frac{1}{4}\overline{L_k}^2(\xi_j)(0)$.

For the proof of (D), we just need to dot-multiply both sides of (3.1.0) by the vector $2\sqrt{-1}\overline{\widetilde{f}}$ and then apply $\overline{L_k^2}$ as before. ∎

§ **3.2**: In this subsection, we first derive a new system of differential equations by making use of Lemma 3.3. Then we use it, together with Lemma 3.3, to achieve an approximate partial linearity theorem for F. We also discuss the distribution introduced by the partial linearity of F and its relation with $\mathcal{E}_p$ introduced at the beginning of §3.1.

From Lemma 3.3 (B), for $k > \kappa_0$, we have $\frac{\partial^2\widetilde{f}}{\partial z_k\partial w}(0) = 0$ for any F satisfying the normalization in (3.2). Actually, it is clear that we even do not need to make $\mu_1 = 1$ to achieve this property. This fact can be easily applied to F_p^{**} to derive a new system of some useful differential equations. In details, we explain it as follows:

Let $I_k(p) = (\xi_1, \cdots, \xi_{n-1})$ and $f_p^{**} = z + \frac{i}{2}wz\mathcal{A}(p) + o_{wt}(3)$ be as before. Then there is a unitary matrix $U_{(p,k)}$, whose kth row is $\frac{I_k}{|I_k|}$ such that

$$U_{(p,k)}\mathcal{A}(p)U_{(p,k)}^{-1} = \mathrm{diag}(\mu_1(p), \cdots, \mu_{\kappa_0}(p), 0, \cdots, 0).$$

Then $\frac{I_k(p)}{|I_k(p)|} = e_k'U_{(p,k)}$, where e_k' is the vector in $\mathbf{C}^{n-1}$ whose component at the jth position is δ_j^k. Also, from the proof of Lemma 3.2, there is an $(N-n)\times(N-n)$ unitary matrix $\widetilde{U_p}$ such that $\phi_p^{**}(zU_{(p,k)}, w)\cdot\widetilde{U_p}^{-1}$ takes the form as in the last expression of (3.2) except that $\mu_1 = 1$. Write

$$\hat{F}_p^{**} = \left(f_p^{**}(zU_{(p,k)}, w)U_{(p,k)}^{-1}, \phi_p^{**}(zU_{(p,k)}, w)\widetilde{U_p}^{-1}, g(zU_{(p,k)}, w)\right), \quad F_p^{***} = \hat{\tau}_p \circ \hat{F}_p^{**} \circ \hat{\sigma}_p \tag{3.6.0}$$

with

$$\hat{\sigma}_p = \frac{(z+\hat{a}w, w)}{1-2i<\overline{\hat{a}}, z> -i|\hat{a}|^2 w}, \quad \hat{\tau}_p = \frac{(z^* - (\hat{a},0)w^*, w^*)}{1+2i<(\overline{\hat{a}},0), z^*> -i|\hat{a}|^2 w^*}$$

where $\hat{a}$ is chosen so that the first κ_0 components of the vector:

$$\frac{\partial^2 f_p^{***}}{\partial w^2}(0) = i\hat{a}U_{(p,k)}\mathcal{A}(p)U_{(p,k)}^{-1} + \frac{\partial^2 f_p^{**}}{\partial w^2}(0)U_{(p,k)}^{-1}$$

are zero. (See Lemma 2.2(B), especially (2.5.4)). Write $\eta = \hat{a}U_{(p,k)}$. We can choose $\hat{a}$ so that $\eta = (\eta_1, \cdots, \eta_{\kappa_0}, 0, \cdots, 0)$. Clearly $\hat{a}$ and η are then uniquely determined; and $\eta_j's$ are holomorphically rational with respect to $\{P_j^l, T^2(f_p^{**})(0)\}$. Moreover, as in (3.4.4), it is easy to conclude that $\eta_j = \frac{i}{\mu_j(0)}T^2(f_{p,j}^{**})(0) + o(|T^2(f_p^{**})|)$. Moreover by (3.1.2) and (2.1.5) (2.1.6), we have

$$T^2(f_{p,j}^{**})(0) = \frac{1}{\lambda(p)}\left(T^2(\widetilde{f})\cdot\overline{L_j\widetilde{f}}^t - \frac{1}{2\lambda}(T\widetilde{f})\cdot\overline{L_j\widetilde{f}}^t\left(T^2(g) - 2iT^2(\widetilde{f})\cdot\overline{\widetilde{f}}^t\right)\right)$$

$$\eta_j = \frac{i}{\lambda(p)\mu_j}\left(T^2(\widetilde{f})\cdot\overline{L_j\widetilde{f}}^t - \frac{1}{2\lambda}(T\widetilde{f})\cdot\overline{L_j\widetilde{f}}^t\left(T^2(g) - 2iT^2(\widetilde{f})\cdot\overline{\widetilde{f}}^t\right)\right) + o(|T^2(f_p^{**}(0)|) \tag{3.6.1}$$

$$\overline{L_k}(\eta_j)(0) = \frac{i}{\mu_j}T^2(\phi_{jk})(0)\overline{\mu_{jk}}.$$

Next notice that $(\phi_p^{***})''_{z_k w}(0) = \frac{1}{|I_k(p)|}\left(\sum_j \xi_j(\phi_p^{**})''_{z_j w}(0) + \sum_{j,l}\xi_j\eta_l(\phi_p^{**})''_{z_j z_l}(0)\right)\widetilde{U_p}^{-1} = 0$; for, by Lemma 2.2 (B), F_p^{***} satisfies the normalization in (3.2) except that $\mu_1 = 1$. Arguing the same way as in the proof of Lemma 3.1, by (2.3.4), we have the following:

(I).

$$\left(\sum_{j',l=1}^{n-1}\xi_{j'}\eta_l L_l L_{j'}(\widetilde{f}) + L_\xi T(\widetilde{f}) - \frac{2i}{\lambda(p)}\overline{T(\widetilde{f})}\cdot L_\xi(\widetilde{f})^t T(\widetilde{f})\right)\cdot\overline{C_j(p)^t} = 0$$

for any j. Here, we write $L_\xi = \sum \xi_j L_j$. Thus there are $\Omega_j(p)'s$ such that

(3.6.2)

$$\sum_{j,l}\xi_j\eta_l L_j L_l(\widetilde{f}) + L_\xi T(\widetilde{f}) - \frac{2i}{\lambda(p)}\overline{T(\widetilde{f})}\cdot L_\xi(\widetilde{f})^t T(\widetilde{f}) = \sum_{j=1}^{n-1}\Omega_j(p)E_j(p) = \sum_{j=1}^{n-1}\Omega_j(p)L_j(\widetilde{f})(p)$$

$$\text{with } \Omega_j(p) = \frac{1}{\lambda(p)}\left(\sum_{j,l}\xi_{j'}\eta_l L_{j'}L_l(\widetilde{f})) + L_\xi T(\widetilde{f}) - \frac{2i}{\lambda(p)}\overline{T(\widetilde{f})}\cdot L_\xi(\widetilde{f})^t T(\widetilde{f})\right)\cdot\overline{L_j(\widetilde{f})^t};$$

(II). $\sum_{l=1}^{n-1}\xi_l P_l^j(p) = 0$ and hence (by (3.4.4))

$$\Omega_j(p) = \frac{\xi_j}{2\lambda(p)}(T^2(g) - 2iT^2(\widetilde{f})\cdot\overline{\widetilde{f}^t}) + \frac{1}{\lambda(p)}\sum_{j',l}\xi_{j'}\eta_l L_{j'}L_l(\widetilde{f})\cdot\overline{L_j(\widetilde{f})}. \tag{3.6.2$'$}$$

Applying $\overline{L_k}$ to (3.6.2) with an application of Claim 3.4, we obtain

$$\begin{aligned}(3.6.3)\qquad &\sum_{j,l=1}^{n-1}\overline{L_k}(\xi_j\eta_l)L_jL_l(\widetilde{f})+2i\sum_{l=1}^{n-1}\eta_lTL_l(\widetilde{f})+2i\sum_{j,l=1}^{n-1}\xi_j\eta_kL_jT(\widetilde{f})+2iT^2(\widetilde{f})\\ &=\sum_{j=1}^{n-1}\overline{L_k}(\Omega_j)L_j(\widetilde{f})+2i\Omega_kT(\widetilde{f})+\overline{L_k}\left(\frac{2i}{\lambda(p)}\overline{T(\widetilde{f})}\cdot L_\xi(\widetilde{f})^tT(\widetilde{f})\right)-\sum\overline{L_k}(\xi_j)L_jT(\widetilde{f}).\end{aligned}$$

Still assume that F is as in Lemma 3.3. Considering the ϕ_{jl}-component in (3.6.3) with an application of (3.6.1), and then evaluating at 0, we can easily conclude that

$$T^2(\phi_{jl})(0)=0\ \text{ for any }\ (j,l)\in\mathcal{S}. \tag{3.6.4}$$

With (3.6.4) and Lemma 3.3 at our disposal, we now are ready to give the following approximate partial linearity theorem for F:

Theorem 3.5: *Suppose that F is a C^2-smooth CR map from a connected open subset M of 0 in $\mathbf{H}_n$ into $\mathbf{H}_N$. Suppose that $Rk_F(p)\equiv\kappa_0<n-1$ is constant for any $p\in M$.*
(A): Then for any $p\in M$, there is a unique affine complex subspace S_p of dimension $(n-\kappa_0)$ transversal to $\mathbf{H}_n$ at p such that there is a linear fractional map $\mathcal{L}_p$ which maps $S_p\cap\mathbf{H}_n$ into $\mathbf{H}_N$ with the following property

$$F(Z)-\mathcal{L}_p(Z)=o(|Z-p|^2)\ \ \textit{for}\ \ Z\in S_p\cap\mathbf{H}_n. \tag{3.7.1}$$

(B): For any affine complex line C_p transversal to $\mathbf{H}_n$ at p, if there is a linear fractional map Ψ_p sending $C_p\cap\mathbf{H}_n$ into $\mathbf{H}_N$ such that $F(Z)-\Psi_p(Z)=o(|Z-p|^2)$ for $Z\in C_p\cap\mathbf{H}_n$, then $C_p\subset S_p$.

Proof of Theorem 3.5: We can assume that F satisfies the normalization condition (3.2) except that $\mu_1=1$. Also, from (2.2.3) and (3.6.0) , there are $\widetilde{\sigma_p^0}\in\mathrm{Aut}(\mathbf{H}_n)$ and $\widetilde{\tau_p^F}\in\mathrm{Aut}(\mathbf{H}_N)$ such that

$$F_p^{***}=\widetilde{\tau_p^F}\circ F\circ\widetilde{\sigma_p^0},\ \ \text{with }\widetilde{\sigma_p^0}(0)=p,\ \widetilde{\tau_p^F}(F(p))=0,\ \text{and} \tag{3.7.2}$$

$$\mathrm{d}(\widetilde{\sigma_p^0})(0)=\begin{pmatrix}U_{(p,k)} & 0\\ 0 & 1\end{pmatrix}\ \ \text{with}\ \ I_k(p)=|I_k(p)|e_k'U_{(p,k)}, \tag{3.7.3}$$

where F_p^{***} satisfies the normalization condition in (3.2). Let $S_0=\{Z=(z_1,\cdots,z_{n-1},w):z_1=\cdots=z_{\kappa_0}=0\}$ be as before and let $\mathcal{L}_0(z_1,\cdots,z_{n-1},w)=(0,\cdots,0,z_{\kappa_0+1},\cdots,z_{n-1},0,\cdots,0,w)$.

By Lemma 3.3 and (3.6.4), if we define $S_p = \widetilde{\sigma_p^0}(S_0)$ and $\mathcal{L}_p = (\widetilde{\tau_p^F})^{-1} \circ \mathcal{L}_0 \circ (\sigma_p^0)^{-1}$, then (3.7.1) holds. To prove the uniqueness of S_p, it suffices for us to prove the statement in (B) with $p = 0$. This can be trivially seen as follows:

Suppose that C_0 is parametrized by $z = (a_1, \cdots, a_{n-1})t = at, w = t$ and suppose $\Psi_0|_{C_0} = \frac{bt}{1+b_0 t} = bt - b_0 bt^2 + o(t^2)$. Since, when restricted to C_0,

$$F = (at, 0, t) + \frac{i}{2}(a_1\mu_1, \cdots, a_{\kappa_0}\mu_{\kappa_0}, 0, \cdots, 0)t^2 + o(t^2),$$

it follows that

$$b = (a, 0, 1), \quad \frac{i}{2}(a_1\mu_1, \cdots, a_{\kappa_0}\mu_{\kappa_0}, 0, \cdots, 0) = -b_0 b.$$

Thus, we conclude that $a_j = 0$ for $j \leq \kappa_0$. Namely, $C_0 \subset S_0$. Finally, notice that $S_p = \widetilde{\sigma_p^0}(S_0)$. The proof of Theorem 3.5 is complete. ∎

For each $p \in M$, let S_p be as in Theorem 3.5. Define $\mathcal{V}_F(p) = CT_p(S_p \cap \mathbf{H}_n)$ and write $\mathcal{V}_F$ for the vector subbundle of CTM, whose fiber at p is $\mathcal{V}_F(p)$. In what follows, for an $(n-1)$-vector valued function $J(p) = (v_1(p), \cdots, v_{n-1}(p))$ along M, we write J^L for the vector field $\sum_{j=1}^{n-1} v_j(p)L_j$. We call that $\mathcal{V}_F$ is integrable in the strong sense if M can be foliated by $\mathcal{V}_F$-integrable C^2 submanifolds of real dimension $2(n-\kappa_0)+1$. The study of the strong integrability of $\mathcal{V}_F$ is closely related to the partial linearity of F. We will not address this issue here. Instead, we will be content with presenting the following proposition, which gives a partial integrability statement for the distribution $\Gamma(\mathcal{V}_F)$.

Proposition 3.6: *Let F be as in Theorem 3.5. Assume the notation we just set up and write $\mathcal{V}_F^{(1,0)} = \mathcal{V}_F \cap T^{(1,0)}M$.*
(I). Then the space $\Gamma(\mathcal{V}_F^{(1,0)})$ of cross sections of $\mathcal{V}^{(1,0)}$ has $\{I_k^L\}_{k=\kappa_0+1,\cdots,n-1}$ as its basis.
(II). For each $k' > \kappa_0$, $\Gamma(\mathcal{V}_F)$ has $\{I_{\kappa_0+1}^L, \overline{I_{\kappa_0+1}^L}, \cdots, I_{n-1}^L, \overline{I_{n-1}^L}, [I_{k'}^L, \overline{I_{k'}^L}]\}$ as its basis. Moreover, for each $k', k'' > \kappa_0$, it holds that

$$[I_{k'}^L, \overline{I_{k''}^L}], [I_{k'}^L, I_{k''}^L], [\overline{I_{k'}^L}, \overline{I_{k''}^L}] \in \Gamma(\mathcal{V}_F). \tag{3.8.1}$$

(III) Let S_p be as in Theorem 3.5. Then $S_p - p$, as an element in the complex Grassmannian manifold $G_{n,n-\kappa_0}(\mathbf{C})$ of $(n-\kappa_0)$-dimensional linear subspaces in $\mathbf{C}^n$, depends continuously on p. Moreover, if $F \in C^k(M)$, then S_p depends C^{k-2}-continuously on p.

Proof of Proposition 3.6: Let $\widetilde{\sigma_{(p,k)}^0}$ be as in (3.7.2). (Here we put the index k to emphasize its dependence on k). Then

$$\mathcal{V}_F^{(1,0)}(p) = \text{span}\{D_0(\widetilde{\sigma_{(p,k)}^0})(e_k'^L|_0)\}_{k=\kappa_0+1,\cdots,n-1},$$

where $D_0(\cdot)$ denotes the differential push-forward map. From the construction of $\widetilde{\sigma^0_{(p,k)}}$, it is clear that we can assume that it takes the form $\widetilde{\sigma^0_{(p,k)}} = \sigma^0_p \circ \mathcal{U}^0_{(p,k)} \circ \hat{\sigma}^0_p$, where $\hat{\sigma}^0_p(0) = 0$, $D_0(\hat{\sigma}^0_p) = \mathrm{Id}$, σ^0_p is as defined in (2.1.0), and

$$\mathcal{U}^0_{(p,k)} = \begin{pmatrix} U_{(p,k)} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad I_k(p) = |I_k(p)| e'_k U_{(p,k)}.$$

Hence,

$$\begin{aligned} D_0(\widetilde{\sigma^0_{(p,k)}})(e_k^{'L}|_0) &= D_0(\sigma^0_p \circ \mathcal{U}^0_{(p,k)})(e_k^{'L}|_0) \\ &= \frac{1}{|I_k(p)|} D_0(\sigma^0_p)(\sum \xi^k_j(p) e_j^{'L}|_0) = \frac{1}{|I_k(p)|} \sum \xi^k_j(p) e_j^{'L}|_p = \frac{I^L_k(p)}{|I_k(p)|}. \end{aligned}$$

Here and in what follows, we start to write $I_k = (\xi^k_j)_{1 \le k \le n-1}$ to emphasize its dependence on the index k.

Next, since $P^l_j = -\overline{P^j_l}$, by (3.4.3) and (3.4.4), for $Z_0 \in S_0$, we have

$$\xi^k_j(Z_0) = \frac{-2i}{\mu_j} \overline{P^k_j(Z_0)} + o(|\sum_{l=1}^{\kappa_0} P^k_l(Z_0)|) = \frac{-2i}{\mu_j} \overline{L_j T(f_k)(Z_0)} + o_{wt}(1). \tag{3.8.2}$$

We conclude that $L_{k'}(\xi^k_j)(0) = 0$. Notice that $\overline{L_{k'}}(\xi^k_j)(0) = 0$ by (3.5.3). We can easily arrive at the following:

$$\mathcal{V}_F(0) = \mathrm{span}\{I^L_{\kappa_0+1}|_0, \overline{I^L_{\kappa_0+1}}|_0, \cdots, I^L_{n-1}|_0, \overline{I^L_{n-1}}|_0, [I^L_{k'}, \overline{I^L_{k'}}]|_0\}. \tag{3.8.3}$$

Moreover, for each $k', k'' > \kappa_0$, it holds that

$$[I^L_{k'}, \overline{I^L_{k''}}]|_0, [I^L_{k'}, I^L_{k''}]|_0, [\overline{I^L_{k'}}, \overline{I^L_{k''}}]|_0 \in \mathcal{V}_F(0). \tag{3.8.4}$$

Notice that $\{I^L_{\kappa_0+1}|_0, \overline{I^L_{\kappa_0+1}}|_0, \cdots, I^L_{n-1}|_0, \overline{I^L_{n-1}}|_0, [I^L_{k'}, \overline{I^L_{k'}}]|_0\}$ is a basis of $\mathcal{V}_F|_0$; for it is a linearly independent system and has the right dimension. This proves the statement in (II) at the origin.

To achieve (3.8.3) (3.8.4) at any other point $p_0(\approx 0) \in M$, we need to go back to the map $F^{***}_{p_0}$ defined in (3.7.2). Write $p' = \widetilde{\sigma^0_{(p_0,k)}}^{-1}(p)$ or $p = \widetilde{\sigma^0_{(p_0,k)}}(p')$. For $p \approx p_0$, notice that $\widetilde{J}^L_k := D_{p'}(\widetilde{\sigma^0_{(p_0,k)}})(e_k^{'L}(p'))$ is a smooth (1,0)-type vector field tangent to $S_{p_0} \cap \mathbf{H}_n$ near p_0, by the way S_{p_0} was constructed. Write $S_{F^{***}_{p_0}}(p')$ for the affine complex subspace of dimension $(n - \kappa_0)$ along which $F^{***}_{p_0}$ is approximated by a linear fractional map as in Theorem 3.5. (Apparently, we can apply Theorem 3.5 to $F^{***}_{p_0}$.) Then it is clear that $\widetilde{\sigma^0_{(p_0,k)}}\left(S_{F^{***}_{p_0}}(p')\right) = S_F(p)$ and thus $D_{p'}(\widetilde{\sigma^0_{(p_0,k)}})(\mathcal{V}^{(1,0)}_{F^{***}_{p_0}}(p')) = \mathcal{V}^{(1,0)}_F(p)$. Hence, there are functions $b_{k,k'}(p)'s$ such that

$$I^L_k(p) = \sum_{k'=\kappa_0+1}^{n-1} b_{k,k'}(p) D_{p'}(\widetilde{\sigma^0_{(p_0,k)}}) \left(I^L_{k'}(p', F^{***}_{p_0})\right), \tag{3.8.5}$$

where $I^L_{k'}(p', F^{***}_{p_0})$ is the corresponding $I^L_{k'}$-vector field associated to the map $F^{***}_{p_0}$ at p'. Also, $b_{k,k'}(p)$ can be uniquely solved by noting the fact that $\{I^L_k(p', F^{***}_{p_0})\}_{k=\kappa_0+1}^{n-1}$ is a linearly independent system for $p' \approx 0$. In fact, it can be seen that

$$b_{k,k'}(p) = B_{k,k'}(I_{k''}(p), I_{k''}(p', F^{***}_{p_0}), p, \overline{p}) \tag{3.8.6}$$

for a certain $B_{k,k'}$ holomorphic in its own variables. By Lemma 3.3 (A) and (3.8.6), $L_j(b_{k,k'}(p))$ and $\overline{L_j}(b_{k,k'}(p))$ are continuous near p_0. Now, write

$$J^L_k(p) = \sum_{k'} b_{k,k'}(p)\widetilde{J^L_{k'}}(p). \tag{3.8.7}$$

Then J^L_k is tangent to $S_{p_0} \cap \mathbf{H}_n$ near p_0 and

$$I^L_k - J^L_k = \sum_j \chi^k_j(p) L_j \tag{3.8.8}$$

with $L_l(\chi^k_j(p))$ and $\overline{L_l}(\chi^k_j(p))$ continuous near p_0 and

$$\chi^k_j(p_0) = J^L_{k'}(\chi^k_j)(p_0) = \overline{J^L_{k'}}(\chi^k_j)(p_0) = 0. \tag{3.8.9}$$

Hence,

$$[I^L_{k'}, \overline{I^L_{k''}}](p_0) = \left([J^L_{k'}, \overline{J^L_{k''}}] + \sum_{j,l'} \overline{J^L_{k''}}(\chi^{k'}_l) L_l - \sum_{j,l} J^L_{k''}(\overline{\chi^{k'}_l})\overline{L_l} + o_{wt}(0) \right)\Big|_{p_0} = [J^L_{k'}, \overline{J^L_{k''}}](p_0)$$

by making use of (3.8.9). Notice that $[J^L_{k'}, \overline{J^L_{k''}}](p_0) \in \mathcal{V}_F(p_0)$ and $[J^L_{k'}, \overline{J^L_{k'}}](p_0) \notin \mathcal{V}^{(1,0)}_F(p_0) \cup \mathcal{V}^{(0,1)}_F(p_0)$, we conclude the first statement in (II) by counting the dimension.

Similarly, we can verify the other identities in Proposition 3.6 (II).

By the results in (I) and (II), we notice that the complex linear space $T^{(1,0)}_p(S_p)$ has basis

$$\{I^L_{\kappa_0+1}(p), \cdots, I^L_{n-1}(p), [I^L_{n-1}, \overline{I^L_{n-1}}](p) - i\mathcal{J}([I^L_{n-1}, \overline{I^L_{n-1}}])(p)\},$$

where $\mathcal{J}$ is the standard complex structure in $\mathbf{C}^n$. We easily conclude the statement in (III).

The proof of Proposition 3.6 is complete. ∎

§ 4: Proof of Theorem 2.3

In this section, with the preparation in §3, we will complete the proof of Theorem 2.3. Since F is now assumed to be C^3, there are two ways to achieve it. One way is to continue the study of the integrability of $\mathcal{V}_F$ and then restrict the equation (3.1.0) to each leaf to reduce the linearity problem to the uniqueness of differential equations. Another method is to obtain the linearity by reducing the problem to a boundary uniqueness problem for holomorphic functions, as in §4 of [HJ1]. Here, we present a discussion on the latter approach, which mainly depends on results obtained in §3.1.

We now proceed to explain how the proof of Theorem 2.3 can be obtained by reducing the problem to the uniqueness problem of holomorphic functions so that we can apply a version of the classical Hopf Lemma due to Burns-Krantz. This approach was motivated by the argument in §4 of [HJ1]. We first start with the following lemma, for whose proof one just needs to copy the argument from [Lemma 5.3, Hu1] with a notice of the C^l-smoothness assumption for the map here.

Lemma 4.0: Let F be a C^l-smooth CR map from a neighborhood M of $0 \in \mathbf{H}_n$ into $\mathbf{H}_N$ with $l \geq 2$. Assume that F satisfies the normalization condition (2.2.1). Then

$$g \in \mathcal{P} + o_{wt}(l+2), \;\; f \in \mathcal{P} + o_{wt}(l+1).$$

For the rest of this section, we will assume that F is C^3-smooth.

We next present the following generalization of Lemma 3.2 of [HJ1].

Lemma 4.1: Let $F = (f, \phi, g)$ be a C^3-smooth CR map from M into $\mathbf{H}_N$, which satisfies the normalization condition (3.2). Then $f \in \mathcal{P} + o_{wt}(4)$, $g \in \mathcal{P} + o_{wt}(5)$. Moreover, we have the following weighted decomposition:

$$f_j^{(4)} = a_j^{(2)}(z)w + a_{02,j}w^2, \;\; g^{(5)} = c^{(1)}(z)w^2, \phi_{jl} = \phi_{jl}^{(2)}(z) + b_{jl}^{(3)}(z) + b_{jl}^{(1)}(z)w + o_{wt}(3) \tag{4.1}$$

with

$$\begin{aligned} &a_{02,j} = 0 \text{ for } j \leq n-1, \;\; c^{(1)}(z) \equiv 0, \;\; \frac{\partial b_{jl}^{(1)}(z)}{\partial z_k} = 0 \text{ for } \;\; k > \kappa_0 \;\; \text{and} \;\; (j,l) \in \mathcal{S}_0; \\ &a_j^{(2)}(z) = 0 \text{ for } j \geq \kappa_0 + 1, \text{ and } a_j(0, \cdots, 0, z_{\kappa_0+1}, \cdots, z_{n-1}) = 0 \text{ for } \;\; j \leq \kappa_0. \end{aligned} \tag{4.2}$$

Proof of Lemma 4.1: Applying Lemma 4.0, we can conclude that $f \in \mathcal{P} + o_{wt}(4)$ and $g \in \mathcal{P} + o_{wt}(5)$.

Collecting terms of weighted degree 5 in the basic equation $\mathrm{Im}(g) = |\widetilde{f}|^2$ for $\mathrm{Im}(w) = |z|^2$, we obtain the equation:

$$\mathrm{Im}\left(g^{(5)} - 2i\overline{z}\cdot f^{(4)}\right) = 2\mathrm{Re}\left(\phi^2\cdot\overline{\phi^{(3)}}\right). \tag{4.3}$$

As in [Lemma 3.2, HJ], it is easy to conclude from (4.3) the expansion in (4.1). Moreover, letting $z_1 = \cdots = z_{\kappa_0} = 0$ in (4.3) and applying the uniqueness of the Chern-Moser Lemma ([Lemma 3.1, CH] (or by a direct calculation), we have that $a_{02,j} = 0$ for $j \geq \kappa_0 + 1$. Collecting coefficients of u^2 in (4.3), we get $c^{(1)}(z) = 2i\sum_{j=1}^{\kappa_0}\overline{a_{02,j}}z_j$. Since $a_{02,j} = 0$ for $j \leq \kappa_0$ by the normalization in (3.2), we see that $g^{(5)} = 0$. Hence, (4.3) can be simplified as follows:

$$\mathrm{Im}\left(2i\sum_{j=1}^{n-1}\overline{z_j}\cdot a_j^{(2)}(z)w + 2i\sum_{(j,l)\in\mathcal{S}_0}\phi_{jl}^{(2)}\cdot\overline{(b_{jl}^{(3)}(z) + wb_{jl}^{(1)}(z))}\right) = 0, \quad w = u + i|z|^2. \tag{4.4}$$

Collecting the coefficients of u in (4.4), we get $\sum_{j=1}^{n-1}\overline{z_j}\cdot a_j^{(2)}(z) = -\sum_{(j,l)\in\mathcal{S}_0}\phi_{jl}^{(2)}(z)\cdot\overline{b_{jl}^{(1)}(z)}$. Thus, we get

$$a_m^{(2)} = -\sum_{(j,l)\in\mathcal{S}_0}\phi_{jl}^{(2)}(z)\overline{\frac{\partial b_{jl}^{(1)}(z)}{\partial z_m}}. \tag{4.5}$$

By the normalization in (3.2), we get $a_m^{(2)}(0,\cdots,0,z_{\kappa_0+1},\cdots,z_{n-1}) = 0$ for any m. Collecting terms in (4.4) without the u-factor, we get

$$\sum_{j=1}^{n-1}\overline{z_j}a_j^{(2)}(z)|z|^2 = \sum_{(j,l)\in\mathcal{S}_0}\phi_{jl}^{(2)}(z)\overline{b_{jl}^{(1)}(z)}|z|^2 + i\sum_{(j,l)\in\mathcal{S}_0}\overline{\phi_{jl}^{(2)}(z)}b_{jl}^{(3)}(z). \tag{4.6}$$

Collecting terms in the above equation with the factor $|z_k|^2\overline{z_k}$ with $k > \kappa_0$, we have

$$a_k^{(2)}(z) = \sum_{(j,l)\in\mathcal{S}_0}\phi_{jl}^{(2)}(z)\overline{\frac{\partial b_{jl}^{(1)}(z)}{\partial z_k}}. \tag{4.7}$$

Combining (4.5) with (4.7), we thus obtain $a_k^{(2)}(z) = 0$ and $\frac{\partial b_{jl}^{(1)}}{\partial z_k} = 0$ for $(j,l)\in\mathcal{S}_0$ and $k > \kappa_0$. ∎

We now let F be as in Theorem 2.3 with constant geometric rank κ_0 over M. Without loss of generality, we further assume that $0 \in M$, $n-1 > \kappa_0 > 0$ and F satisfies the normalization in (3.2). Also, we keep the notations which we have set up. Write

$$\mathbf{q} = (0,\cdots,0,q_1,\cdots,q_{n-\kappa_0-1}) \in \mathbf{C}^{n-1}.$$

Let $F_c^q = \tau_c^q \circ F \circ \sigma_c^q$, where the two automorphisms $\sigma_c^q \in \mathrm{Aut}(\mathbf{H}_n)$ and $\tau_c^q \in \mathrm{Aut}(\mathbf{H}_N)$ are given by

$$\sigma_c^q(z,w) = \left(\frac{z + w\mathbf{q}}{1 - 2i\overline{\mathbf{q}}z - i|\mathbf{q}|^2 w}, \frac{w}{1 - 2i\overline{\mathbf{q}}z - i|\mathbf{q}|^2 w}\right), \tag{4.8}$$

$$\tau_c^q(z^*,w^*) = \left(\frac{z^* - w^*(\mathbf{q},0')}{1 + 2i < (\overline{\mathbf{q}},0'), z^* > -i|\mathbf{q}|^2 w^*}, \frac{w^*}{1 + 2i < (\overline{\mathbf{q}},0'), z^* > -i|\mathbf{q}|^2 w^*}\right). \tag{4.9}$$

Corollary 4.2: Let F be as in (3.2) and let σ_c^q, τ_c^q be as in (4.8)-(4.9). Define $F_c^q := \tau_c^q \circ F \circ \sigma_c^q$, which we write as $F_c^q = (f_c^q, \phi_c^q, g_c^q) = (\widetilde{f}_c^q, g_c^q) = (f_{c,1}^q, \cdots, \phi_{c,jl}^q, \cdots, g_c^q)$. Then F_c^q still satisfies the normalization condition (3.2). Moreover, we have $g_c^q = w + o_{wt}(5)$, $\frac{\partial^2 f_c^q}{\partial w^2}(0) = 0$, $\frac{\partial^2 \phi_c^q}{\partial w \partial z_k}(0) = 0$ for $k > \kappa_0$, $(f_c^q)^{(4)} = (a_c^q)^{(2)}(z)w^2$ with $(a_{c,j}^q)^{(2)}(0, \cdots, 0, z_{\kappa_0+1}, \cdots, z_{n-1}) = 0$ for $j \leq \kappa_0$, and $(a_{c,j}^q)^{(2)} = 0$ for $j \geq \kappa_0 + 1$.

Proof of Corollary 4.2: By Lemma 2.2 (A) (C), Lemma 3.3 (C), (2.5.4) and a direct verification, we conclude that F_c satisfies the normalization condition (3.2). Hence, Corollary 4.2 follows from Lemma 4.1. ∎

Let $S_0 = \{(z_1, \cdots, z_{n-1}, w) : (z_1, \cdots, z_{\kappa_0}) = 0\}$ and

$$S_0' = \{(z_1^*, \cdots, z_{N-1}^*, w^*) : z_1^* = \cdots = z_{\kappa_0}^* = z_n^* = \cdots = z_{N-1}^* = 0\}.$$

Then S_0 and S_0' are complex subspaces of dimension $n - \kappa_0$ in $\mathbf{C}^n$ and $\mathbf{C}^N$, respectively. They are also transversal to the corresponding Heisenberg hypersurfaces. With the above preparation, we are ready to prove the following:

Lemma 4.3: Let F be a C^3-smooth CR map from a neighborhood of $0 \in \mathbf{H}_n$ into $\mathbf{H}_N$, that has constant geometric rank $\kappa_0 > 1$ and satisfies the normalization condition (3.2). Assume the above notation. Then $F(S_0 \cap M) \subset S_0' \cap \mathbf{H}_N$.

Proof of Lemma 4.3: For $q = (q_1, \cdots, q_{n-\kappa_0-1}) \in \mathbf{C}^{n-\kappa_0-1}$ with $|q| >> 1$, we let F_c^q be defined as in Corollary 4.2.

Claim 4.4: $g_c^q(0,w) = w + o(w^3)$.

Proof of Claim 4.4: For a fixed $k > \kappa_0$, let $I_{c,k}^q = \{(\xi_{c,j}^q)\}$ be the corresponding vector associated to the map F_c^q. Notice that F_c^q satisfies the normalization condition in (3.2). Hence, by

Lemmas 3.3, 4.1 and Corollary 4.2, it follows that $g_c^q = w + o_{wt}(5)$, $\xi_{c,j}^q = O_{wt}(2)$, $\overline{L_k}(\xi_{c,j}^q) = o_{wt}(0)$ for $j \leq \kappa_0$. We thus conclude from (3.4.2) the following:

$$-8T^2(g_c^q) = \sum_{l=1}^{n-1} \overline{L_k^2}(\Lambda_{c,l}^q) 2i\overline{z_l} + 4i\overline{L_k}(\Lambda_{c,l}^q) + o_{wt}(2), \tag{4.10}$$

where $\Lambda_{c,l}^q = \sum_{j=1}^{n-1} \frac{4\sqrt{-1}}{\lambda_c^q} \xi_{c,j}^q \xi_{c,l}^q \overline{(\widetilde{f_c^q})'_w} \cdot L_j(\widetilde{f_c^q})$. In the remaining argument for this claim, we drop the subscript c and superscript q to simplify the notation. Also, for two functions $A(Z)$ and $B(Z)$ with $Z = (z, \overline{z}, u)$, we say $A = B$ mod(terms other than $Z^{\alpha_1}\overline{Z^{\beta_1}}, \cdots, Z^{\alpha_l}\overline{Z^{\beta_l}}$) if in the weighted expansions of $A(Z)$ and $B(Z)$ up to order $\max_j\{\deg_{wt}(Z^{\alpha_j}\overline{Z^{\beta_j}})\}$, the coefficients for the $Z^{\alpha_j}\overline{Z^{\beta_j}}$-terms ($j = 1, \cdots, l$) are the same.

Recall by (3.5.1) that $\overline{L_k}(\Lambda_l) = II_1 + II_2 + II_3 + II_4$, where

$$II_1 = \sum_{j=1}^{n-1} \frac{-4i}{\lambda^2} \overline{L_k}(\lambda)(\overline{\widetilde{f'_w}} \cdot L_j \widetilde{f}^t)\xi_j\xi_l, II_2 = \sum_{j=1}^{n-1} \frac{4i}{\lambda}(\overline{L_k(\widetilde{f'_w})} \cdot L_j \widetilde{f}^t)\xi_j\xi_l,$$

$$II_3 = \frac{-8}{\lambda}(\overline{\widetilde{f'_w}} \cdot (\widetilde{f'_w})^t)\xi_l, II_4 = \sum_{j=1}^{n-1} \frac{4i}{\lambda}(\overline{\widetilde{f'_w}} \cdot L_j \widetilde{f}^t)\overline{L_k}(\xi_j\xi_l).$$

Since we assumed that F is C^3, it can be easily seen that $\overline{L_l}(\xi_j)$ and $L_l(\xi_j)$ are C^1 near the origin. Moreover, by (2.2.1), Lemma 3.3 (A) and Lemma 3.3 (C), we have $\lambda = 1 + o_{wt}(1)$, $\xi_j = O_{wt}(2)$, $\overline{L_k}(\xi_j) = O_{wt}(1)$. With these at our disposal and making use of Lemma 3.3, one can easily verify that

$$\begin{aligned} &II_1, II_3, II_4 = o_{wt}(2) \text{ mod(terms other than } |z_k|^2,\ u), \\ &II_2 = 4i\overline{L_k((f_k)'_w)} + o_{wt}(2) \text{ mod(terms other than } |z_k|^2,\ u). \end{aligned}$$

Notice that $\overline{L_j}(L_k((f_k)'_w) \in \mathcal{P} + o_{wt}(1), T(L_k((f_k)'_w)) \in \mathcal{P} + o_{wt}(0)$. Also applying L_j to the f_k-component of the first equation in (3.4.1), one sees that $L_j(L_k((f_k)'_w) = L_j(\overline{L_k}(\Lambda_k)) + o_{wt}(1) \in \mathcal{P} + o_{wt}(1)$. This shows by [Lemma 5.2, Hu1] that $L_k((f_k)'_w)) \in \mathcal{P} + o_{wt}(2)$.

Next, write $\gamma = \frac{\partial^3 f_k}{\partial z_k \partial w^2}(0)$. We thus obtain $II_2 = 4i\overline{\gamma}(u - 3i|z_k|^2)$ mod(terms other than $|z_k|^2,\ u$). Hence, it follows that

$$-8T^2(g) = -16(\overline{\gamma}(u - 3i|z_k|^2)) + 32i\overline{\gamma}|z_k|^2 + o_{wt}(2) \text{ mod(terms other than } |z_k|^2,\ u). \tag{4.11}$$

Write $c_{03} = \frac{1}{6}\frac{\partial^3 g}{\partial w^3}(0)$. From (4.10), we have $T^2(g) \in \mathcal{P} + o_{wt}(2)$. Hence, by a similar argument as above, it follows that

$$c_{03}(u + i|z_k|^2) = 2\overline{\gamma}u - 10i\overline{\gamma}|z_k|^2$$

which gives immediately that $c_{03} = 0$. ∎

Notice that $g_c^q(0', w)$ is defined over the upper-half plane for $|q| >> 1$. Once we know that $g_c^q(0', w) = w + o(w^3)$, by a generalized Hopf lemma due to Burns-Krantz [BK], we can conclude that $g_c^q(0', w) = w$ for $|q| >> 1$. In fact, for $|q| >> 1$, we can define the harmonic function h on the upper-half plane of $\mathbf{C}$ by: $h(w) = \mathrm{Im}\left(\frac{1}{w} - \frac{1}{g_c^q(0',w)}\right)$. Then it is clear that $h(w) = o(|w|)$ as $w \to 0$ and $\lim_{w(\in H^+)\to x\in(\mathcal{R}\cup\infty)} h(w) \geq 0$. Hence 0 is the minimum value of $h(w)$. By the classical Hopf lemma, it follows that $h(w) \equiv 0$. Namely, $g_c^q(0', w) \equiv w$.

Next, we also have $\widetilde{f_c^q}(0, w) \equiv 0$ by the boundary equation $\mathrm{Im}(g_c^q) = |\widetilde{f_c^q}|^2$. Let q vary. We conclude that F maps an open subset of $S_0 \cap \mathbf{H}_n$ into $S_0' \cap \mathbf{H}_N$. By a theorem of Alexander [Alx], it follows that F must be linear fractional when restricted to $S_0 \cap \mathbf{S}_n \cap \Omega$. Considering F_p^{***} in (3.6.0) instead of F, we conclude that for each $p(\approx 0) \in \mathbf{H}_n$, there is a unique affine subspace S_p of dimension $n - \kappa_0$ such that $F|_{S_p\cap\Omega}$ is a linear fractional map. (The uniqueness follows from Lemma 2.2(D) or Theorem 3.5). Also, by Proposition 3.6 (III), $S_p - p$, as an element in the complex Grassmannian manifold $G_{n,n-\kappa_0}(\mathbf{C})$ of $(n-\kappa_0)$-dimensional linear spaces in $\mathbf{C}^n$, depends continuously on p (or $C^{l-2}-$smoothly on p, if F is assumed to be C^l). (If we just need the dependence in a neighborhood of a certain point $p^* \approx p$, we then do not need Proposition 3.6. Indeed, we can obtain it directly from the construction of $\widetilde{\sigma_p^0}$ (for instance, see (3.7.2)) with a notice of the fact that $S_p = \widetilde{\sigma_p^0}(S_0)$.)

Returning to the proof of Theorem 2.3, we can apparently assume that $0 \in M$ and F satisfies the normalization in (3.2). Hence, from Proposition 3.6 and the just mentioned argument, the proof of Theorem 2.3 follows. ∎

§5. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1 and thus the proof of Corollary 1.2.

Let F be a proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^N$, that is C^2-smooth up to the boundary. Assume that $N < n + P(n, \kappa)$. For each $p \in \partial\mathbf{B}^n$. We can find Φ_p and Ψ_p, biholomorphic linear fractional map from $\mathbf{B}^n$ to $\mathbf{S}^n$ and $\mathbf{B}^N$ to $\mathbf{S}^N$, respectively, such that $\Phi_p(p) = 0$ and $\Psi_p(F(p)) = 0$. Then we define the geometric rank $Rk_F(p)$ of F at p to be the geometric rank of the map $\Psi_p \circ F \circ \Phi_p^{-1}$ at 0. By Lemma 2.2 (B), $Rk_F(p)$ is a well-defined lower semi-continuous function over $\partial\mathbf{B}^n$. Let $\kappa_0 = \max_{p\in\partial\mathbf{B}^n} Rk_F(p)$. Then $E := \{p : p \in \partial\mathbf{B}^n,\ Rk_F(p) = \kappa_0\}$ is an open subset of $\partial\mathbf{B}^n$ and $\kappa_0 < \kappa$ by Lemma 2.2 (B). Now, when F is further assumed to be C^3, for $p_0 \in E$, applying Theorem 2.3 to $\Psi_p \circ F \circ \Phi_p^{-1}$, we conclude that for each $p \in \partial\mathbf{B}^n$ close to p_0, there is a unique affine complex space S_p, which transversally passes through $\partial\mathbf{B}^n$ at p and

is of complex dimension $n-\kappa_0$ such that the restriction of F to $S_p \cap \mathbf{B}^n$ is linear. By Theorem 2.3, the collection of all these S_p must fill in an open subset U_{p_0} close to p_0 in $\mathbf{B}^n$. Apparently, shrinking the size of U_{p_0} if necessary, we see that for each $Z = (z_1, \cdots, z_n) \in U_{p_0}$, there is an affine complex space S_Z of dimension $n-\kappa_0$ passing through Z such that the restriction of F to S_Z is a linear fractional map. Since all affine complex subspaces through Z much intersect $\beta\mathbf{B}^n$ near p, it follows from Lemma 2.2 (D) that F can not be linear when restricted to any affine subspace through Z of dimension larger than $n-\kappa_0$.

Now, to complete the proof of Theorem 1.1, we need only to prove the following:

Lemma 5.1: Let F be a proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^N$ with $n \geq 2$. Assume that there is a point $Z_0 \in \mathbf{B}^n$ such that for each $Z \approx Z_0$, there is an affine complex subspace S_Z of complex dimension $n-\kappa_0 > 0$ such that $F\big|_{S_Z}$ is linear fractional. Then F is $(n-\kappa_0)$- linear over $\mathbf{B}^n$.

Proof of Lemma 5.1: Define

$$\mathcal{P} := \{(Z,S) \in \mathbf{B}^n \times G_{n,n-\kappa_0}(\mathbf{C}),\ F \text{ is linear fractional when restricted to } Z+S\}.$$

We claim that $\mathcal{P}$ is a complex analytic variety. For this purpose, we need to verify that (i) $\mathcal{P}$ is closed; (ii) $\mathcal{P}$ is locally defined by holomorphic functions. To prove $\mathcal{P}$ is closed, let $\{(Z_j, S_j)\} \in \mathcal{P}$ be such that $Z_j \to Z \in \mathbf{B}^n$ and $S_j \to S \in G_{n,n-\kappa_0}(\mathbf{C})$. After applying an automorphism of $\mathbf{B}^n$ if necessary, we can assume, without loss of generality, that $Z = 0$ and $S = \mathrm{Span}\{e_1, \cdots, e_{n-\kappa_0}\}$. Here e_j is the vector in $\mathbf{C}^n$, whose component at the lth-position is δ_j^l. Let $S_j = \mathrm{Span}\{e_1^j, \cdots, e_{n-\kappa_0}^j\}$ be such that $e_l^j \to e_l$ as $j \to \infty$. Since F is linear fractional when restricted to S_j, we can write $F(Z_j + \sum_{j=1}^{n-\kappa_0} t_j e_l^j) = \frac{F(Z_j)+\sum a_j(Z_j,S_j)t_j}{1+\sum_j b_j(Z_j,S_j)t_j}$. Apparently, a_j, b_j depend continuously on Z_j and S_j. Hence, letting $j \to \infty$, we see that $F(\sum_j t_j e_j)$ is linear fractional on $t = (t_1, \cdots, t_{n-\kappa_0})$. We remark that by a result of Alexander, F must be also a biholomorphic map from S into its image which is also an affine space of the same dimension.

Next, we let $(Z_0, S_0) \in \mathcal{P}$ and we want to show that $\mathcal{P}$ near (Z_0, S_0) is defined by holomorphic equations. As above, we can also assume that $Z_0 = 0$ and $S_0 = \mathrm{span}\{e_1, \cdots, e_{n-\kappa_0}\}$. We will use the standard local coordinates for the Grassmannian $G_{n,n-\kappa_0}(\mathbf{C})$ near S_0. Namely, for any S near S_0 we associate it uniquely with the coordinates (ξ_{jl}) where j runs from 1 to $n-\kappa_0$ and l runs from $n+1-\kappa_0$ to n such that $S = \mathrm{span}\{e_1(S), \cdots, e_{n-\kappa_0}(S)\}$. Here $e_j(S) = (0, \cdots, 1, \cdots, 0, \xi_{j(n+1-\kappa_0)} \cdots, \xi_{jn})$. Notice that $(Z,S)(\approx (0,S_0)) \in \mathcal{P}$ if and only if

$$F(Z + \sum_j t_j e_j(S)) = \frac{F(Z) + \sum A_j(Z,S)t_j}{1 + \sum_j b_j(Z,S)t_j} \tag{5.1}$$

for certain N-tuples $A_j's$ and scalar $b_j's$, which depend on (Z, S). Write

$$F(Z + \sum_j t_j e_j(S)) = \sum_\alpha C_\alpha(Z, \xi_{jl}) t^\alpha.$$

Then C_α depends holomorphically on (Z, ξ_{jl}). Multiplying $(1 + \sum_j b_j(Z, S) t_j)$ of both sides of (5.1) and then considering the Taylor expansion in t at the origin, we see that (5.1) holds if and only if:

$$C_\alpha + \sum_{j=1}^{n-\kappa_0} b_j C_{\alpha - e'_j} \equiv 0 \text{ for } |\alpha| \geq 1, \tag{5.2}$$

$C_0 = F(Z)$, $C_{e'_j} = D_{t_j} F(z + \sum_j t_j e_j(S))|_{t=0}$ and $A_j = F(Z) b_j + C_{e'_j}$. Here e'_j is the vector in $\mathbf{C}^{n-\kappa_0}$ defined as for e_j. As we mentioned before, since $F|_S$ must be a linear embedding, we see that $\{C_{e'_j}\}_{j=1}^{n-\kappa_0}$ are linearly independent vectors. Hence, we can holomorphically solve $b'_j s$ in (5.2) in terms of C_α with $|\alpha| = 1, 2$. Hence (5.2) can be completely written as a system of holomorphic equations in (Z, ξ_{jl}). Together with the closeness of $\mathcal{P}$, we conclude that $\mathcal{P}$ is a complex analytic variety in $\mathbf{B}^n \times G_{n,n-\kappa_0}(\mathbf{C})$. Let π be the natural projection from $\mathcal{P}$ into $\mathbf{B}^n$. Then π is clearly proper and thus $\pi(\mathcal{P})$ is a subvariety of $\mathbf{B}^n$. Since $\pi(\mathcal{P})$ contains an open subset of the ball, we conclude that $\pi(\mathcal{P}) = \mathbf{B}^n$. Hence, we showed that for each point Z in the ball, there is a complex subspace S of dimension $(n - \kappa_0)$ such that the restriction of F to $S + Z$ is linear. Since a linear fractional map sends affine lines to affine lines we conclude the proof of Lemma 5.1. ∎

Proof of Theorem 1.1: Let κ be as in Theorem 1.1 and κ_0 be the geometric rank of the map F. By Lemma 3.2, $n - \kappa_0 \geq n - \kappa + 1$. Hence, combining Theorem 2.3 with Lemma 5.1, we complete the proof of Theorem 1.1. ∎

The argument presented above, together with Lemma 5.3, can be clearly used to give the following local result of Theorem 1.1:

Corollary 5.2: *Let M be a connected open subset of $\mathbf{H}_n$ and let F be a C^3 CR map from M into $\mathbf{H}_N$ with $N \geq n > 1$. Assume that F extends holomorphically to a sub-domain Ω of $\mathbf{S}^n$ which has M as part of its smooth boundary. Let $\kappa_0 = \max_{p \in M} Rk_F(p)$. Then $\{p \in M : \ Rk_F(p) = \kappa_0\}$ is an open dense subset of M. When $\kappa_0 < n - 1$, F is $(n - \kappa_0)$-linear over Ω. Furthermore, assume that F has constant geometric rank κ_0 in a connected open subset M' of M. Then there is a sufficiently small subdomain Ω' of Ω with M' as part of its smooth boundary satisfying the following property: For each $Z \in \Omega' \setminus E$ with E a certain proper complex analytic variety in Ω,*

there is a unique complex subspace S_Z of dimension $n-\kappa_0$ such that the restriction of F to S_Z+Z is linear fractional. Also S_Z, as elements in $G_{n,n-\kappa_0}(\mathbf{C})$, depends holomorphically on $Z \in \Omega \setminus E$. Moreover, S_Z extends holomorphically to Ω' and continuously up to M'

Lemma 5.3: Let M be a connected open subset of $\mathbf{H}_n$. Let F be a C^2 CR map from M into $\mathbf{H}_N$ with $N \geq n > 1$ and with constant geometric rank $\kappa_0 < n-1$. Assume that F extends holomorphically to a sub-domain Ω of $\mathbf{S}^n$, which has M as part of its smooth boundary. Assume that F is $(n-\kappa_0)$-linear over Ω. Let $p_0 \in M$. Then for $Z(\in \Omega \setminus E) \approx p_0$ with E a certain complex analytic variety of positive codimension, there is a unique complex subspace S_Z of dimension $(n-\kappa_0)$ such that F, when restricted to $S_Z + Z$, is linear fractional. Moreover S_Z, as elements in $G_{n,n-\kappa_0}(\mathbf{C})$, depends holomorphically on $Z(\approx p_0) \in \Omega \setminus E$ and extends holomorphically across E.

Proof of Lemma 5.3: Without loss of generality, we assume that $p_0 = 0$. For each $p \in M$, write S_p^a for the unique affine subspace through p of dimension $n-\kappa_0$, along which F is approximately linear as in Theorem 3.5. (To be consistent with the notation S_Z we use here, we add the superscript a to emphasize S_p^a is affine.) We can also assume that S_0^a is the subspace defined by $z_1 = \cdots = z_{\kappa_0} = 0$. Since for any $Z(\in \Omega) \approx 0$, $S_Z + Z$ must cut M near the origin, by Theorem 3.5 (B) and Proposition 3.6 (III), S_Z as an element in $G_{n,n-\kappa_0}(\mathbf{C})$ must be also very close to S_0^a. Let $\mathcal{P}_r$ be as defined in (5.0) with $\mathbf{B}^n$ being replaced by $\Omega \cap \{|z| < r\}$ for $r << 1$. Still write π for the projection from $\mathcal{P}_r$ to $\Omega_r = \Omega \cap \{|z| < r\}$. Then by the assumption, π is a surjective proper map. Since when $r << 1$, for each $Z \in \Omega \cap \{|z| < r\}$, $\pi^{-1}(Z)$, being close to S_0, can be embedded into $\mathbf{C}^{\kappa_0(n-\kappa_0)}$, $\pi^{-1}(Z)$ must be a finite set. Present $S_Z(\approx S_0^a)$ by its coordinates $(\xi_{jl})_{1\leq j\leq n-\kappa_0, n+1-\kappa_0 \leq l \leq n}$ as in the last part of the proof of Lemma 5.1. Let $\mathcal{V}$ be an irreducible component of dimension n of $\mathcal{P}_r$. For each $Z \in \Omega_r$, write $\pi^{-1} \cap \mathcal{V} = \{S^{(j)}(Z)\}_{j=1,\cdots,m}$ with m fixed for a generic choice of Z. Let $\sigma_{m,k}$ be the standard symmetric function in m variables of degree k. Then for each fixed (j,l), $\sigma_{m,k}(\xi_{jl}(S^{(1)}(Z)), \cdots, \xi_{jl}(S^{(m)}(Z)))$ is a holomorphic function with boundary value at $p \in M$: $\sigma_{m,k}(\xi_{jl}(S_p^a - p), \cdots, \xi_{jl}(S_p^a - p))$. Hence, it follows easily that there is only one irreducible component of maximum dimension in $\mathcal{P}$, which must also be single-sheeted and thus is biholomorpic to Ω_r through the projection map π away from a proper subvariety. Notice that the other irreducible components of $\mathcal{P}_r$ must have projection in Ω_r of positive codimension. By Proposition 3.6 (III) and the Riemann removable singularity theorem, we see the proof of Lemma 5.3. ∎

References

[Alx] H. Alexander, Proper holomorphic maps in $\mathbf{C}^n$, Indiana Univ. Math. Journal 26, 137-146 (1977).

[BER] M. S. Baouendi, P. Ebenfelt and L. Rothschild, Real Submanifolds in Complex Spaces and Their Mappings, Princeton Univ. Mathematics Series 47, Princeton University, New Jersey, 1999.

[BHR] S. Baouendi, X. Huang and L. Rothschild, Regularity of CR mappings between algebraic hypersurfaces, Invent. Math. 125, 13-36 (1996).

[BT] S. Baouendi and F. Treves, A property of the functions and distributions annihilated by a locally integrable system of complex vector fields. Ann. of Math. (2) 113 (1981), no. 2, 387–421.

[BK] D. Burns and S. Krantz, Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary, Jour. of AMS 7, 661-676 (1994).

[CaMo] H. Cao and N. Mok, Holomorphic immersions between compact hyperbolic space forms, Invent. Math. (100), 49–61, 1990.

[CM] S. S. Chern and J. K. Moser, Real hypersurfaces in complex manifolds, Acta Math. 133, 219-271 (1974).

[CS1] J. Cima and T. J. Suffridge, A reflection principle with applications to proper holomorphic mappings, Math Ann. 265, 489-500 (1983).

[CS2] J.Cima and T. J. Suffridge, Boundary behavior of rational proper maps, Duke Math. J. 60, 135-138 (1990).

[Da1] J. D'Angelo, Several Complex Variables and the Geometry of Real Hypersurfaces, CRC Press, Boca Raton, 1993.

[Da2] J. D'Angelo, Proper holomorphic mappings between balls of different dimensions, *Mich. Math. J.* 35, 83-90 (1988).

[Da3] J. D'Angelo, Inequalities from Complex Analysis, Carus Mathematical Monograph No. 28, Mathematics Association of America, 2002.

[DC] J. D'Angelo and D. Catlin, A stabilization theorem for Hermitian forms and applications to holomorphic mappings, Math Research Letters 3, 149-166 (1996).

[DL] J. D'Angelo and D. Lichtblau, Spherical space forms, CR mappings, and proper maps between balls, J. Geom. Anal. 2 (1992), no. 5, 391–415.

[Fa1] J. Faran, Maps from the two ball to the three ball, Invent. Math. 68, 441-475 (1982).

[Fa2] J. Faran, On the linearity of proper maps between balls in the lower dimensional case, Jour. Diff. Geom. 24, 15-17 (1986).

[EHZ1] P. Ebenfelt, X. Huang and D. Zaitsev, Rigidity of CR submanifolds of spheres, preprint,

2002.

[EHZ2] P. Ebenfelt, X. Huang and D. Zaitsev, The equivalence problem and rigidity for hypersurfaces embedded into hyperquadrics, Preprint, 2002. (*Amer. Jour. of Math.*, to appear).

[Fo1] F. Forstneric, Extending proper holomorphic mappings of positive codimension, Invent. Math., 95, 31-62 (1989).

[Fo2] F. Forstneric, A survey on proper holomorphic mappings, Proceeding of Year in SCVs at Mittag-Leffler Institute, Math. Notes 38 (1992), Princeton University Press, Princeton, N.J.

[HS] M. Hakim and N. Sibony, Fonctions holomorphes bornes sur la boule unite de $\mathbf{C}^n$. Invent. Math. 67 (1982), no. 2, 213–222.

[Hu1] X. Huang, On a linearity problem of proper holomorphic mappings between balls in complex spaces of different dimensions, Jour. of Diff. Geom. Vol (51) No. 1, 13-33 (1999).

[Hu2] X. Huang, On some problems in several complex variables and Cauchy-Riemann Geometry, Proceedings of ICCM (edited by S. T. Yau and L. Yang; December, 1998), AMS/IP studies in Advanced Mathematics, Vol 20, 383-396, 2001.

[Hu3]: X. Huang, Schwarz reflection principle in complex spaces of dimension two, Comm. in PDE 21 (1996), 1781-1828.

[Hu4]: X. Huang, Regularity of holomorphic correspondences and applications to the mapping problem, Preprint publically circulated in August-October of 1995. (This is the early version of [Hu3], whose main content was later published as Part II of [Hu3]).

[HJ1]: X. Huang and S. Ji, Mapping $\mathbf{B}^n$ into $\mathbf{B}^{2n-1}$, Invent. Math. 145, 219-250, 2001.

[HJ2]: X. Huang and S. Ji, Holomorphic mappings with partial linearity, in preparation.

[Low] E. Low, Embeddings and proper holomorphic mappings of strictly pseudoconvex domains into polydiscs and balls, Math Z. 190, 401-410(1985).

[Mok] Ngaiming Mok, Metric rigidity theorems on Hermitian locally symmetric manifolds. Series in Pure Mathematics, 6. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[MSY] N. Mok, Y. T. Siu and S. K. Yeung, Geometric superrigidity, Invent. Math. 113, 57-83, 1993.

[Mos] J. Moser, Analytic surfaces in $\mathbf{C}^2$ and their local hull of holomorphy. Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985), 397–410.

[Po] H. Poincaré, Les fonctions analytiques de deux variables et la représentation conforme, Ren. Cire. Mat. Palermo, II. Ser. 23, 185-220 (1907).

[Sp] M. Spivak, A comprehensive introduction to differential geometry. Vol. IV. Second edition. Publish or Perish, Inc., Wilmington, Del., 1979.

[Ste] B. Stensones, Proper holomorphic mappings between the balls, Jour. of Geom. Analysis 6,

317-339 (1996).
[Ta] N. Tanaka, On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables, J. Math. Soc. Japan 14, 397-429(1962).
[Tu] Zhen-Han Tu, Rigidity of proper holomorphic mappings between nonequidimensional bounded symmetric domains. Math. Z. 240 (2002), no. 1, 13–35.
[We1] S. Webster, On mapping an n-ball into an (n+1)-ball in the complex space, Pac. J. Math. 81, 267-272 (1979).
[We2] S. Webster, The rigidity of C-R hypersurfaces in a sphere. Indiana Univ. Math. J. 28 (1979), no. 3, 405–416.
[Yue] C. Yue, Dimension and rigidity of quasi-Fuchsian representations. Ann. of Math. (2) 143 (1996), no. 2, 331–355.

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