A generalization of a theorem of Moser

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Abstract

Let $M \subset \mathbb{C}^{n+1}$ $(n \geq 2)$ be a real analytic submanifold defined by an equation of the form: $w = |z|^2 + O(|z|^3)$, where we use $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ for the coordinates of $\mathbb{C}^{n+1}$. We first derive a pseudo-normal form for M near 0. We then use it to prove that $(M, 0)$ is holomorphically equivalent to the quadric $(M_\infty : w = |z|^2,\ 0)$ if and only if it can be formally transformed to $(M_\infty, 0)$. We also use it to give a necessary and sufficient condition when $(M, 0)$ can be formally flattened. The results generalize the work of Moser for the case of $n = 1$.

1 Introduction

Let $M \subset \mathbb{C}^{n+1}$ $(n \geq 1)$ be a submanifold. For a point $p \in M$, we define $CR(p)$ to be the CR dimension of M at p, namely, the complex dimension of the space $T_p^{(0,1)}M$. A point $p \in M$ is called a CR point if $CR(q) = CR(p)$ for $q(\in M) \approx p$. Otherwise, p is called a CR singular point of M. The local equivalence problem in Several Complex Variables is to find a complete set of holomorphic invariants of M near a fixed point $p \in M$. The investigation normally has quite different nature in terms of whether p being a CR point or a CR singular point. The CR case was first considered by Poincaré and Cartan. A complete set of invariants in the strongly pseudoconvex hypersurface case was give by Chern-Moser in [CM] (see the survey paper of Baouendi-Ebenfelt-Rothschild [BER1] and the lecture notes of the first author [Hu1] for many references along these lines). The study for the CR singular points first appeared in the paper of Bishop [Bis]. Further investigations on the precise holomorphic structure of M near a non-degenerate CR singular point, in the critical dimensional case of $dim_{\mathbb{R}} M = n+1$, can be found in the work of Moser-Webster [MW] and in the work of Moser [Mos], Gong [Gon1-2], Huang-Yin [HY], etc. (The reader can find many references in [Hu1] on this matter.)

Recently, there appeared several papers, in which CR singular points in the non-critical dimensional case were considered (see [Sto], [DTZ], [Cof], to name a few). In [Sto], among other things, Stolovitch introduced a set of generalized Bishop invariants for a non-degnerate general CR singular point, and established some of the results of Moser-Webster [MW] to the

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case of $dim_{\mathbb{R}} M > dim_{\mathbb{C}} \mathbb{C}^{n+1}$. In [DTZ], Dolbeault-Tomassini-Zaitsev introduced the concept of the elliptic flat CR singular points and studied global filling property by complex analytic varieties for a class of compact submanifold of real codimension two in $\mathbb{C}^{n+1}$ with exactly two elliptic flat CR singular points.

In this paper, we study the local holomorphic structure of a manifold M near a CR singular point p, for which we can find a local holomorphic change of coordinates such that in the new coordinates system, $p = 0$ and M near p is defined by an equation of the form: $w = |z|^2 + O(|z|^3)$. Here we use $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ for the coordinates of $\mathbb{C}^{n+1}$. Such a non-degenerate CR singular point has an intriguing nature that its quadric model has the largest possible symmetry. We will first derive a pseudo-normal form for M near p (see theorem 2.3). As expected, the holomorphic structure of M near p is influenced not only by the nature of the CR singularity, but also by the fact that (M, p) partially inherits the property of strongly pseudoconvex CR structures for $n > 1$. Unfortunately, as in the case of $n = 1$ first considered by Moser [Mos], our pseudo-normal form is still subject to the simplification of the complicated infinite dimensional formal automorphism group of the quadric $aut_0(M_\infty)$, where M_∞ is defined by $w = |z|^2$. Thus, our pseudo-normal form can not be used to solve the local equivalence problem. However, with the rapid iteration procedure, we will show in §4 that if all higher order terms in our pseudonormal form vanish, then M is biholomorphically equivalent to the model M_∞. Namely, we have the following:

Theorem 1: *Let $M \subset \mathbb{C}^{n+1}$ ($n \geq 1$) be a real analytic submanifold defined by an equation of the form: $w = |z|^2 + O(|z|^3)$. Then $(M, 0)$ is holomorphically equivalent to the quadric $(M_\infty, 0)$ if and only if it can be formally transformed to $(M_\infty, 0)$.*

One of the fundamental differences of our consideration here from the case of $n = 1$ is that a generic $(M, 0)$ can not be formally mapped into the Levi-flat hypersurface $Im(w) = 0$. As another application of the pseudo-normal form to be obtained in §2, we will give a necessary and sufficient condition when $(M, 0)$ can be formally flattened (see Theorem 3.5).

Theorem 1, in the case of $n = 1$, is due to Moser [Mos]. Indeed, our proof of Theorem 1 uses the approach of Moser in [Mos] and Gong in [Gon2], which is based on the rapidly convergent power series method. Convergence results along the lines of Theorem 1 near other type of CR singular points can be found in the earlier papers of Gong [Gon1] and Stolovitch [Sto].

2 A formal pseudo-normal form

We use $(z, w) = (z_1, \cdots, z_n, w)$ for the coordinates in $\mathbb{C}^{n+1}$ with $n \geq 2$ in all that follows. We first recall some notation and definitions already discussed in the previous papers of Stolovitch [St] and Dolbeault-Tomassini-Zaitsev [DTZ].

Let $(M, 0)$ be a formal submanifold of codimenion two in $\mathbb{C}^{n+1}$ with $0 \in M$ as a CR singular point and $T_0^{(1,0)} M = \{w = 0\}$. Then, M can be defined by a formal equation of the form:

$$w = q(z, \overline{z}) + o(|z|^2), \tag{2.1}$$

where $q(z,\overline{z})$ is a quadratic polynomial in $(z,\overline{z})$. We say that $0 \in M$ is a not-completely-degenerate CR singular point if there is no change of coordinates in which we can make $q \equiv 0$. Following Dolbeault-Tomassini-Zaitsev, we further say that 0 is a not-completely-degenerate flat CR singular point if we can make q real-valued after a linear change of variables.

Assume that 0 is a not-completely-degenerate flat CR singular point with $q(z,z) = A(z,\overline{z}) + B(z,\overline{z}) \in \mathbb{R}$ for each z. Here $A(z,\overline{z}) = \sum_{\alpha,\beta=1}^{n} a_{\alpha\overline{\beta}} z_\alpha \overline{z_\beta}$, $B(z,\overline{z}) = 2Re(\sum_{\alpha,\beta=1}^{n} b_{\alpha\beta} z_\alpha z_\beta)$. Then the assumption that $A(z,\overline{z})$ is definite is independent of the choice of the coordinates system. Suppose that A is definite. Then making use of the classical Takagi theorem, one can find a linear change of coordinates in (z,w) such that in the new coordinates, in the defining equation for $(M,0)$ of the form in (2.1), one has that

$$q(z,\overline{z}) = \sum_{\alpha=1}^{n} \{|z_\alpha|^2 + \lambda_\alpha(z_\alpha^2 + \overline{z_\alpha}^2)\},$$

where $0 \le \lambda_\alpha < \infty$ with $0 \le \lambda_1 \le \cdots \le \lambda_n < \infty$. In terms of Stolovitch, we call $\{\lambda_1, \cdots, \lambda_n\}$ the set of generalized Bishop invariants. When $0 \le \lambda_\alpha < 1/2$ for all α, we say that 0 is an elliptic flat CR singular point of M. Notice that $0 \in M$ is an elliptic flat CR singular point if and only if in a certain defining equation of M of the form as in (2.1), we can make $q(z,\overline{z}) > 0$ for $z \ne 0$. (Hence the definition coincides with the notion of elliptic flat Complex points in [DTZ].) When $\lambda_\alpha > 1/2$ for all α, we say $0 \in M$ is a hyperbolic flat CR singular point. Notice that, in the other case, we can always find a two dimenional linear subspace of $\mathbb{C}^{n+1}$ whose intersection with M has a parabolic complex tangent at 0. For a more general related notion on elliptcity and hyperbolicity, we refer the reader to the paper of Stolovitch [Sto].

In terms of the terminology above, the manifold in Theorem 1 has vanishing generalized Bishop invariants at the CR singular point. In [Gon1] [Sto], one finds the study on the related convergence problem in the other situations, where, among other non-degeneracy conditions, all the generalized Bishop invariants are assumed to be non-zero. The method studying CR singular points with vanishing Bishop invariants is different from that used in the non-vanishing Bishop invariants case (see [MW] [Mos] [Gon2] [Sto] [HY]).

We now return to the manifolds with only vanishing generalized Bishop invariants.

Let $E(z,\bar{z})$ (respectively, $f(z,w)$) be a formal power series in $(z,\bar{z})$ (respectively, in (z,w)) without constant term. We say $Ord\,(E(z,\bar{z})) \ge k$ if $E(tz,t\bar{z}) = O(t^k)$. Similarly, we say $Ord_{wt}\,(f(z,w)) \ge k$ if $f(tz,t^2w) = O(t^k)$. Set the weight of $z,\bar{z}$ to be 1 and that of w to be 2. For a polynomial $h(z,w)$, we define its weighted degree, denoted by $deg_{wt}h$, to be the degree counted in terms the weighted system just given. Write $E^{(t)}(z,\bar{z})$ and $f^{(t)}(z,w)$ for the sum of monomials with weighted degree t in the expansion of E and f at 0, respectively.

Write $u_k = \sum_{i=1}^{k} |z_i|^2$ for $1 \le k \le n$ and $v_k = \sum_{i=1}^{k-1} |z_i|^2 - |z_k|^2$ for $2 \le k \le n$. We also write $u = u_n = |z|^2$. In what follows, we make a convention that the sum $\sum_{p=j}^{l} a_p$ is defined to be 0 if $j > l$.

We start with the following elementary algebraic lemma:

Lemma 2.1: $Span_{\mathbb{C}}\{|z_1|^2, \cdots, |z_n|^2\} = Span\{u, v_2, \cdots, v_n\}$. Moreover, for each index i with $1 \leq i \leq n$, $|z_i|^2$ can be uniquely expressed as the following linear combination of $u,\ v_2, \cdots,\ v_n$:

$$\begin{cases} |z_1|^2 = 2^{1-n}\left(u + \sum\limits_{h=2}^{n} 2^{n-h} v_h\right), \\ |z_i|^2 = 2^{-(n+1-i)}\left(u + \sum\limits_{h=i+1}^{n} 2^{n-h} v_h - 2^{n-i} v_i\right) \text{ for } 2 \leq i \leq n. \end{cases} \tag{2.2}$$

Proof of Lemma 2.1: By a direct computation, we have

$$\begin{aligned}
&2^{1-n}\left(u + \sum_{h=2}^{n} 2^{n-h} v_h\right) = 2^{1-n}\left(\sum_{i=1}^{n}|z_i|^2 + \sum_{h=2}^{n} 2^{n-h}(\sum_{i=1}^{h-1}|z_i|^2 - |z_h|^2)\right) \\
&\quad = 2^{1-n}\left((1 + \sum_{h=2}^{n} 2^{n-h})|z_1|^2 + \sum_{j=2}^{n-1}(1 + \sum_{h=j+1}^{n} 2^{n-h} - 2^{n-j})|z_j|^2\right) \\
&\quad = 2^{1-n}(2^{n-1}|z_1|^2) = |z_1|^2; \\
&2^{-(n+1-i)}\left(u + \sum_{h=i+1}^{n} 2^{n-h} v_h - 2^{n-i} v_i\right) \\
&\quad = 2^{-(n+1-i)}\left(\sum_{i=1}^{n}|z_i|^2 + \sum_{h=i+1}^{n} 2^{n-h}(\sum_{j=1}^{h-1}|z_j|^2 - |z_h|^2) - 2^{n-i}(\sum_{j=1}^{i-1}|z_j|^2 - |z_i|^2)\right) \\
&\quad = 2^{-(n+1-i)}\left(\sum_{j=1}^{i-1}(1 + \sum_{h=i+1}^{n} 2^{n-h} - 2^{n-i})|z_j|^2 + (1 + \sum_{h=i+1}^{n} 2^{n-h} + 2^{n-i})|z_i|^2\right. \\
&\qquad \left. + \sum_{j=i+1}^{n}(1 + \sum_{h=j+1}^{n} 2^{n-h} - 2^{n-j})|z_j|^2\right) = |z_i|^2, \text{ for } i \geq 2.
\end{aligned}$$

Hence, we see that $\mathrm{span}_{\mathbb{C}}\{|z_1|^2, \cdots, |z_n|^2\} = \mathrm{span}_{\mathbb{C}}\{u, v_2, \cdots, v_n\}$. The uniqueness assertion in the lemma now is obvious. ∎

For a formal (or holomorphic) transformation $f(z, w)$ of $(\mathbb{C}^n, 0)$ to itself, we write

$$\begin{cases} f(z, w) = (f_1(z, w), \cdots, f_n(z, w)), \\ f_k(z, w) = \sum_{(i_1, \cdots, i_n)} f_{k,(I)}(w) z^I,\ I = (i_1, \cdots, i_n) \text{ and } z^I = z_1^{i_1} \cdots z_n^{i_n}. \end{cases} \tag{2.3}$$

Let $E(z, \bar{z})$ be a formal power series with $E(0) = 0$. We next prove the following:

Lemma 2.2: *$E(z, \bar{z})$ has the following expansion:*

$$E(z, \bar{z}) = \sum_{\{i_k \cdot j_k = 0,\ k = 1 \cdots, n\}} E_{(I,J)}(u, v_2, \cdots, v_n) z^I \bar{z}^J = \sum_{\{i_k \cdot j_k = 0,\ k = 1, \cdots, n\}} E^{(K)}_{(I,J)} z^I \bar{z}^J u^{k_1} {v_2}^{k_2} \cdots {v_n}^{k_n}. \tag{2.4}$$

Here and in what follows, we write $I = (i_1, \cdots, i_n)$*,* $J = (j_1, \cdots, j_n)$*,* $K = (k_1, \cdots, k_n)$*,* $z^I = z_1^{i_1} \cdots z_n^{i_n}$ *and* $\overline{z}^J = \overline{z}_1^{j_1} \cdots \overline{z}_n^{j_n}$*. Moreover, the coefficients* $E_{(I,J)}^{(K)}$ *are uniquely determined by* E*.*

Proof of Lemma 2.2: Since $\{|z_i|^2\}_{i=1}^n$ and $\{u, v_2, \cdots, v_n\}$ are the unique linear combinations of each other by Lemma 2.1, one sees the existence of the expansion in (2.4). Also, to complete the proof of Lemma 2.3, it suffices for us to prove the following statement:

$$\sum_{(I,J,K)\in A(N,N^*)} E_{(I,J)}^{(K)} z^I \overline{z}^J |z_1|^{2k_1} \cdots |z_n|^{2k_n} = 0 \text{ if and only if } E_{(I,J)}^{(K)} \equiv 0.$$

Here, we define $A(N, N^*) = \{(I, J, K) \in \mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}^n,\ i_l \cdot j_l = 0,\ i_l, j_l, k_l \geq 0 \text{ for } 1 \leq l \leq n,\ \sum_{l=1}^n (i_l + k_l) = N,\ \sum_{l=1}^n (j_l + k_l) = N^*\}$. Let $P = (p_1, \cdots, p_n)$ and $Q = (q_1, \cdots, q_n)$ with $p_1, \cdots, p_n, q_1, \cdots, q_n$ non-negative integers be such that $|P| = N, |Q| = N^*$. We define $A(N, N^*; P, Q) = \{(I, J, K) \in A(N, N^*) :\ i_l \cdot j_l = 0,\ i_l, j_l, k_l \geq 0, i_l + k_l = p_l,\ j_l + k_l = q_l, \text{ for } 1 \leq l \leq n\}$. Now, suppose that $\sum_{(I,J,K)\in A(N,N^*)} E_{(I,J)}^{(K)} z^I \overline{z}^J |z_1|^{2k_1} \cdots |z_n|^{2k_n} = 0$. We then get

$$\sum_{(I,J,K)\in A(N,N^*;P,Q)} E_{(I,J)}^{(K)} \equiv 0, \text{ for each } P,\ Q \text{ with } |P| = N,\ |Q| = N^*.$$

We next claim that there is at most one element in $A(N, N^*; P, Q)$. Indeed, $(I, J, K) \in A(N, N^*; P, Q)$ if and only if $i_l + k_l = p_l,\ j_l + k_l = q_l,\ i_l \cdot j_l = 0$, for $1 \leq l \leq n$. Now, if $i_l = 0$, then $k_l = p_l$. Since $j_l = q_l - p_l \geq 0$, thus this happens only when $q_l \geq p_l$. If $j_l = 0$, then $k_l = q_l$. Since $i_l = p_l - q_l \geq 0$, we see that this can only happen when $p_l \geq q_l$. Hence, we see that i_l, j_l are uniquely determined by p_l and q_l when $p_l \neq q_l$. When $p_l = q_l$, it is easy to see that $i_l = j_l = 0,\ k_l = q_l = p_l$. We thus conclude the argument for the claim. This completes the proof of Lemma 2.3. ■

We now let $M \subset \mathbb{C}^{n+1}$ be a formal submanifold defined by:

$$w = |z|^2 + E(z, \bar{z}) \tag{2.5}$$

where E is a formal power series in $(z, \bar{z})$ with $Ord(E) \geq 3$. We will subject (2.5) to the following formal power series transformation in (z, w):

$$\begin{cases} z' = F = z + f(z, w) & Ord_{wt}(f) \geq 2 \\ w' = G = w + g(z, w) & Ord_{wt}(g) \geq 3. \end{cases} \tag{2.6}$$

Write $e_j \in \mathbb{Z}^n$ for the vector whose component is 1 at the j^{th}-position and is 0 elsewhere. We next give a formal pseudo-normal form for $(M, 0)$ in the following theorem:

Theorem 2.3: *There exits a unique formal transformation of the form in (2.6) with the* *normalization*

$$\begin{cases} f_{i,(0)}(u)=0,\ \ 1\le i\le n;\\ f_{i,(e_j)}(u)=0\ \mathit{for}\ 1\le j<i\le n;\\ \ f_{1,(e_1)}(u)=0,\ \ Im\left(f_{i,(e_i)}(u)\right)=0\ \mathit{for}\ 2\le i\le n,\end{cases}\tag{2.7}$$

that transforms M to a formal submanifold defined in the following pseudo-normal form:

$$w'=|z'|^2+\varphi(z',\overline{z'}).\tag{2.8}$$

Here $\varphi=O(|z'|^3)$ and in the following unique expansion of φ,

$$\varphi=\sum_{i_l\cdot j_l=0,l=1,\cdots,n}\varphi_{(I,J)}z^I\overline{z}^J=\sum_{i_k\cdot j_k=0,\ k=1,\cdots,n}\varphi_{(I,J)}^{(K)}z^I\overline{z}^Ju^{k_1}{v_2}^{k_2}\cdots{v_n}^{k_n}.\tag{2.9}$$

we have, for any $k\ge 0,\ l\ge 1,\ \tau\ge 2$, the following normalization condition:

$$\begin{cases}\varphi_{(0,0)}^{(\tau e_1)}=0\ ;\\ Re(\varphi_{(0,0)}^{(le_1+e_i)})=0,\ \mathit{for}\ 2\le i\le n\ ;\\ \varphi_{(e_i,e_j)}^{(le_1)}=0,\ \mathit{for}\ i>j\ ;\\ \varphi_{(I,0)}^{(le_1)}=\varphi_{(0,I)}^{(le_1)}=\varphi_{(0,I)}^{(ke_1+e_j)}=0,\ \mathit{for}\ |I|\ge 1;\\ \varphi_{(I,e_h)}^{(ke_1)}=0,\ \mathit{for}\ h\ge 1,|I|\ge 2,i_h=0;\\ \varphi_{(0,I)}^{(0)}=\overline{\varphi_{(I,0)}^{(0)}},|I|>2.\end{cases}\tag{2.10}$$

Proof of Theorem 2.3: We need to prove that the following equation, with unknowns in $(f,g,\ \varphi)$, can be uniquely solved under the normalization conditions in (2.7) and (2.10):

$$w+g(z,w)=\sum_{i=1}^n\left(z_i+f_i(z,w)\right)\left(\bar{z}_i+\overline{f_i(z,w)}\right)+\varphi\left(z+f(z,w),\overline{z}+\overline{f(z,w)}\right).\tag{2.11}$$

Collecting terms of degree t in the above equation, we obtain for each $t\ge 3$ the following:

$$E^{(t)}(z,\bar{z})+g^{(t)}(z,u)=2Re\sum_{i=1}^n\left(\overline{z_i}f_i^{(t-1)}(z,u)\right)+\varphi^{(t)}(z,\bar{z})+I^{(t)}(z,\bar{z}),\tag{2.12}$$

where $I^{(t)}(z,\bar{z})$ is a homogeneous polynomial of degree t depending only on $g^{(\sigma)}$, $f^{(\sigma-1)}$, $\varphi^{(\sigma)}$ for $\sigma<t$. Thus, by an induction argument, we need only to uniquely solve the following equation under the above given normalization:

$$\Gamma(z,\bar{z})+g(z,u)=2Re\left(\sum_{i=1}^n\left(\overline{z_i}f_i(z,u)\right)\right)+\varphi(z,\bar{z}).\tag{2.13}$$

Indeed, if we can uniquely solve (2.13), then, we can start with (2.12) with $t=3$ and $\Gamma=E^{(3)}$. We then get $(F^{(2)},G^{(3)})$. Now, we transform M by $H_2=(z,w)+(F^{(2)},G^{(3)})$. Then the new manifold is normalized up to weighted order 3. Let $H = (F, G) = (z + O_{wt}(3), w + O_{wt}(4))$ be a normalized map and consider (2.12) with $t = 4$. We can then uniquely determine $(F^{(3)}, G^{(4)})$. Transforming the manifold by the map $H_2 = (z, w) + (F^{(3)}, G^{(4)})$, we get one which is normalized up to order 4. Now, by an induction, we can prove the existence part of Theorem 2.3. The uniqueness part of the Theorem follows also from the unique solvability of (2.13).

Expand Γ, φ as in (2.4) and (2.9) and expand f, g as in (2.3). Making use of Lemma 2.2 and comparing the coefficients in (2.13) of $z^I\overline{z}^J$ with $i_l \cdot j_l = 0,\ l = 1, \cdots, n$, we get the following system:

$$z^0\overline{z}^0: \quad -g_{(0)} + \sum_{i=1}^{n} 2Re\left(|z_i|^2 f_{i,(e_i)}\right) + \varphi_{(0,0)} = \Gamma_{(0,0)}; \tag{2.14}$$

$$z_j,\ \overline{z_j}: \quad \begin{cases} -g_{(e_j)} + \overline{f_{j,(0)}} + \sum_{i=1}^{n} |z_i|^2 f_{i,(e_i+e_j)} + \varphi_{(e_j,0)} = \Gamma_{(e_j,0)} \\ f_{j,(0)} + \sum_{i=1}^{n} |z_i|^2 \overline{f_{i,(e_i+e_j)}} + \varphi_{(0,e_j)} = \Gamma_{(0,e_j)} \end{cases} \quad \text{for } 1 \le j \le n; \tag{2.15}$$

$$z_i\overline{z_j}: \quad \begin{cases} f_{j,(e_i)} + \overline{f_{i,(e_j)}} + \varphi_{(e_i,e_j)} = \Gamma_{(e_i,e_j)} \\ \overline{f_{j,(e_i)}} + f_{i,(e_j)} + \varphi_{(e_j,e_i)} = \Gamma_{(e_j,e_i)} \end{cases} \quad \text{for } i \ne j; \tag{2.16}$$

$$z_i\overline{z}^J,\ z^J\overline{z_i}: \quad \begin{cases} \overline{f_{i,(J)}} + \varphi_{(e_i,J)} = \Gamma_{(e_i,J)} \\ f_{i,(J)} + \varphi_{(J,e_i)} = \Gamma_{(J,e_i)} \end{cases} \quad \text{for } |J| \ge 2, j_i = 0; \tag{2.17}$$

$$z^I,\ \overline{z}^I: \quad \begin{cases} -g_{(I)} + \sum_{i=1}^{n} \left(|z_i|^2 f_{i,(I+e_i)}\right) + \varphi_{(I,0)} = \Gamma_{(I,0)} \\ \sum_{i=1}^{n} \left(|z_i|^2 \overline{f_{i,(I+e_i)}}\right) + \varphi_{(0,I)} = \Gamma_{(0,I)} \end{cases} \quad \text{for } |I| \ge 2; \tag{2.18}$$

$$z^I\overline{z}^J: \quad \varphi_{(I,J)} = \Gamma_{(I,J)} \text{ for } |I|, |J| \ge 2,\ i_l \cdot j_l = 0,\ l = 1, \cdots, n. \tag{2.19}$$

Here we demonstrate in details how the system (2.18) is uniquely solved. The others are done similarly (and, in fact, more easily). We first substitute (2.2) to (2.18) and then collect coefficients of the zeroth order term, linear terms and higher order terms in $v_2, \cdots, v_n$, respectively, while taking u as a parameter. We obtain, by Lemma 2.2, the following:

$$\sum_k \Gamma_{(I,0)}^{(ke_1)} u^k = -g_{(I)}(u) + 2^{1-n} u f_{1,(I+e_1)} + \sum_{i=2}^{n} 2^{i-1-n} u f_{i,(I+e_i)} + \sum_k \varphi_{(I,0)}^{(ke_1)} u^k; \tag{2.20}$$

$$\sum_k \Gamma_{(I,0)}^{(ke_1+e_j)} u^k = 2^{1-j} f_{1,(I+e_1)} + \sum_{i=2}^{j-1} 2^{i-1-j} f_{i,(I+e_i)} - 2^{-1} f_{j,(I+e_j)} + \sum_k \varphi_{(I,0)}^{(ke_1+e_j)} u^k,\ j \ge 2; \tag{2.21}$$

$$\varphi_{(I,0)}^{(k_1e_1+k_2e_2+\cdots+k_ne_n)} = \Gamma_{(I,0)}^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}, k_2 + \cdots + k_n \ge 2; \tag{2.22}$$

$$\sum_k \Gamma^{(ke_1)}_{(0,I)} u^k = 2^{1-n} u\overline{f_{1,(I+e_1)}} + \sum_{i=2}^n 2^{i-1-n} u\overline{f_{i,(I+e_i)}} + \sum_k \varphi^{(ke_1)}_{(0,I)} u^k; \tag{2.23}$$

$$\sum_k \Gamma^{(ke_1+e_j)}_{(0,I)} u^k = 2^{1-j}\overline{f_{1,(I+e_1)}} + \sum_{i=2}^{j-1} 2^{i-1-j}\overline{f_{i,(I+e_i)}} - 2^{-1}\overline{f_{j,(I+e_j)}} + \sum_k \varphi^{(ke_1+e_j)}_{(0,I)} u^k,\ j \geq 2; \tag{2.24}$$

$$\varphi^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(0,I)} = \Gamma^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(0,I)},\ k_2+\cdots+k_n \geq 2. \tag{2.25}$$

Using the normalization in φ and letting $u=0$ in (2.20) (2.23), we get $\Gamma^{(0)}_{(I,0)} = -g_{(I)}(0) + \varphi^{(0)}_{(I,0)}$ and $\Gamma^{(0)}_{(0,I)} = \varphi^{(0)}_{(0,I)}$. By the normalization $\varphi^{(0)}_{(I,0)} = \overline{\varphi^{(0)}_{(0,I)}}$, we get $\varphi^{(0)}_{(I,0)} = \overline{\Gamma^{(0)}_{(I,0)}}$ and

$$g_{(I)}(0) = \overline{\Gamma^{(0)}_{(0,I)}} - \Gamma^{(0)}_{(I,0)}. \tag{2.26}$$

Sum up (2.24) with $j = 2,\cdots,n$ and then add it to (2.23). By the the normaliztaion condition $\varphi^{(le_1)}_{(0,I)} = \varphi^{(ke_+e_j)}_{(I,0)} = 0$ for $k \geq 0, l \geq 1$, we obtain the following:

$$f_{1,(I+e_1)}(u) = \sum_{k\geq 1} \overline{\Gamma^{(ke_1)}_{(0,I)}} u^{k-1} + \sum_{k\geq 0}\sum_{j=2}^n \overline{\Gamma^{(ke_1+e_j)}_{(0,I)}} u^k. \tag{2.27}$$

Subtracting the complex conjugate of (2.24) from (2.21), we obtain

$$\varphi^{(ke_1+e_j)}_{(I,0)} = \Gamma^{(ke_1+e_j)}_{(I,0)} - \overline{\Gamma^{(ke_1+e_j)}_{(0,I)}},\ j \geq 2,\ k \geq 0. \tag{2.28}$$

From (2.20) and (2.24), we can similarly get

$$g_{(I)}(u) = \sum_{k=0}^\infty \left(\overline{\Gamma^{(ke_1)}_{(0,I)}} - \Gamma^{(ke_1)}_{(I,0)}\right) u^k, \quad |I| \geq 2. \tag{2.29}$$

Back to the equation (2.24), we can inductively get:

$$f_{j,(I+e_j)}(u) = \sum_{k\geq 1} \overline{\Gamma^{(ke_1)}_{(0,I)}} u^{k-1} + \sum_{k\geq 0} \left(\sum_{i=0}^{n-j-1} \overline{\Gamma^{(ke_1+e_{n-i})}_{(0,I)}} - \overline{\Gamma^{(ke_1+e_j)}_{(0,I)}}\right) u^k \text{ for } 2 \leq j \leq n. \tag{2.30}$$

Similarly, we get from (2.14) the following

$$g_{(0)}(u) = \sum_{k\geq 2} \left(-\Gamma^{(ke_1)}_{(0,0)} u^k\right) - Re\left(\sum_{k\geq 1;\ j=2,\cdots,n} \Gamma^{(ke_1+e_j)}_{(0,0)} u^{k+1}\right); \tag{2.31}$$

$$f_{h,(e_h)}(u) = \frac{1}{2}\sum_{k\geq 1}\left(-\sum_{j=2}^{h-1} Re(\Gamma^{(ke_1+e_j)}_{(0,0)} u^k) - 2Re(\Gamma^{(ke_1+e_h)}_{(0,0)} u^k)\right),\ h \geq 2; \tag{2.32}$$

$$\varphi_{(0,0)} = \Gamma_{(0,0)} - \sum_{k\geq 2} \Gamma^{(ke_1)}_{(0,0)} u^k - Re(\sum_{k\geq 1, j=2,\cdots,n} \Gamma^{(ke_1+e_j)}_{(0,0)} u^k v_j). \tag{2.33}$$

From (2.15), we obtain the following:

$$f_{1,(e_1+e_j)}(u) = \sum_{k\geq 1} \overline{\Gamma^{(ke_1)}_{(0,e_j)}} u^{k-1} + \sum_{k\geq 0}\sum_{i=2}^{n} \overline{\Gamma^{(ke_1+e_i)}_{(0,e_j)}} u^k; \tag{2.34}$$

$$f_{i,(e_j+e_i)}(u) = \sum_{k\geq 1} \overline{\Gamma^{(ke_1)}_{(0,e_j)}} u^{k-1} + \sum_{k\geq 0} \left(\sum_{l=0}^{n-i-1} \overline{\Gamma^{(ke_1+e_{n-l})}_{(0,e_j)}} - \overline{\Gamma^{(ke_1+e_i)}_{(0,e_j)}} \right) u^k \text{ for } 2 \leq i \leq n; \tag{2.35}$$

$$g_{(e_j)}(u) = \sum_{k=1}^{\infty} \left(\overline{\Gamma^{(ke_1)}_{(0,e_j)}} - \Gamma^{(ke_1)}_{(e_j,0)} \right) u^k; \tag{2.36}$$

$$\varphi^{(ke_1+e_l)}_{(e_j,0)} = \Gamma^{(ke_1+e_l)}_{(e_j,0)} - \overline{\Gamma^{(ke_1+e_l)}_{(0,e_j)}},\ l \geq 2,\ k \geq 0; \tag{2.37}$$

$$\varphi^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(e_j,0)} = \Gamma^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(e_j,0)},\ k_2 + \cdots + k_n \geq 2; \tag{2.38}$$

$$\varphi^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(0,e_j)} = \Gamma^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(0,e_j)},\ k_2 + \cdots + k_n \geq 2. \tag{2.39}$$

From (2.16), we get

$$f_{i,(e_j)}(u) = \sum_{k=1}^{\infty} \Gamma^{(ke_1)}_{(e_j,e_i)} u^k,\ i < j; \tag{2.40}$$

$$\varphi^{ke_1}_{(e_i,e_j)} = \Gamma^{(ke_1)}_{(e_i,e_j)} - \overline{\Gamma^{(ke_1)}_{(e_j,e_i)}},\ i < j,\ k \geq 1; \tag{2.41}$$

$$\varphi^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(e_i,e_j)} = \Gamma^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(e_i,e_j)},\ \text{for } \mathrm{k}_2 + \cdots + \mathrm{k}_\mathrm{n} \geq 1. \tag{2.42}$$

From (2.17), we obtain

$$f_{i,(J)}(u) = \sum_{k\geq 0} \Gamma^{(ke_1)}_{(J,e_i)} u^k,\quad 1 \leq i \leq n, \tag{2.43}$$

$$\varphi^{(ke_1)}_{(e_i,J)} = \Gamma^{(ke_1)}_{(e_i,J)} - \overline{\Gamma^{(ke_1)}_{(J,e_i)}},\quad 1 \leq i \leq n, k \geq 0, \tag{2.44}$$

$$\varphi^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(J,e_i)} = \Gamma^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(J,e_i)},\ \text{for } \mathrm{k}_2 + \cdots + \mathrm{k}_\mathrm{n} \geq 1, \tag{2.45}$$

$$\varphi^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(e_i,J)} = \Gamma^{(k_1e_1+k_2e_2+\cdots+k_ne_n)}_{(e_i,J)},\ \text{for } \mathrm{k}_2 + \cdots + \mathrm{k}_\mathrm{n} \geq 1. \tag{2.46}$$

where $|J| \geq 2$ and $j_i = 0$.

Summarizing the solutions just obtained, we have the following formula: (One can also directly verify that they are indeed the solutions of (2.13) with the normalization conditions given in (2.7) and (2.10))

$$
\begin{aligned}
F_1(z,u) &= z_1 + f_1(z,u) = z_1 + \sum_{k\geq 0, j_1=0, |J|\geq 1} z^J \Gamma^{(ke_1)}_{(J,e_1)} u^k + \sum_{|I|\geq 1} z^{I+e_1} S^{(1)}_I, \\
F_h(z,u) &= z_h + f_h(z,u) = z_h + \tfrac{1}{2} z_h \sum_{k\geq 1} \Big(-\sum_{j=2}^{h-1} Re(\Gamma^{(ke_1+e_j)}_{(0,0)} u^k) - 2Re(\Gamma^{(ke_1+e_h)}_{(0,0)} u^k)\Big) \\
&\quad + \sum_{k\geq 1, i>h} z_i \Gamma^{(ke_1)}_{(e_i,e_h)} u^k + \sum_{k\geq 0, j_h=0, |J|\geq 2} z^J \Gamma^{(ke_1)}_{(J,e_h)} u^k + \sum_{|I|\geq 1} z^{I+e_h} S^{(h)}_I, \quad n\geq h\geq 2, \\
G(z,u) &= u + g(z,u) = u + \left(-\sum_{k\geq 2} \Gamma^{(ke_1)}_{(0,0)} u^k - Re\Big(\sum_{k\geq 1, j=2,\cdots,n} \Gamma^{(ke_1+e_j)}_{(0,0)} u^{k+1}\Big)\right) \\
&\quad + \sum_{k\geq 0, |I|\geq 1} z^I u^k \left(\overline{\Gamma^{(ke_1)}_{(0,I)}} - \Gamma^{(ke_1)}_{(I,0)} \right), \\
\varphi &= \Gamma(z,\bar z) + g(z,u) - 2Re\left(\sum_{i=1}^{n} (\overline{z_i} f_i(z,u)) \right),
\end{aligned}
\tag{2.47}
$$

where

$$
\begin{cases}
S^{(1)}_I = \sum_{k\geq 1} \overline{\Gamma^{(ke_1)}_{(0,I)}} u^{k-1} + \sum_{k\geq 0} \sum_{i=2}^{n} \overline{\Gamma^{(ke_1+e_i)}_{(0,I)}} u^k, \\
S^{(h)}_I = \sum_{k\geq 1} \overline{\Gamma^{(ke_1)}_{(0,I)}} u^{k-1} + \left(\sum_{k\geq 0} \sum_{i=0}^{n-h-1} \overline{\Gamma^{(ke_1+e_{n-i})}_{(0,I)}} u^k \right) - \sum_{k\geq 0} \overline{\Gamma^{(ke_1+e_h)}_{(0,I)}} u^k, \text{ for } 2\leq h\leq n.
\end{cases}
\tag{2.48}
$$

This completes the proof of Theorem 2.3. ∎

Let $(M,0)$ be as in (2.5). We say that $(M^*,0)$ is a formal pseudo-normal form for $(M,0)$ if $(M^*,0)$ is formally equivalent to $(M,0)$ and M^* is defined by $w=|z|^2+\varphi$ with φ satisfying the normalizations in (2.13). We notice that pseudo-normal forms of $(M,0)$ are not unique. Furthermore, we have the following observations:

Remark 2.4: **(A)**.The pseudo-normal form obtained in Theorem 2.3 contains information reflecting both the singular CR structure and partial strongly pseudoconvex CR structure at the point under study. For instance, the following submanifold in $\mathbb{C}^3$ is given in a pseudo-normal form:

$$
M:\ w = |z|^2 + 2Re \sum_{j_1+j_2\geq 3} \left(a_{j_1 j_2} z_1^{j_1} z_2^{j_2} \right) + \sum_{j_1\geq 2, j_2\geq 2} b_{j_1 \overline{j_2}} z_1^{j_1} \overline{z_2}^{j_2}. \tag{2.49}
$$

Here the harmonic terms $Re\sum_{j_1+j_2\geq 3}\left(a_{j_1j_2}z_1^{j_1}z_2^{j_2}\right)$ are presented due to the nature of CR singularity of M at 0, which may be compared with the Moser pseudo-normal form in [Mos] in the pure CR singularity setting. Typical mixed terms like $\sum_{j_1\geq 2, j_2\geq 2} b_{j_1\overline{j_2}} z_1^{j_1}\overline{z_2}^{j_2}$ are associated with the partial CR structure near 0, which can be compared with the Chern-Moser normal form in the pure CR setting [CM].

(B). Suppose that M is defined by a formal equation of the form: $w=|z|^2+E(z,\overline{z})$ with $Ord(E)\geq 3$ and $\overline{E(z,\overline{z})}=E(z,\overline{z})$. In the normalized map $H(z,w)=(F(z,w),G(z,w))$

transforming M into its normal form in Theorem 2.3, the w-component $G(z,u)$ is only a function in u and is formally real-valued, by the formula in (2.47). This is due to the fact that the Γ in (2.47) obtained from each induction stage in the process of the proof of Theorem 2.3 is formally real-valued. Hence, the φ in the pseudo-normalization of M obtained in Theorem 2.3 is also formally real-valued. However, fundamentally different from the two dimensional case, this is no longer true for a general M. Indeed, we will see in Theorem 3.5 that M can be formally flattened if and only if its pseudo-normal form is given by a formal real-valued function.

3 Normalization of holomorphic maps by automorphisms of the quadric

In this section, we first compute the isotropic automorphism group of the model space $M_\infty \subset \mathbb{C}^{n+1}$ defined by the equation: $w = \sum_{i=1}^n |z_i|^2$. Write $Aut_0(M_\infty)$ for the set of biholomorphic self-maps of $(M_\infty, 0)$. We have the following:

Proposition 3.1: *$Aut_0(M_\infty)$ consists of the transformations given in the following (3.1) or (3.2) :*

$$\begin{cases} z' = b(w)\dfrac{wa(w) - \frac{\langle z,\bar a(w)\rangle}{\langle a(w),\bar a(w)\rangle}a(w) + \sqrt{1-wa(w)\bar a(w)}\big(z - \frac{\langle z,\bar a(w)\rangle}{\langle a(w),\bar a(w)\rangle}a(w)\big)}{1-\langle z,\bar a(w)\rangle}U(w) \\ w' = b(w)\bar b(w)w \end{cases} \tag{3.1}$$

$$(z',w') = \big(b(w)zU(w), b(w)\bar b(w)w\big)\,. \tag{3.2}$$

where $a = (a_1,\cdots,a_n)$, $\sum_{j=1}^n a_j(0)\bar a_j(0) < 1$, $\langle z,\bar a\rangle = \sum\limits_{i=1}^n \bar a_i z_i$, $b(0)\neq 0$, $a(0)\neq 0$, $U(Re(w))$ is a unitary matrix and $a(w), b(w), U(w)$ are holomorphic in w.

Proof of Proposition 3.1: Write $w = x+\sqrt{-1}y$. Let $(F,G)\in Aut_0(M_\infty)$. Then $Im(G(z,|z|^2))\equiv 0$ for $z\approx 0$. Since M_∞ bounds a family of balls near 0 defined by

$$B_r = \left\{(z,w)\in\mathbb{C}^{n+1} : w = x+\sqrt{-1}y, y=0, x=r^2\geq |z^2|\right\}.$$

We see that $Im(G(z,x))\equiv 0$ for $z\approx 0$ and $x(\in\mathbb{R})\approx 0$. Therefore, $G(z,w) = G(w) = cw+o(w)$ $(c>0)$ is independent of z and takes real value when $w=x$ is real. Now $F(z,r^2)$ must be a biholomorphic map from $|z|^2 < r^2$ to $|z|^2 < G(r^2)$ for any $r>0$. Using the explicit expression for automorphisms of the unit ball (see [Rud]), we obtain either:

$$F(z,r^2) = \sqrt{G(r^2)}\frac{a(r) - \frac{\langle \frac{z}{r},\bar a(r)\rangle}{\langle a(r),\bar a(r)\rangle}a(r) + v\left(\frac{z}{r} - \frac{\langle \frac{z}{r},\bar a(r)\rangle}{\langle a(r),\bar a(r)\rangle}a(r)\right)}{1-\langle\frac{z}{r},\bar a(r)\rangle}U(r) \tag{3.3}$$

where $U(r)$ is a unitary matrix and $v = \sqrt{1-a(r)\bar a(r)}$, $a\neq 0$, or we have

$$F(z,r^2) = \sqrt{G(r^2)}(\frac{z}{r})U(r). \tag{3.4}$$

Write $G(x) = xb(x)\bar{b}(x)$ with $b(0) \neq 0$ and $b(w)$ holomorphic in w. In the case of (3.4), $F(z,x) = b(x)zU(r)e^{\sqrt{-1}\theta(x)}$ is real analytic, where $\theta(x)$ is real-valued real analytic function in x . Hence, $b(x)U(r)e^{\sqrt{-1}\theta(x)}$ is the Jacobian matrix of F in z. Since both $e^{\sqrt{-1}\theta(x)}$ and $b(x)(\neq 0)$ are real analytic for $x \approx 0$, we conclude that $U(r)$ is real analytic in x. Hence, $U(w)$ is also holomorphic in w. Still write $U(x)$ for $U(x)e^{i\theta(x)}$. We see the proof of Proposition 3.1 in the case of (3.2).

Suppose that $a \neq 0$. Still write $G(w) = wb(w)\bar{b}(w)$ with $b(0) \neq 0$. We have

$$F(z,r^2) = b(r^2)\frac{ra(r) - \frac{\langle z,\bar{a}(r)\rangle}{\langle a(r),\bar{a}(r)\rangle}a(r) + v\left(z - \frac{\langle z,\bar{a}(w)\rangle}{\langle a(r),\bar{a}(r)\rangle}a(r)\right)}{1 - \langle z, \frac{\bar{a}(r)}{r}\rangle}e^{i\theta}U(r)$$

Since $f(z,w)$ is holomorphic in (z,w) and $f(0,w) = b(w)\sqrt{w}a(\sqrt{w})U^*(\sqrt{w})$ with $U^* = e^{i\theta}U$, we see that $\sqrt{w}a(\sqrt{w})U^*(\sqrt{w})$ is holomorphic in w . In particular, $|a(\sqrt{w})|^2$ is real analytic in w. Moreover,

$$\frac{\partial F}{\partial z_i}(0,w) = b(w)\left(\frac{|a|^2 - v - 1}{|a|^2}\bar{a}_i a + ve_i\right)U^*(\sqrt{w})$$

is analytic. Since $\sqrt{w}a(\sqrt{w})U^*(\sqrt{w})$ is real analytic, we see that

$$\left(\frac{|a|^2 - v - 1}{|a|^2}\bar{a}_i a + ve_i\right)U^*(\sqrt{w})\overline{U^*(\sqrt{w})}^t\overline{a(\sqrt{w})}^t r = \left((|a|^2 - v - 1) + v\right)r\bar{a}_i$$

is real analytic, too. Here $(\cdot)^t$ denotes the matrix transpose. Since $(|a|^2 - v - 1) + v = |a|^2 - 1$ is real analytic, we conclude that both ra_i and a_i/r are real analytic in w. Since both $\sqrt{w}a(\sqrt{w})U^*(\sqrt{w})$ and ra_i are real analytic, we see that $U^*(\sqrt{w})$ is real analytic. Still denote a for a/r, we further obtain the following with the given properties stated in the Proposition:

$$\begin{cases} F(z,w) = b(w)\frac{wa(w) - \frac{\langle z,\bar{a}(w)\rangle}{\langle a(w),\bar{a}(w)\rangle}a(w) + \sqrt{1 - wa(w)\bar{a}(w)}\left(z - \frac{\langle z,\bar{a}(w)\rangle}{\langle a(w),\bar{a}(w)\rangle}a(w)\right)}{1 - \langle z,\bar{a}(w)\rangle}U^*(w) \\ G(w) = b(w)\bar{b}(w)w. \end{cases}$$

This completes the proof of Proposition 3.1. ■

Remark 3.2: In Proposition 3.1, if we let $a(w), b(w), U(w)$ be formal power series in w with $a(0)$, $b(0) \neq 0$ and $\langle a(0), \overline{a}(0)\rangle < 1$, $U(x) \cdot U(x)^t = I$, then (3.1) and (3.2) give formal automorphisms of M_∞, which are not convergent. Write the set of automorphisms obtained in this way as $aut_0(M_\infty)$. One may prove that $aut_0(M_\infty)$ consists of all the formal automorphisms of $(M_\infty, 0)$.

We now suppose that $H = (F,G)$ is a formal equivalence self-map of $(\mathbb{C}^{n+1}, 0)$, mapping a formal submanifold of the form $w = |z|^2 + O(|z|^3)$ to a submanifold of the form $w = |z|^2 + O(|z|^3)$. The following lemma shows that we can always normalize H by composing it from the left with an element from $aut_0(M_\infty)$ to get a normalized mapping. This fact will be used in the proof of Theorem 1. In what follows, we set $v(g,a) = \sqrt{1 - g \cdot a(g) \cdot \bar{a}(g)}$.

Lemma 3.3: *There exists a unique automorphism $T \in aut_0(M_\infty)$ such that $T \circ H$ satisfies the normalized condition in (2.7). When H is biholomorphic, $T \in Aut_0(M_\infty)$*

Proof of Lemma 3.3: First, it is easy to see that by composing an automorphism of the form $w' = |c|^2 w,\ z' = czU$, we can assume that $F = z + O_{wt}(2)$ and $G = w + O_{wt}(3)$ (see [Hu1]). Here c is a non-zero constant and U is a certain $n \times n$-unitary matrix.

Let $b(w) = 1, a_j = \alpha_j(w), a_1 = \cdots = a_{j-1} = a_{j+1} = \cdots = a_n = 0$, and $U = I$ in (3.1). We get the following automorphism of M_∞

$$T_j = \left(\frac{v(w,\alpha_j)z_1}{1-\bar{\alpha}_j z_j}, \cdots, \frac{v(w,\alpha_j)z_{j-1}}{1-\bar{\alpha}_j z_j}, \frac{z_j - w\alpha_j}{1-\bar{\alpha}_j z_j}, \frac{v(w,\alpha_j)z_{j+1}}{1-\bar{\alpha}_j z_j}, \cdots, \frac{v(w,\alpha_j)z_n}{1-\bar{\alpha}_j z_j}, w \right).$$

Write

$$\begin{cases} H_j = ({}_{(j)}F, {}_{(j)}G) = T_j \circ T_{j-1} \circ \cdots \circ T_1 \circ H,\ H_0 = H; \\ \alpha_j = \frac{{}_{(j-1)}F_{j,(0)}(u)}{{}_{(j-1)}G_{(0)}(u)} \circ \left({}_{(j-1)}G_{(0)}(u)\right)^{-1}. \end{cases} \tag{3.5}$$

Then a direct computation shows that $({}_{(j)}F)_{i,(0)}(u) = 0$ for $1 \le i \le j$. In particular, we have $({}_{(j)}F)_{i,(0)}(u) = 0$ for all $1 \le i \le n$.
Still write H for H_n. Next, for $i < j$, let $b(w) = 1, a = 0$, and let

$$U^i_j = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & \cos(\theta^i_j) & 0 & -\sin(\theta^i_j) & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & \sin(\theta^i_j) & 0 & \cos(\theta^i_j) & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}$$

in (3.1), where $\cos(\theta^i_j)$ is at the i^{th} row and the j^{th} column. Then we get an automorphism T^i_j. Set

$$H^i_j = ({}^i_jF, {}^i_jG) = T^i_j \circ \cdots \circ T^i_{i+1} \circ T^{i-1}_n \circ \cdots T^{i-1}_i \circ \cdots \circ T^1_n \circ \cdots \circ T^1_2 \circ H,$$
$$\theta^i_j = \begin{cases} \tan^{-1}\left(\frac{({}^{i-1}_nF)_{j,(e_i)}}{({}^{i-1}_nF)_{i,(e_i)}} \right) \circ \left({}^{i-1}_nG_{(0)}(w)\right)^{-1}, & j = i+1; \\ \tan^{-1}\left(\frac{({}^i_{j-1}F)_{j,(e_i)}}{({}^i_{j-1}F)_{i,(e_i)}} \right) \circ \left({}^i_{j-1}G_{(0)}(w)\right)^{-1}, & j \ne i+1. \end{cases} \tag{3.6}$$

Then we can inductively prove that H^i_j satisfies

$$({}^i_jF)_{(0)} = 0\ ,\ ({}^i_jF)_{k,(e_l)} = 0 \text{ for } l = i, i+1 \le k \le j \text{ or } l < i, l+1 \le k \le n.$$

In particular, we see that H^{n-1}_n satisfies $({}^{n-1}_nF)_{(0)} = 0$, $({}^{n-1}_nF)_{i,(e_j)} = 0$ for $1 \le j < i \le n$.
Still write H for H^{n-1}_n and set $H' = T \circ H = (F', G')$ with

$$T = \left(d(w)z, d(w)\bar{d}(w)w\right)\ ,\ d = \frac{1}{F_{1,(e_1)}(w)} \circ \left(G_{(0)}(w)\right)^{-1}.$$

Then H' satisfies

$$(F')_{(0)} = 0\ ,\ (F')_{1,(e_1)} = 1\ ,\ (F')_{i,(e_j)} = 0 \text{ for } 1 \le j < i \le n.$$

At last, a composition from the left with the rotation map as follows:

$$\hat{T} = (z_1, \beta_2 z_2, \cdots, \beta_n z_n, w),\ \beta_i = \frac{(\bar{F}')_{i,(e_i)}(w)}{\sqrt{(F')_{i,(e_i)}(w)\cdot(\bar{F}')_{i,(e_i)}(w)}} \circ \left(G'_{(0)}(w)\right)^{-1}$$

makes H' satisfy the normalization condition (2.7). This proves the existence part of the lemma.

Next, suppose both $H = (F, G) = (z + O_{wt}(2), w + O_{wt}(3))$ and $\hat{H} = (\hat{F}, \hat{G}) = T \circ H = (z + O_{wt}(2), w + O_{wt}(3))$ satisfy the normalization condition (2.7). Here T is an automorphism of M_∞. Then T must be of the form in (3.2), for $T(0, w) = 0$. Hence,

$$T = (b(w)zU(w), b(w)\bar{b}(w)w).$$

By the normalization condition (2.7) on H, $\hat{H}$, we have

$$\begin{pmatrix} 1 & & 0 \\ & \hat{F}_{2,(e_2)} & \\ & & \ddots \\ & * & & \hat{F}_{n,(e_n)} \end{pmatrix} = b\left(G_{(0)}(w)\right) \begin{pmatrix} 1 & & 0 \\ & F_{2,(e_2)} & \\ & & \ddots \\ & * & & F_{n,(e_n)} \end{pmatrix} U(G_0(w)). \tag{3.7}$$

with $U(x)$ unitary and $Im(\hat{F}_{i,(e_i)}(0, u)) = Im(F_{i,(e_i)}(0, u)) = 0$. Considering the norm of the first row of the right hand side, we get $b(G_{(0)}(w)) \cdot \bar{b}(G_{(0)}(w)) = 1$ in case $G_{(0)}(w) = \overline{G_{(0)}(w)}$. Since $G_0(w) = w + o(w)$, this implies that $b(w)\bar{b}(w) \equiv 1$ and thus $T = (b(w)zU(w), w)$. Write

$$b(w)U(w) = \widetilde{U}(w) = \begin{pmatrix} u_{11} & \cdots & u_{nn} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn}. \end{pmatrix}.$$

We notice that $\widetilde{U}$ is a lower triangular matrix and is unitary when $w = x$. Thus we have $u_{ii}(w)\overline{u_{ii}}(w) = 1$ and $u_{ij} = 0$ for $i \neq j$. Notice that

$$u_{11} \equiv 1,\ \hat{F}_{i,(e_i)}(w) = u_{ii}(w) \cdot F_{i,(e_i)}(w) \text{ for } 2 \leq \text{i} \leq \text{n}.$$

Since $\hat{F}_{i,(e_i)}(x), F_{i,(e_i)}(x) = 1 + o(x)$ are real, we get $u_{ii}(x) = 1$. This proves the uniqueness part of the lemma. ■

Lemma 3.4: *Suppose that H with $H(0) = 0$ is an equivalence map from $w = |z|^2 + \varphi(z, \bar{z})$ to $w' = |z'|^2 + \varphi'(z', \bar{z'})$. Here φ and φ' are normalized as in (2.10). Let s, s' be the lowest order of vanishing in φ and φ', respectively, then $s = s'$.*

Proof of Lemma 3.4: We seek for a contradiction if $s \neq s'$. Assume, for instance, that $s < s'$. Let T be an automorphism of M_∞ with $T \circ H$ being normalized as in (2.7). Suppose that T transforms $w' = |z'|^2 + \varphi'(z', \bar{z'})$ to $w'' = |z''|^2 + \varphi''(z'', \overline{z''})$ with s'' the lowest vanishing order for

φ''. We claim that $s' = s''$. Suppose not. We assume, without loss of generality, that $s' < s''$. Write the linear part of T (in (z', w')) as $(z'' = z'B + Dw', w'' = dw')$ with $B \in GL(n, \mathbb{C})$, $d \neq 0$. Then a direct computation shows that

$$\varphi''^{(s')}(z'B, \overline{z'B}) = d \cdot \varphi'^{(s')}(z', z').$$

This is a contradiction.

Now, $T \circ H$ transforms $w = |z|^2 + \varphi$ to $w'' = |z''|^2 + \varphi''$ with $T \circ H$, φ being normalized as in (2.7) and (2.10), respectively. Also $s < s''$. we see that $T \circ H$ transforms $w = |z|^2 + \varphi^{(s)}$ to $w = |z|^2$, moduling $O(|(z_1, \cdots, z_n)|^{s+1})$. This contradicts the uniqueness part of Theorem 2.3. The proof of Lemma 3.4 is complete. ■

We say that a formal submanifold $(M, 0)$ of real dimension $2n$ defined by (2.5) can be formally flattened if there is a formal change of coordinates $(z', w') = H(z, w)$ with $H(0) = 0$ such that in the new coordinates $(M, 0)$ is defined by a formal function of the form $w' = E^*(z', \overline{z'})$ with $E^*(z', \overline{z'}) = \overline{E^*(z', \overline{z'})}$. We also say a pseudo-normal form of $(M, 0)$ given by $w = |z|^2 + \varphi(z, \overline{z})$ with φ satisfying the normalizations in (2.10) is a flat pseudo-normal form if φ is formally real-valued. An immediate application of Lemma 3.3 and Remark 2.4 (b) is that if $(M, 0)$ has a flat pseudo-normal form, then all of its other pseudo-normal forms are flat. Indeed, for a given pseudo-normal form of $(M, 0)$, there is a formal equivalence map H mapping it into $Imw = 0$. Now, by Lemma 3.3, we can compose H with an element T of $aut_0(M_\infty)$ to normalize H. Next, since T maps any flattened submanifold to a flattened submanifold, there is a formal transformation H^* such that $H^* \circ T \circ H$ maps the pseudo-normal form given at the beginning to a flat one. On the other hand, since $H^* \circ T \circ H$ satisfies the normalizations in (2.7), by Theorem 2.3, we see that $H^* \circ T \circ H = id$ and two pseudo-normal forms are the same. Summarizing the above, we proved the following:

Theorem 3.5: *Let $(M, 0)$ be a formal submanifold defined by an equation of the form: $w = |z|^2 + E(z, \overline{z})$ with $E = O(|z|^3)$. Then the following statements are equivalent:*
(I). $(M, 0)$ can be flattened
(II). $(M, 0)$ has a flat pseudo-normal form. Namely, M has a pseudo-normal form given by an equation of the form: $w' = |z'|^2 + \varphi(z', \overline{z'})$ with φ satisfying the normalizations in (2.10) and the reality condition $\varphi(z', \overline{z'}) = \overline{\varphi(z', \overline{z'})}$.
(III). Any pseudo-normal form of $(M, 0)$ is flat.

Remark 3.6: By Theorem 3.5, we see that M defined in (2.49) can be formally flattened if and only if $b_{i\bar{j}} = \overline{b_{j\bar{i}}}$ for all i, j.

4 Proof of Theorem 1

We now give a proof of Theorem 1 by using the rapidly convergent power series method. We let M be defined by

$$w = \Phi(z, \bar{z}) = |z|^2 + E(z, \bar{z}) \tag{4.1}$$

where $E(z, \xi)$ is holomorphic near $z = \xi = 0$ with vanishing order ≥ 3. Assume that $H = (F, G) = (z + f, w + g)$ is a formal map satisfying the normalization condition in (2.6). We define

$$R = (r_1, r_2, \cdots, r_n) = (2^{-\frac{n-2}{2}} r, 2^{-\frac{n-2}{2}} r, 2^{-\frac{n-3}{2}} r \cdots, 2^{-\frac{1}{2}} r, r). \tag{4.2}$$

Then $|R|^2 = 2^{-(n-2)} r^2 + \sum_{h=2}^{n} (2^{-\frac{n-i}{2}} r)^2 = 2r^2$. Define the domains:

$$\begin{array}{l} \Delta_r = \{(z, w) : |z_i| < r_i, |w| < 2r^2\}, \\ D_r = \{(z, \xi) : |z_i| < r_i, |\xi_i| < r_i \text{ for } 1 \leq i \leq n\}. \end{array} \tag{4.3}$$

When $E(z, \xi)$ is defined over $\overline{D_r}$, we set the norm of $E(z, \bar{z})$ on D_r by

$$\|E\|_r = \sup_{(z,\xi) \in D_r} |E(z, \xi)|. \tag{4.4}$$

Also for a holomorphic map $h(z, w)$ defined on $\overline{\Delta_r}$, we define

$$|h|_r = \sup_{(z,w) \in \Delta_r} |h(z, w)|. \tag{4.5}$$

After a scaling transformation $(z, \xi, w) \longrightarrow (az, a\xi, a^2 w)$, we may assume that E is holomorphic on $\overline{D_1}$ with $|E|_1 \leq \eta$ for a given small $\eta > 0$.

Suppose that H maps M to the quadric $w' = |z'|^2$. Then we have the following equation:

$$E(z, \bar{z}) + g(z, \Phi) = 2Re\Big(\sum_{i=1}^{n} \bar{z}_i f_i(z, \Phi)\Big) + |f(z, \Phi)|^2. \tag{4.6}$$

We consider the following linearized equation of (4.6) with (f, g, φ) as its unknowns:

$$E(z, \bar{z}) = -g(z, u) + 2Re\Big(\sum_{i=1}^{n} \bar{z}_i f_i(z, u)\Big) + \varphi(z, \bar{z}), \tag{4.7}$$

where φ satisfies (2.10). The unique solution of (4.7) is given in the formula (2.47). However, we will make a certain truncation to (f, g, φ) to faciliate the estimates. Suppose that $Ord(E) \geq d \geq 3$. Set

$$\begin{cases} f = \hat{f} + O_{wt}(2d-3), \ deg_{wt}(\hat{f}) \leq 2d-4, \\ g = \hat{g} + O_{wt}(2d-2), \ deg_{wt}(\hat{g}) \leq 2d-3. \end{cases} \tag{4.8}$$

Define

$$\hat{F} = z + \hat{f} \ , \ \hat{G} = w + \hat{g}, \ \hat{H} = (\hat{F}, \hat{G}).$$

Write $\Theta = (\hat{F}, \hat{G})$ and write

$$\hat{\varphi}(z, \bar{z}) = E(z, \bar{z}) + \hat{g}(z, u) - 2Re\Big(\sum_{i=1}^{n} \bar{z}_i \hat{f}_i(z, u)\Big). \tag{4.9}$$

Then $\hat{\varphi}(z,\bar{z})-\varphi(z,\bar{z}) = O(|z|^{2d-2})$. Assume that $M' = \Theta(M)$ is defined by $w' = |z'|^2+E'(z',\bar{z'})$. Choose r', σ, ϱ, r to be such that

$$\frac{1}{2} < r' < \sigma < \varrho < r \le 1,\ \varrho = \frac{1}{3}(2r'+r),\ \sigma = \frac{1}{3}(2r'+\varrho).$$

As in the paper of Moser [Mos], the following lemma will be fundamental for applying the rapid iteration procedure of Moser to prove Theorem 1.

Lemma 4.1: *Let $M : w = |z|^2 + E(z,\bar{z})$ be as in Theorem 1. Suppose that $Ord(E) \ge d$. Let $\hat{H}$ and E' be defined above. Then $Ord(E') \ge 2d-2$.*

Proof of Lemma 4.1: Making use of (4.9), we have

$$E'(z',\overline{z'}) = \Big(\hat{g}(z,\Phi)-\hat{g}(z,u)\Big) - 2Re\Big(\sum_{i=1}^{n}\overline{z_i}(\hat{f}_i(z,\Phi)-\hat{f}_i(z,u))\Big) - |\hat{f}(z,\Phi)|^2 + \hat{\varphi}(z,\bar{z}). \tag{4.10}$$

Since $Ord(E) \ge d$, by (2.47) and (2.48), we see that $Ord(\hat{f}) \ge d-1$ and $Ord(\hat{g}) \ge d$. Hence, we have

$$\begin{aligned} &Ord\Big(\hat{g}(z,\Phi)-\hat{g}(z,u)\Big) \ge \min\{(d-1)+d, 2d-2\} = 2d-2,\\ &Ord\Big(\hat{f}_i(z,\Phi)-\hat{f}_i(z,u)\Big) \ge \min\{(d-2)+d, 2d-3\} = 2d-3,\\ &Ord\Big(\Big|\hat{f}(z,\Phi)\Big|^2\Big) \ge 2(d-1) = 2d-2. \end{aligned}$$

Thus $Ord(E'-\hat{\varphi}) \ge 2d-2$. By Lemma 3.3 and the assumption that $w = |z|^2+E$ is formally equivalent to $w = |z|^2$, we have $s = \infty$. Hence we have $Ord(\varphi) \ge 2d-2$. The lemma follows.■

Before proceeding to the estimates of the solution given in (2.47), we need the following lemma:

Lemma 4.2: *If E is holomorphic in $\overline{D_r}$, then we have*

$$|E^{(ke_1)}_{(I,T)}| \le \tfrac{(k+2)^n\|E\|_r}{R^{I+T}\cdot(2r^2)^k}\ ,\ |E^{(ke_1+e_j)}_{(I,T)}| \le \tfrac{2^n(k+2)^n\|E\|_r}{R^{I+T}(2r^2)^{k+1}}.$$

Proof of Lemma 4.2: We here give the estimates for $|E^{(ke_1)}_{(0,I)}|$, $|E^{(ke_1+e_j)}_{(0,I)}|$. The others can be done similarly. Suppose that $E = \sum a_{i_1\cdots i_n j_1\cdots j_n} z_1^{i_1}\cdots z_n^{i_n}\overline{z_1}^{j_l}\cdots\overline{z_1}^{j_l}$. Then by (2.3), we have

$$\begin{aligned} E_{(0,I)} &= \sum_J a_{j_1\cdots j_n(i_1+j_1)\cdots(i_n+j_n)}|z_1|^{2j_1}\cdots|z_n|^{2j_n}\\ &= \sum_J a_{J(I+J)}\left(2^{1-n}(u+\sum_{i=2}^{n}2^{n-i}v_i)\right)^{j_1}\Pi_{h=2}^{n}\left(2^{h-n-1}(u+\sum_{i=h+1}^{n}2^{n-i}v_i-v_h)\right)^{j_h}\\ &= \sum_J a_{J(I+J)}2^{-\left((n-1)j_1+\sum\limits_{h=2}^{n}(n-h+1)j_h\right)}\Big(u^{|J|}+\sum_{k=2}^{n}2^{n-k}\left(\sum_{h=1}^{k-1}j_h-j_k\right)u^{|J|-1}v_k\\ &\quad +O\left(|(v_2,\cdots,v_n)|^2\right)\Big). \end{aligned}$$

Thus we obtain

$$
\begin{aligned}
E^{(ke_1)}_{(0,I)} &= \sum_{|J|=k} a_{J(I+J)} 2^{-\left((n-1)j_1+\sum\limits_{h=2}^{n}(n-h+1)j_h\right)},\\
E^{(ke_1+e_l)}_{(0,I)} &= \sum_{|J|=k+1} a_{J(I+J)} 2^{-\left((n-1)j_1+\sum\limits_{h=2}^{n}(n-h+1)j_h\right)} 2^{n-l}\left(\sum_{h=1}^{l-1} j_h - j_l\right).
\end{aligned}
\tag{4.11}
$$

By the Cauchy estimates, we get

$$
\begin{aligned}
|E^{ke_1}_{(0,I)}| &= |\sum_{|J|=k} a_{J(I+J)} 2^{-\left((n-1)j_1+\sum\limits_{h=2}^{n}(n-h+1)j_h\right)}|\\
&\le \sum_{|J|=k} \frac{\|E\|_r}{R^{I+2J}} 2^{-\left((n-1)j_1+\sum\limits_{h=2}^{n}(n-h+1)j_h\right)}\\
&= \sum_{|J|=k} \frac{\|E\|_r}{R^I} \frac{2^{-((n-1)j_1+(n-1)j_2+\cdots+j_n)}}{(2^{2-n}r^2)^{j_1}\cdot(2^{2-n}r^2)^{j_2}\cdots(r^2)^{j_n}}\\
&\le \frac{(k+1)^n\|E\|_r}{R^I\cdot(2r^2)^k},\\
|E^{ke_1+e_l}_{(0,I)}| &= |\sum_{|J|=k+1} a_{J(I+J)} 2^{-\left((n-1)j_1+\sum\limits_{h=2}^{n}(n-h+1)j_h\right)} 2^{n-l}\left(\sum_{h=1}^{l-1} j_h - j_l\right)|\\
&\le \sum_{|J|=k+1} \frac{\|E\|_r}{R^{I+2J}} 2^{-\left((n-1)j_1+\sum\limits_{h=2}^{n}(n-h+1)j_h\right)} 2^{n-l}|\sum_{h=1}^{l-1} j_h - j_l|\\
&\le \sum_{|J|=k+1} \frac{\|E\|_r}{R^I\cdot(2r^2)^{k+1}} 2^n(k+1)\\
&= \frac{2^n(k+2)^n\|E\|_r}{R^I\cdot(2r^2)^{k+1}}.
\end{aligned}
$$

Here we have used the fact that

$$
\sharp\{(j_1,j_2,\cdots,j_n)\in\mathbb{Z}^n : j_h\ge 0 \text{ for } 1\le h\le n,\ j_1+j_2+\cdots+j_n=k\}\le (k+1)^{n-1}.
$$

This completes the proof of Lemma 4.2. ■

To carry out the rapid iteration procedure, we need the following estimates of the solution given by (2.47) for the equation (4.7).

Proposition 4.3: *Suppose that $w=|z|^2+E(z,\bar z)$ is formally equivalent to M_∞ with E holomorphic over $\overline{D_r}$ and $Ord(E)\ge d$. Then the solution given in (2.47) satisfies the following estimates:*

$$
\begin{aligned}
|\hat f_h|_\varrho, |\hat g|_\varrho &\le \frac{C(n)(2d)^{2n}\|E\|_r}{r-\varrho}\left(\frac{\varrho}{r}\right)^{d-1},\\
|\nabla \hat f_h|_\varrho, |\nabla\hat g|_\varrho &\le \frac{C(n)(2d)^{2n}\|E\|_r}{(r-\varrho)^3}\left(\frac{\varrho}{r}\right)^{\frac{d-1}{2}},\\
|\hat\varphi|_\varrho &\le \frac{(2d)^{2n}\|E\|_r}{(r-\varrho)^{2n}}\left(\frac{\varrho}{r}\right)^{2d-2},
\end{aligned}
\tag{4.12}
$$

where $C(n) = 3^3 n(n+1)2^{n+3}$.

Proof of Proposition 4.3: Notice that by the definition of $\hat{f}$ given in (4.8), we have $Ord(\hat{f}) \geq d-1$ and $deg_{wt} \leq 2d-4$. In terms of (2.47), we can write, for $2 \leq h \leq n$,

$$\hat{f}_h = A_1 + A_2 + A_3 + A_4,$$

where

$$\begin{aligned}
A_1 &= \sum_{d\leq 2k+2\leq 2d-3} \tfrac{1}{2} z_h \left(-\textstyle\sum_{j=2}^{h-1} Re(E^{(ke_1+e_j)}_{(0,0)} u^k) - 2Re(E^{(ke_1+e_h)}_{(0,0)} u^k)\right), \\
A_2 &= \sum_{i>h, d\leq 2k+2\leq 2d-3} z_i E^{(ke_1)}_{(e_i,e_h)} u^k, \ \ A_3 = \sum_{|J|\geq 1, d\leq |J|+2k+1\leq 2d-3} z^J E^{(ke_1)}_{(J,e_h)} u^k, \\
A_4 &= \sum_{|I|\geq 1, d\leq |I|+2k\leq 2d-3} z^{I+e_h} u^{k-1} \overline{E^{(ke_1)}_{(0,I)}} + \sum_{|I|\geq 1, d\leq |I|+2k+2\leq 2d-3} \sum_{i=0}^{n-h-1} z^{I+e_h} u^k \overline{E^{(ke_1+e_{n-i})}_{(0,I)}} \\
&\quad - \sum_{|I|\geq 1, d\leq |I|+2k+2\leq 2d-3} z^{I+e_h} u^k \overline{E^{(ke_1+e_h)}_{(0,I)}} \\
&:= B_1 + B_2 + B_3.
\end{aligned}$$

By Lemma 4.2, we have, for B_1, the following

$$\begin{aligned}
\|B_1\|_\varrho &= \| \sum_{|I|\geq 1, d\leq |I|+2k\leq 2d-3} z^{I+e_h} u^{k-1} \overline{E^{(ke_1)}_{(0,I)}} \|_\varrho \\
&\leq \sum_{|I|\geq 1, d\leq |I|+2k\leq 2d-3} (R')^{I+e_h} (2\varrho^2)^{k-1} \tfrac{(k+2)^n \|E\|_r}{R^I \cdot (2r^2)^k} \\
&\leq \sum_{|I|\geq 1, d\leq |I|+2k\leq 2d-3} \left(\tfrac{\varrho}{r}\right)^{|I|+2k-1} \tfrac{(k+2)^n \|E\|_r}{2r} \\
&\leq \sum_{|I|\geq 1, d\leq |I|+2k\leq 2d-3} \left(\tfrac{\varrho}{r}\right)^{|I|+2k-1} (2d)^n \|E\|_r \\
&\leq \tfrac{(2d)^{2n} \|E\|_r}{r-\varrho} \left(\tfrac{\varrho}{r}\right)^{d-1}.
\end{aligned}$$

Here and in what follows, we write $R' = (2^{-\frac{n-2}{2}}\varrho, 2^{-\frac{n-2}{2}}\varrho, 2^{-\frac{n-3}{2}}\varrho \cdots, 2^{-\frac{1}{2}}\varrho, \varrho)$. Wee have also used the fact that

$$\sharp\{(i_1, i_2, \cdots, i_n, k) \in \mathbb{Z}^{n+1} : i_h, k \geq 0 \text{ for } 1 \leq h \leq n, \sum_{h=1}^n i_h + 2k = 2d-1\} \leq (2d)^n.$$

For B_2, we have

$$\begin{aligned}
\|B_2\|_\varrho &= \| \sum_{|I|\geq 1, d\leq |I|+2k+2\leq 2d-3} \sum_{i=0}^{n-h-1} z^{I+e_h} u^k \overline{E^{(ke_1+e_{n-i})}_{(0,I)}} \|_\varrho \\
&\leq \sum_{|I|\geq 1, d\leq |I|+2k+2\leq 2d-3} (R')^{I+e_h} (2{r'}^2)^k \cdot n \tfrac{2^n (k+2)^n \|E\|_r}{R^I \cdot (2r^2)^{k+1}} \\
&\leq \sum_{|I|\geq 1, d\leq |I|+2k+2\leq 2d-3} \left(\tfrac{\varrho}{r}\right)^{|I|+2k+1} \tfrac{n 2^n (k+2)^n \|E\|_r}{2r} \\
&\leq \tfrac{n 2^n (2d)^{2n} \|E\|_r}{r-\varrho} \left(\tfrac{\varrho}{r}\right)^{d-1}.
\end{aligned}$$

Similarly, we have $\|B_3\|_\varrho \leq \frac{2^n(2d)^{2n}\|E\|_r}{r-\varrho}\left(\frac{\varrho}{r}\right)^{d-1}$. Thus we obtain:

$$\|A_4\|_\varrho \leq \frac{n2^{n+1}(2d)^{2n}\|E\|_r}{r-\varrho}\left(\frac{\varrho}{r}\right)^{d-1}.$$

In the same manner, we have

$$\|A_1\|_\varrho, \|A_2\|_\varrho, \|A_3\|_\varrho \leq \frac{n \cdot 2^{n+1}(2d)^{2n}\|E\|_r}{r-\varrho}(\frac{\varrho}{r})^{d-1},$$

Hence we get

$$|\hat{f}_h|_\varrho \leq \frac{n \cdot 2^{n+3}(2d)^{2n}\|E\|_r}{r-\varrho}(\frac{\varrho}{r})^{d-1}.$$

Now letting $\tau = \frac{r+2\varrho}{3}$ and using the Cauchy estimates, we have the following estimate of the derivatives of F:

$$\begin{aligned} |(\hat{f}_h)'_{z_i}|_\varrho &\leq \tfrac{\tau|\hat{f}_h|_\tau}{(\tau-\varrho)^2} \leq \tfrac{3^3 n\cdot 2^{n+3}(2d)^{2n}\|E\|_r}{(r-\varrho)^3}(\tfrac{\varrho}{r})^{\frac{d-1}{2}}, \\ |(\hat{f}_h)'_w|_\varrho &\leq \tfrac{2\tau^2|\hat{f}_h|_\tau}{(2\tau^2-2(\varrho)^2)^2} \leq \tfrac{3^3 n\cdot 2^{n+3}(2d)^{2n}\|E\|_r}{(r-\varrho)^3}(\tfrac{\varrho}{r})^{\frac{d-1}{2}}. \end{aligned} \tag{4.13}$$

Here we have used the fact that

$$(\frac{\tau}{r})^2 \leq \frac{\varrho}{r} \text{ for } \frac{1}{2} < \varrho < \tau < r \leq 1\ ,\ \tau = \frac{r+2\varrho}{3}. \tag{4.14}$$

The inequality (4.13) shows that $|\nabla \hat{f}_h|_\varrho \leq \frac{3^3 n(n+1)\cdot 2^{n+3}(2d)^{2n}\|E\|_r}{(r-r')^3}(\frac{\varrho}{r})^{\frac{d-1}{2}}$. The corresponding estimates on $\hat{f}_1$ and $\hat{g}$ can be achieved similarly.

We next estimate $\hat{\varphi}$. Notice that $-\hat{g}(z,u) + 2Re\left(\sum_{h=1}^n \bar{z}_i \hat{f}_i(z,u)\right)$ is only used to cancel terms of with weight $< 2d-2$ in E. By (4.9), we have the following:

$$\begin{aligned} &\|\hat{\varphi}\|_\varrho = \|\sum_{t\geq 2d-2} E^{(t)}\|_\varrho \\ &= \|\sum_{|I|+|J|\geq 2d-2} a_{i_1\cdots i_n j_1\cdots j_n} z_1^{i_1}\cdots z_n^{i_n}\overline{z_1}^{j_1}\cdots\overline{z_n}^{j_n}\|_\varrho \\ &\leq \sum_{|I|+|J|\geq 2d-2} \|E\|_r(\tfrac{R'}{R})^{I+J} \\ &\leq \sum_{|I|+|J|=2d-2,|K|,|L|\geq 0} \|E\|_r(\tfrac{\varrho}{r})^{|I|+|J|}\cdot(\tfrac{\varrho}{r})^{k_1}\cdots(\tfrac{\varrho}{r})^{k_n}\cdot(\tfrac{\varrho}{r})^{l_1}\cdots(\tfrac{\varrho}{r})^{l_n} \\ &\leq \sum_{|I|+|J|=2d-2} \|E\|_r(\tfrac{\varrho}{r})^{2d-2}\cdot(\tfrac{1}{1-\frac{\varrho}{r}})^{2n} \\ &\leq \tfrac{(2d)^{2n}\|E\|_r}{(r-\varrho)^{2n}}(\tfrac{\varrho}{r})^{2d-2}. \end{aligned}$$

Here we have used the fact that

$$\sharp\{(i_1,\cdots,i_n,j_1,\cdots,j_n)\in\mathbb{Z}^{2n} : i_h, j_h \geq 0 \text{ for } 1\leq h\leq n, \sum_{h=1}^n (i_h+j_h) = k\} \leq (k+1)^{2n}.$$

This finishes the proof of Proposition 4.3. ■

Proposition 4.4: *Let $E, r, \varrho, C(n)$ be as in Proposition 4.3. Then there exists a constant $\delta > 0$ such that for*

$$\frac{C(n)(2d)^{2n}\|E\|_r}{(r-\varrho)^3}\left(\frac{\varrho}{r}\right)^{\frac{d-1}{2}} < \delta, \tag{4.15}$$

$\Psi(z', w') := \Theta^{-1}(z', w')$ is well defined in $\overline{\triangle_\sigma}$. Moreover, it holds that $\Psi(\triangle_{r'}) \subset \triangle_\sigma$, $\Psi(\triangle_\sigma) \subset \triangle_\varrho$, $E'(z, \xi)$ is holomorphic in $\overline{\triangle_\sigma}$ and

$$\|E'\|_{r'} \leq C_d\|E\|_r^2 + \widetilde{C_d}\|E\|_r. \tag{4.16}$$

Here

$$C_d = \frac{(2n+1)\cdot 3^3 C(n)(2d)^{2n}}{(r-r')^3}\left(\frac{r'}{r}\right)^{\frac{d-1}{4}} + \left(\frac{r'}{r}\right)^{d-1} n \cdot \left(\frac{3C(n)(2d)^{2n}}{r-r'}\right)^2, \quad \widetilde{C_d} = \frac{3^{2n}\cdot(2d)^{2n}}{(r-r')^{2n}}\left(\frac{r'}{r}\right)^{d-1}.$$

Proof of Proposition 4.4: We need to show that for each $(z', w') \in \overline{\triangle_\sigma}$, we can uniquely solve the system:

$$\begin{cases} z' = z + \hat{f}(z, w) \\ w' = w + \hat{g}(z, w) \end{cases}$$

with $(z, w) \in \triangle_\varrho$. By (4.12), choosing δ sufficiently small such that $|\nabla \hat{f}|_\varrho + |\nabla \hat{g}|_\varrho < \frac{1}{2n+4}$ and $|\hat{f}|_\varrho + |\hat{g}|_\varrho < \frac{1}{2n+4}.(r-\varrho)$. Define $(z^{[1]}, w^{[1]}) = (z', w') \in \triangle_\sigma$ and $(z^{[j]}, w^{[j]})$ inductively by

$$\begin{cases} z^{[j+1]} = z' - \hat{f}(z^{[j]}, w^{[j]}) \\ w^{[j+1]} = w' - \hat{g}(z^{[j]}, w^{[j]}). \end{cases}$$

By a standard argument on the Picard iteration procedure, we can get a unique $(z, w) \in \triangle_\varrho$ satisfying $\Psi^{-1}(z, w) = (z', w')$, which gives that $\Psi(\triangle_\sigma) \subset \triangle_\varrho$. Similarly, we have $\Psi(\triangle_{r'}) \subset \triangle_\sigma$. Hence we conclude that E' is holomorphic in $\triangle_\sigma$. Moreover,

$$\|E'\|_{r'} \leq \|Q\|_\sigma \tag{4.17}$$

where

$$Q = (\hat{g}(z, \Phi) - \hat{g}(z, u)) - 2Re\left(\sum_{i=1}^n \overline{z_i}(\hat{f}_i(z, \Phi) - \hat{f}_i(z, u))\right) - |\hat{f}(z, \Phi)|^2 + \hat{\varphi}(z, \bar{z}). \tag{4.18}$$

Notice that

$$\begin{aligned} |(\hat{g}(z, \Phi) - \hat{g}(z, u)|_\sigma &\leq |\nabla \hat{g}|_\varrho \cdot \|E\|_r \leq \frac{C(n)(2d)^{2n}\|E\|_r^2}{(r-\varrho)^3}\left(\frac{\varrho}{r}\right)^{\frac{d-1}{2}} \\ &\leq \frac{3^3 C(n)(2d)^{2n}\|E\|_r^2}{(r-r')^3}\left(\frac{r'}{r}\right)^{\frac{d-1}{4}}. \end{aligned} \tag{4.19}$$

Here we have used the fact that $(\frac{\varrho}{r})^2 < \frac{r'}{r}$. (This can be achieved by the same token as for (4.14).) We also have

$$
\begin{aligned}
&|(\hat{f}_i(z,\Phi) - \hat{f}_i(z,u)|_\sigma \leq \tfrac{3^3 C(n)(2d)^{2n}\|E\|_r^2}{(r-r')^3}(\tfrac{r'}{r})^{\frac{d-1}{4}} \text{ for } 1 \leq \text{i} \leq \text{n}.\\
&|\hat{f}(z,\Phi)|_\sigma^2 \leq n \cdot \left(\tfrac{C(n)(2d)^{2n}\|E\|_r}{r-\sigma}(\tfrac{\sigma}{r})^{d-1}\right)^2 \leq n \cdot \left(\tfrac{3C(n)(2d)^{2n}\|E\|_r}{r-r'}\right)^2 (\tfrac{r'}{r})^{d-1} \\
&\|\hat{\varphi}\|_\sigma \leq \tfrac{(2d)^{2n}\|E\|_r}{(r-\sigma)^{2n}}(\tfrac{\sigma}{r})^{2d-2} \leq \tfrac{3^{2n}(2d)^{2n}\|E\|_r}{(r-r')^{2n}}(\tfrac{r'}{r})^{d-1}.
\end{aligned}
\tag{4.20}
$$

By (4.18)-(4.20), we obtain:

$$
\|E'\|_{r'} \leq \left\{ \tfrac{(2n+1)\cdot 3^3 C(n)(2d)^{2n}}{(r-r')^3}(\tfrac{r'}{r})^{\frac{d-1}{4}} + (\tfrac{r'}{r})^{d-1} n \cdot \left(\tfrac{3C(n)(2d)^{2n}}{r-r'}\right)^2 \right\} \|E\|_r^2 + \tfrac{3^{2n}\cdot(2d)^{2n}}{(r-r')^{2n}}(\tfrac{r'}{r})^{d-1}\|E\|_r
$$

This completes the proof of Proposition 4.4. ■

Now we turn to the proof of Theorem 1. Set $r_\upsilon, \varrho_\upsilon, \sigma_\upsilon$ as follows:

$$
r_\upsilon = \frac{1}{2}\left(1 + \frac{1}{\upsilon+1}\right) \ , \ \varrho_\upsilon = \frac{1}{3}(2r_\upsilon + r_{\upsilon+1}) \ , \ \sigma_\upsilon = \frac{1}{3}(2r_\upsilon + \varrho_\upsilon).
$$

We will apply the previous estimates with $r = r_\upsilon, \varrho = \varrho_\upsilon, \sigma = \sigma_\upsilon, r' = r_{\upsilon+1}, \Psi = \Psi_\upsilon, \cdots$, with $\upsilon = 0, 1, \cdots,$. Then we have the following (see [(4.5), Moser]):

$$
(r_\upsilon - r_{\upsilon+1})^{-1} = 2(\upsilon+1)(\upsilon+2), \ \frac{r_{\upsilon+1}}{r_\upsilon} = 1 - \frac{1}{(\upsilon+2)^2} \tag{4.21}
$$

Define a sequence of real analytic submanifolds

$$
M_k \ : \ w = |z|^2 + E_k(z,\bar{z})
$$

by $M_0 = M$, $M_{\upsilon+1} = \Psi_\upsilon^{-1}(M_\upsilon)$ for all $\upsilon = 0, 1, 2, \cdots$, where Ψ_υ is the biholomorphic mapping taking $\triangle_{\sigma_\upsilon}$ into $\triangle_{\varrho_\upsilon}$. And let

$$
d_\upsilon = Ord(E_\upsilon) \ , \ \Phi_\upsilon = \Psi_0 \circ \Psi_1 \circ \cdots \circ \Psi_\upsilon.
$$

Since $s = \infty$, we find that

$$
Ord(E_\upsilon) = d_\upsilon \geq 2^\upsilon + 2 \text{ for } \upsilon \geq 0.
$$

We next state the following elementary fact:

Lemma 4.5: Suppose that there is a constant C and number $a > 1$ such that $d_\upsilon \geq Ca^\upsilon$. Then for any integer $m_1, m_2, m_3 > 0$,

$$
\lim_{\upsilon\to\infty} \upsilon^{m_3} d_\upsilon^{m_1}(1 - \frac{1}{\upsilon^{m_2}})^{d_\upsilon} = 0.
$$

Then one can prove, by using (4.21) and Lemma 4.5, that

$$\lim_{v\to\infty} C_{d_v} = 0 \ , \ \lim_{v\to\infty} \widetilde{C_{d_v}} = 0.$$

Hence C_{d_v} and $\widetilde{C_{d_v}}$ are bounded. Set $C_{d_v}, \widetilde{C_{d_v}} < C$, where C is a fixed positive constant. Also, one can verify that the hypothesis in (4.15) holds for all $v \geq 0$, by choosing $\eta_0^* = \|E_0\|_{r_0}$ sufficiently small. Indeed, we can even have $\|E_v\|_{r_v} \leq \epsilon 2^{-v}$ for all $v \geq 0$ and any given $1 > \epsilon > 0$.

Choose N large enough such that $C_{d_v}, \widetilde{C_{d_v}} \leq \frac{1}{4}$ when $v \geq N$. Suppose $C > 1$ and choose E_0 such that $\eta_0^* = \epsilon(2C)^{-2N} < 1$. Then we have the following

(I) When $v \leq N$, we have

$$\|E_v\|_{r_v} \leq C(\|E_{v-1}\|_{r_{v-1}} + 1)\|E_{v-1}\|_{r_{v-1}} \leq 2C \cdot \|E_{v-1}\|_{r_{v-1}} \leq (2C)^v \|E_0\|_{r_0} \leq \epsilon(2C)^{v-2N} \leq \epsilon 2^{-N}.$$

(II) When $v > N$, we have

$$\|E_v\|_{r_v} \leq \tfrac{1}{4} \cdot 2 \cdot \|E_{v-1}\|_{r_{v-1}} \leq (\tfrac{1}{2})^{v-N} \|E_N\|_{r_N} \leq \epsilon 2^{-v}.$$

Now, choose ϵ sufficiently small. Then it follows from (4.12) and Proposition 4.4 that $\|d\Psi_v^{-1}\|_{\triangle_{\varrho_v}} \leq 1 + C_0\|E_v\|_{r_v} \leq 1 + C_0\epsilon 2^{-v}$ for some constant C_0. Notice that Ψ_v maps $\triangle_{\sigma_v}$ into $\triangle_{\varrho_v}$. By Cramer's rule, we have $\|d\Psi_v\|_{\triangle_{\sigma_v}} \leq 1 + \epsilon C_1 2^{-v}$ for some constant C_1. Now the convergence of Φ_v in $\triangle_{\frac{1}{2}}$ follows from the fact that

$$\Pi_{v=0}^{\infty} \|d\Psi_v\|_{\triangle_{\sigma_v}} \leq \Pi_{v=0}^{\infty}(1 + \epsilon C_1 2^{-v}) < \infty,$$

which completes the proof of Theorem 1. ■

Remark 4.6: We notice that the formal map in Theorem 1 sending $(M, 0)$ to its quadric $(M_\infty, 0)$ may not be convergent as $aut_0(M_\infty)$ contains many non-convergent elements. This is quite different from the setting for CR manifolds, where formal maps are always convergent under certain not too degenerate assumptions. We refer the reader to the survey article [BER1] for discussions and references on this matter.

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