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A Bishop surface with a vanishing Bishop invariant

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Abstract

We give a solution to the equivalence problem for Bishop surfaces with the Bishop invariant $\lambda = 0$. As a consequence, we answer, in the negative, a problem that Moser asked in 1985 after his work with Webster in 1983 and his own work in 1985. This will be done in two major steps: We first derive the formal normal form for such surfaces. We then show that two real analytic Bishop surfaces with $\lambda = 0$ are holomorphically equivalent if and only if they have the same formal normal form (up to a trivial rotation). Our normal form is constructed by an induction procedure through a completely new weighting system from what is used in the literature. Our convergence proof is done through a new hyperbolic geometry associated with the surface.

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As an immediate consequence of the work in this paper, we will see that the modular space of Bishop surfaces with the Bishop invariant vanishing and with the Moser invariant $s < \infty$ is of infinite dimension. This phenomenon is strikingly different from the celebrated theory of Moser-Webster for elliptic Bishop surfaces with non-vanishing Bishop invariants where the surfaces only have two and one half invariants. Notice also that there are many real analytic hyperbolic Bishop surfaces, which have the same Moser-Webster formal normal form but are not holomorphically equivalent to each other as shown by Moser-Webster and Gong. Hence, Bishop surfaces with the Bishop invariant $\lambda = 0$ behave very differently from hyperbolic Bishop surfaces and elliptic Bishop surfaces with non-vanishing Bishop invariants.

1 Introduction and statements of main results

In this paper, we study the precise holomorphic structure of a real analytic Bishop surface near a complex tangent point with the Bishop invariant vanishing. A Bishop surface is a generically embedded real surface in the complex space of dimension two. Points on a Bishop surface are either totally real or have non-degenerate complex tangents. The holomorphic structure near a totally real point is trivial. Near a point with a complex tangent, namely, a point with a non-trivial complex tangent space of type $(1,0)$, the consideration could be much more subtle. The study of this problem was initiated by the celebrated paper of Bishop in 1965 [Bis], where for a point p on a Bishop surface M with a complex tangent, he defined an invariant λ now called the Bishop invariant. Bishop showed that there is a holomorphic change of variables, that maps p to 0, such that M, near $p = 0$, is defined in the complex coordinates $(z, w) \in \mathbb{C}^2$ by

$$w = z\overline{z} + \lambda(z^2 + \overline{z}^2) + o(|z|^2), \tag{1.1}$$

where $\lambda \in [0, \infty]$. When $\lambda = \infty$, (1.1) is understood as $w = z^2 + \overline{z}^2 + o(|z|^2)$. It is now a standard terminology to call p an elliptic, hyperbolic or parabolic point of M, according to whether $\lambda \in [0, 1/2)$, $\lambda \in (1/2, \infty]$ or $\lambda = 1/2$, respectively.

Bishop discovered an important geometry associated with M near an elliptic complex tangent p by proving the existence of a family of holomorphic disks attached to M shrinking down to p. He also proposed several problems concerning the uniqueness and regularity of the geometric object obtained by taking the union of all locally attached holomorphic disks. These problems, including their higher dimensional cases, were completely answered through the combining efforts of many people. (See [Hu3], [BG], [KW1], [KW2], [MW], [Mos], [HK]).

Bishop invariant is a quadratic invariant, capturing the basic geometric character of the surface. The celebrated work of Moser-Webster [MW] first investigated the more subtle higher order invariants. Different from Bishop's approach of using the attached holomorphic disks, Moser-Webster's starting point is the existence of a more dynamically oriented object: an intrinsic pair of involutions on the complexification of the surface near a non-exceptional complex tangent. Here, recall that the Bishop invariant is said to be non-exceptional if $\lambda \neq 0, 1/2, \infty$ or if $\lambda\nu^2 - \nu + \lambda = 0$ has no roots of unity in the variable ν. Moser-Webster proved that, near a non-exceptional complex tangent, M can always be mapped, at least, by a formal transformation to the normal form defined in the complex coordinates $(z, w = u + iv) \in \mathbb{C}^2$ by:

$$u = z\overline{z} + (\lambda + \epsilon u^s)(z^2 + \overline{z}^2)\ ,\ v = 0\ ,\ \epsilon \in \{0, 1, -1\}\ ,\ s \in \mathbb{Z}^+. \tag{1.2}$$

Moser-Webster also provided a convergence proof of the above mentioned formal transformation in the non-exceptional elliptic case: $0 < \lambda < 1/2$. However, the intriguing elliptic case with $\lambda = 0$ has to be excluded from their theory. Instead, Moser in [Mos] carried out a study for $\lambda = 0$ from a more formal power series point of view. Moser derived the following formal pseudo-normal form for M with $\lambda = 0$:

$$w = z\overline{z} + z^s + \overline{z}^s + 2Re\{\sum_{j \geq s+1} a_j z^j\}. \tag{1.3}$$

Here s is the simplest higher order invariant of M at a complex tangent with a vanishing Bishop invariant, which we call the Moser invariant. Moser showed that when $s = \infty$, M is then holomorphically equivalent to the quadric $M_\infty = \{(z, w) \in \mathbb{C}^2 : w = |z|^2\}$.

Moser's formal pseudo-normal form is still subject to the simplification of a very complicated infinitely dimensional group $aut_0(M_\infty)$, the formal self-transformation group of M_∞. And it was left open from the work of Moser [Mos] to derive any higher order invariant other than s from the Moser pseudo-normal form. At this point, we mention that $aut_0(M_\infty)$ contains many non-convergent elements. Based on this, Moser asked two basic problems concerning a Bishop surface near a vanishing Bishop invariant in his paper [Mos]. The first one is on the analyticity of the geometric object formed by the attached disks up to the complex tangent point. This was answered in the affirmative in [HK]. Hence, the work of [HK], together with that of Moser-Webster [MW], shows that, as far as the analyticity of the local hull of holomorphy is concerned, all elliptic Bishop surfaces are of the same character. The second problem that Moser asked concerns the higher order invariants. Notice that by the Moser-Webster normal form, an analytic elliptic Bishop surface with $\lambda \neq 0$ is holomorphically equivalent to an algebraic one and possesses at most two more higher order invariants. Moser asked if M with $\lambda = 0$ is of the same character as that for elliptic surfaces with $\lambda \neq 0$. Is the equivalence class of a Bishop surface with $\lambda = 0$ determined by an algebraic surface obtained by truncating the Taylor expansion of its defining equation at a sufficiently higher order level? Gong showed in [Gon2] that under the equivalence relation of a smaller class of transformation group, called the group of holomorphic symplectic transformations, M with $\lambda = 0$ does have an infinite set of invariants. However, under this equivalence relation, elliptic surfaces with non-vanishing invariants also have infinitely many invariants. Gong's work later on (see, for example, [Gon2-3] [AG]) demonstrates that as far as many dynamical properties are concerned, exceptional or non-exceptional hyperbolic, or even parabolic complex tangents are not much different from each other.

In this paper, we derive a formal normal form for a Bishop surface near a vanishing Bishop invariant, by introducing a quite different weighting system. This new weighting system fits extremely well in our setting and may have applications in the study of many other related problems. We will obtain a complete set of invariants under the action of the formal transformation group. We show, in particular, that the modular space for Bishop surfaces with a vanishing Bishop invariant and with a fixed (finite) Moser invariant s is an infinitely dimensional manifold in a Fréchet space. This then immediately provides an answer, in the negative, to Moser's problem concerning the determination of a Bishop surface with a vanishing Bishop invariant from a finite truncation of its Taylor expansion. Furthermore, it can also be combined with some already known arguments to show that most Bishop surfaces with $\lambda = 0,\ s \neq \infty$ are not holomorphically equivalent to algebraic surfaces. Hence, one sees a striking difference of elliptic Bishop surfaces with a vanishing Bishop invariant from elliptic Bishop surfaces with non-vanishing Bishop invariants. The general phenomenon that the infinite dimensionality of the modular space has the consequence that any subclass formed by a countable union of finite dimensional spaces is of the first category in the modular space seems already clear even to Poincaré [Po]. In the CR geometry category, we refer the reader to a paper of Forstneric [For] in which the infinite dimensionality of the modular space of generic CR manifolds is used to show that CR manifolds holomorphically equivalent to algebraic ones form a very thin set among all real analytic CR manifolds. Similar to what Forstneric did in [For], our argument to show the generic non-algebraicity from the infinite dimensionality of the modular space also uses the Baire category theorem.

It remains to be an open question to answer whether the new normal form obtained in this paper for a real analytic Bishop surface with $\lambda = 0,\ s < \infty$ is always convergent. However, we will show that two Bishop surfaces with $\lambda = 0$ and $s < \infty$ are holomorphically equivalent if and only if their formal normal forms are the same up to a trivial rotation of the form: $(z, w) \longmapsto (e^{i\theta}z, w)$ with $e^{i\theta s} = 1$. Hence, the formal normal form that we will derive provides a solution to the equivalence problem also in the holomorphic category. We will achieve this goal by proving that any formal map between two real analytic Bishop surfaces with $\lambda = 0, s < \infty$ is convergent. Remark that there are many non-convergent formal maps transforming real analytic Bishop surfaces with a vanishing Bishop invariant and with $s = \infty$ to the model surface M_∞ defined before. (See [MW] [Mos] [Hu2]). Hence, our convergence theorem reveals a non-trivial role that the Moser invariant has played in the study of the precise holomorphic structure of a Bishop surface with $\lambda = 0$. At this point, we would like to mention that there are many other different type of problems where one studies the convergence problem for formal power series, though very different methods and approaches need to be employed in different settings. Just to name a few, we here mention the work in [BER], [Mir], [We], [Sto] and the references therein.

Our convergence argument uses the Moser-Webster [MW] polarization, as in the non-vanishing Bishop invariant case treated by Moser-Webster. However, different from the Moser-Webster situation, we do not have a pair of involutions, which were the starting point of the Moser-Webster theory. The main idea in the present paper for dealing with the convergence problem is to find a new surface hyperbolic geometry, by making use of the flattening theorem of Huang-Krantz [HK].

We next state our main results, in which we will use some terminology to be defined in the next section:

Theorem 1.1. *Let M be a formal Bishop surface which has an elliptic complex tangent at 0 with its Bishop invariant $\lambda = 0$ and its Moser invariant $s \geq 3$ and $s < \infty$. Then there exists a formal equivalence map:*

$$(z', w') = F(z, w) = (\widetilde{f}(z, w), \widetilde{g}(z, w)), \quad F(0, 0) = (0, 0)$$

such that in the (z', w') coordinates, $M' = F(M)$ is represented near the origin by a formal equation of the following normal form:

$$w' = z'\bar{z'} + z'^s + \bar{z'}^s + \varphi(z') + \overline{\varphi(z')}$$

where

$$\varphi(z') = \sum_{k=1}^{\infty} \sum_{j=2}^{s-1} a_{ks+j} z'^{ks+j}.$$

Such a formal transform is unique up to a composition from the left with a rotation of the form:

$$(z'', w'') = R_\theta(z', w') := (e^{\sqrt{-1}\theta} z', w'), \quad \text{where } \theta \text{ is a constant with } e^{\sqrt{-1}s\theta} = 1.$$

Namely, if there is another formal equivalence map $(z'', w'') = F^*(z, w)$ with $F^*(0) = 0$ that maps M into the following normal form:

$$w'' = z''\bar{z''} + z''^s + \bar{z''}^s + \varphi^*(z'') + \overline{\varphi^*(z'')} \text{ with}$$

$$\varphi^*(z'') = \sum_{k=1}^{\infty} \sum_{j=2}^{s-1} a^*_{ks+j} z''^{ks+j}.$$

Then

$$F^* = R_\theta \circ F \text{ for a certain } \theta \text{ with } e^{\sqrt{-1}\theta s} = 1 \text{ and } a_{ks+j} = e^{\sqrt{-1}j\theta} a^*_{ks+j}.$$

Theorem 1.2. *Let M_1 and M_2 be real analytic Bishop surfaces with $\lambda = 0$ and $s \neq \infty$ at 0. Suppose that M_1 has a formal normal form:*

$$w' = z'\bar{z'} + z'^s + \bar{z'}^s + 2Re\{\sum_{k=1}^{\infty} \sum_{j=2}^{s-1} a_{ks+j} z'^{ks+j}\};$$

and suppose that M_2 has a formal normal form:

$$w' = z'\bar{z'} + z'^s + \bar{z'}^s + 2Re\{\sum_{k=1}^{\infty} \sum_{j=2}^{s-1} b_{ks+j} z'^{ks+j}\}.$$

Then $(M_1, 0)$ is biholomorphic to $(M_2, 0)$ if and only if there is a constant θ with $e^{s\theta\sqrt{-1}} = 1$, such that $a_{ks+j} = e^{\theta j\sqrt{-1}} b_{ks+j}$ for any $k \geq 1$ and $j = 2, \cdots, s-1$.

Theorem 1.1 and Theorem 1.2 give a solution to the equivalence problem for Bishop surfaces with $\lambda = 0$ and $s < \infty$. Theorem 1.1 is used to prove the following Theorem 1.3. Theorems 1.1 and 1.3 provide, in the negative, a solution to a problem that Moser asked on [pp 399, Mos].

Theorem 1.3. *Most real analytic elliptic Bishop surfaces with the Bishop invariant $\lambda = 0$ and the Moser invariant $s < \infty$ at 0 are not equivalent to algebraic surfaces in $\mathbb{C}^2$.*

Define $\mathcal{Z}_s$ for the group of transformations consisting of maps of the form $\{\psi_\theta : (z, w) \mapsto (e^{i\theta}z, w),\ e^{is\theta} = 1\}$. Then the following corollary is a consequence of Theorems 1.1 and 1.2:

Corollary 1.4. *(a). Suppose M_{nor} is a formal Bishop surface near the origin defined by*

$$w = z\overline{z} + z^s + \overline{z}^s + 2Re\{\sum_{k=1}^{\infty}\sum_{j=2}^{s-1} a_{ks+j} z^{ks+j}\}.$$

Then the group of the origin preserving formal self-transformations of M_{nor}, denoted by $aut_0(M_{nor})$, is a subgroup of $\mathcal{Z}_s$. Moreover, $\psi_\theta \in aut_0(M_{nor})$ if and only if

$$a_{ks+j} = 0 \ \ \textit{for any } k \textit{ and } j \textit{ with } k \geq 1,\ 2 \leq j \leq s-1,\ \ e^{\sqrt{-1}j\theta} \neq 1.$$

(b). $aut_0(M_s) = \mathcal{Z}_s$, where M_s is defined by $w = z\overline{z} + z^s + \overline{z}^s$.

(c). Any subgroup of $\mathcal{Z}_s$ can be realized as the formal automorphism group of a certain algebraic surface M_{nor}.

(d). Let M be a formal Bishop surface with a vanishing Bishop invariant and $s < \infty$ at 0. Then $aut_0(M)$ is isomorphic to a subgroup of $\mathcal{Z}_s$.

(e). Let M be a real analytic Bishop surface with a vanishing Bishop invariant and the Moser invariant $s < \infty$ at 0. Suppose that $aut_0(M)$ is isomorphic to $\mathcal{Z}_s$. Then $(M, 0)$ is biholomorphic to $(M_s, 0)$, where M_s, as before, is defined by $w = z\overline{z} + z^s + \overline{z}^s$.

(f). Let M be a real analytic elliptic Bishop surface with $\lambda = 0$ and s a prime number at 0. Then $aut_0(M)$ is a trivial group unless $(M, 0)$ is biholomorphic to $(M_s, 0)$.

The convergence statement in Theorem 1.2 is obtained by proving the following:

Theorem 1.5. *Let M and M' be real analytic Bishop surfaces near 0 with the Bishop invariant vanishing and the Moser invariant finite. Suppose that $F : (M, 0) \longrightarrow (M', 0)$ is a formal equivalence map. Then F is biholomorphic near 0.*

Idea for the proof of Theorem 1.1: We give the main idea behind the complicated argument for the proof of Theorem 1.1. Let M be as in Theorem 1.1. We want to find a formal biholomorphic map sending M into a formal normal form. We also need to prove that such a map is unique up to a trivial rotation. This then leads us to study an infinite system of homogeneous equations by truncating the original equation. Now, the homogeneous linearized normalization equations (see §3) have non-trivial kernel spaces, due to the fact that $aut_0(M_\infty)$ is of infinite dimension. The non-uniqueness part of the lower degree solutions needs to be uniquely determined in the higher order equations. Unfortunately, these lower order terms get into the scene in the higher order truncation non-linearly. Hence, the normalization problem in this setting is a non-linear normalization problem, which is quite different from the consideration in the literature (see Chern-Moser [CM] and Moser in [Mos]), where the normalization equation is always truncated into an infinite system of linear equations. The new idea to overcome this difficulty is to consider a new model $w = |z|^2 + z^s$ instead of the quadric, which reduces the automorphism group to the finite group $\mathcal{Z}_s$. Now, to treat $|z|^2$ equally with the term z^s, it forces us to define the weight of $\overline{z}$ to be $s-1$ and thus $|z|^2+z^s$ is a weighted homogeneous polynomial of degree s. Indeed, under the new weighting system and with a complicated induction argument, we will be able to trace precisely how the lower order terms get involved non-linearly: The kernel space of degree $2t+1$ is used and determined at the truncated equation of degree $ts+1$ and the kernel space of degree $2t+2$ is used and determined at the truncated equation of degree $ts+s$. This approach seems to be powerful in handling the normalization problem, where the model has a big automorphism group. It may find applications in the study of many other related problems.

One of the new features of this part of the paper is that we are studying a normalization problem whose linear truncation at each weighted degree level turns out to be a semi-non-linear equation. In this sense, our normalization problem seems to be quite different from what has been studied in the literature.

Idea for the proof of Theorem 1.5: We next say a few words about the complicated argument for the proof of Theorem 1.5. Let M be a real analytic Bishop surface as in Theorem 1.5. (Assume that M has been normalized up to a certain order, say order s.) By a result of Huang-Krantz [HK], M can be assumed to be in $\mathbb{C} \times \mathbb{R}$. Consider its Moser-Webster complexification $\mathcal{M}$, which is a complex surface in $\mathbb{C}^4$. There is a natural projection from $\mathcal{M}$ into $\mathbb{C}^2$, which is generically s to one. The projection is branched along a one dimensional complex analytic variety, whose intersection with $\mathbb{C}\times\mathbb{R}$ gives $s-$curves $z = A_j(u)$ with $j = 0, \cdots, s-1$. Here, we use (z, u) for the coordinates of $\mathbb{C} \times \mathbb{R}$ and each $A_j(u)$ has a convergent power series expansion in $u^{1/s}$. These curves are invariant under a biholomorphic transformation and are formally invariant in a certain sense under a formal invertible transformation. For each $0 < u \ll 1$, $(A_j(u), u)'s$ are roughly equally distributed s-points on the circle with center at the origin and of radius $C(s)u^{(s-1)/s}$ in a simply connected Riemann surface $D(u) \times \{u\}$ attached to M. ($D(u)$ is roughly a disk centered at the origin with radius $\sqrt{u}$.) The hyperbolic geometry derived from $A_j(u)'s$ with the Poincaré metric over $D(u)$, as well as its counterpart from M', can be used to control the (normalized) formal map F from M to M'. This, in particular, provides us a convergence proof for the map in Theorem 1.5.

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2 A uniqueness theorem for formal maps

In what follows, we use (z, w) or (z', w') for the coordinates of $\mathbb{C}^2$. Let $A(z, \overline{z})$ be a formal power series in $(z, \overline{z})$ without constant term. We say that the order of $A(z, \bar{z})$ is k if $A(z, \overline{z}) = \sum_{j+l=k} A_{j\bar{l}} z^j \overline{z}^l + o(|z|^k)$ with at least one of the $A_{j\bar{l}} \in \mathbb{C}$ $(j + l = k)$ not equal to 0. In this case, we write $\mathrm{Ord}(A(z, \overline{z})) = k$. We say $\mathrm{Ord}(A(z, \overline{z})) \geq k$ if $A(z, \overline{z}) = O(|z|^k)$. When $A \equiv 0$, we say that the order of A is ∞.

Consider a formal real surface M in $\mathbb{C}^2$ near the origin. Suppose that 0 is a point of complex tangent for M. Then, after a linear change of variables, we can assume that $T_0^{(1,0)} M = \{w = 0\}$. If there is no change of coordinates such that M is defined by an equation of the form $w = O(|z|^3)$, we then say that 0 is a point on M with a non-degenerate complex tangent. In this case, Bishop showed that there is a change of coordinates in which M is defined by ([Bis] [Hu1]):

$$w = z\overline{z} + \lambda(z^2 + \overline{z}^2) + O(|z|^3). \tag{2.1}$$

Here $\lambda \in [0, \infty]$ and when $\lambda = \infty$, the equation takes the form: $w = z^2 + \overline{z}^2 + O(|z|^3)$. λ is the first absolute invariant of M at 0, called the Bishop invariant. Bishop invariant is a quadratic invariant, resembling to the Levi eigenvalue in the hypersurface case. When $\lambda \in [0, 1/2)$, we say that M has an elliptic complex tangent at 0. In this paper, we are only interested in the case of an elliptic complex tangent. We need only to study the case of $\lambda = 0$; for, in the case of $\lambda \in (0, 1/2)$, the surface has been well understood by the work of Moser-Webster [MW]. When $\lambda = 0$, Moser-Webster and Moser showed in [MW] [Mos] that there is an integer $s \geq 3$ or $s = \infty$ such that M is defined by

$$w = z\bar{z} + z^s + \bar{z}^s + E(z, \bar{z}), \tag{2.2}$$

where E is a formal power series in $(z, \overline{z})$ with $\mathrm{Ord}(E) \geq s+1$. When $s = \infty$, we understand the above equation as $w = z\overline{z}$, namely, M is formally equivalent to the quadric $M_\infty = \{w = z\overline{z}\}$. s is the next absolute invariant for M, called the Moser invariant. The case for $s = \infty$ is also well-understood through the work of Moser [Mos]. Hence, in all that follows, our M will have $\lambda = 0$ and a fixed $s < \infty$.

A formal map $z' = F(z,w),\ w' = G(z,w)$ without constant terms is called a formal equivalence transformation (or simply, a formal transformation) if $\frac{\partial(F,G)}{\partial(z,w)}(0,0)$ is invertible. When a formal map has no constant term, we also say that it preserves the origin.

Lemma 2.1. *Let M be defined as in (2.2). Suppose that $z' = F(z,w), w' = G(z,w)$ is a formal equivalence transformation preserving the origin and sending M into M', where M' is defined by $w' = z'\overline{z}' + z'^s + \overline{z'^s} + E^*(z', \overline{z'})$ with $Ord(E^*) \geq s+1$. Then*
(i): $F = az + bw + O(|(z,w)|^2),\ G = cw + O(|w|^2 + |zw| + |z|^3)$ where $c = |a|^2, a \neq 0$.
(ii): Suppose that M and M' are further defined by $w = H(z,\overline{z}) = z\overline{z} + z^s + \overline{z}^s + o(|z|^s)$ and $w' = H^*(z', \bar{z}') = z'\overline{z'} + z'^s + \overline{z'}^s + o(|z'|^s)$, respectively, where $s \geq 3$. Then

$$(F,G) = (e^{i\theta}z + O(|z|^2 + |w|), w + O(|w|^2 + |zw| + |z|^3)) \text{ where } \theta \text{ is a constant with } e^{is\theta} = 1.$$

(iii): In (i), when $\overline{E(z,\overline{z})} = E(z,\overline{z}) + o(|z|^N)$ and $\overline{E^*(z',\overline{z}')} = E^*(z',\overline{z}') + o(|z|^N)$ with $N \geq s$, we then have

$$G(z,w) = \sum_{1\leq j\leq [N/2]} a_j w^j + \sum_{j+2k\geq N+1} b_{jk} z^j w^k \text{ with } \overline{a_j} = a_j \text{ for } j \in [1,[N/2]].$$

In particular, when $\overline{E(z,\overline{z})} = E(z,\overline{z})$ and $\overline{E^*(z',\overline{z}')} = E^*(z',\overline{z}')$, then G satisfies the following reality condition:

$$G(z,w) = G(w) \text{ and } \overline{G(w)} = G(\overline{w}).$$

Proof of Lemma 2.1: (i) is the content of Lemma 3.2 of [Hu1]. To prove (ii), we write $(F,G) = (az + f, cw + g)$, where by (i), we can assume that

$$f(z,w) = O(|z|^2 + |w|) \quad , \quad g(z,w) = O(|w|^2 + |zw| + |z|^3).$$

Notice that

$$f(0, H(0,\bar{z})) = O(\bar{z}^s) \quad , \quad \overline{f}(\bar{z}, \bar{H}(\bar{z},0)) = O(\bar{z}^2) \quad , \quad g(0, H(0,\bar{z})) = o(\bar{z}^s).$$

Applying the defining equation of M' , we have on M the following:

$$\begin{aligned} cw + g(z,w) =& |a|^2|z|^2 + \bar{a}\bar{z}f(z,w) + az\overline{f}(\bar{z},\bar{w}) + f(z,w)\overline{f}(\bar{z},\bar{w}) \\ &+ (az + f(z,w))^s + \left(\bar{a}\bar{z} + \overline{f}(\bar{z},\bar{w})\right)^s + o(|z|^s). \end{aligned}$$

Regarding z and $\bar{z}$ as independent variables in the above equation and then letting $z = 0, w = H(0,\overline{z}) = \overline{z^s} + o(\overline{z^s}),\ \bar{w} = \overline{H}(\overline{z},0) = \overline{z^s} + o(\overline{z^s})$, we obtain

$$c\bar{z}^s + o(\bar{z}^s) = (\bar{a}\bar{z})^s + o(\bar{z}^s).$$

Hence, it follows that $c = \bar{a}^s$. Together with $c = |a|^2 \neq 0$ and $s \geq 3$, we get

$$c = 1\ ,\ a = e^{i\theta}\ ,\ \text{where } \theta \text{ is a constant with } \ e^{is\theta} = 1.$$

This completes the proof of Lemma 2.1 (ii).

Now we turn to the proof of (iii). Notice that

$$G(z,w) = |F(z,w)|^2 + \big(F(z,w)\big)^s + \overline{F(z,w)}^s + E^*(F(z,w), \overline{F(z,w)}) \text{ for } (z,w) \in M.$$

Since E, E^* are assumed to be real valued up to order N, we have

$$G(z,w) = \overline{G(z,w)} + o(|z|^N) \quad \text{when } w = |z|^2 + z^s + \overline{z^s} + E(z,\overline{z}).$$

Write

$$G(z,w) = \sum_{\alpha+2\beta>0}^{\infty} a_{\alpha\beta} z^{\alpha} w^{\beta}.$$

When $\alpha + 2\beta \leq N$, we will prove inductively that $a_{\alpha\beta} = \overline{a_{\alpha\beta}}$ for $\alpha = 0$ and $a_{\alpha\beta} = 0$ otherwise. First, for each positive integer m, write $E(z,\overline{z}) = E_{(m)}(z,\overline{z}) + E_m(z,\overline{z})$ with $E_{(m)}(z,\overline{z})$ a polynomial of degree at most $m-1$ and $E_m(z,\overline{z}) = O(|z|^m)$. Since $E_{(N+1)}(z,\overline{z})$ is real-valued by the hypothesis, we then get the following:

$$\sum_{\alpha+2\beta>0}^{N} a_{\alpha\beta} z^{\alpha} w^{\beta} = \sum_{\alpha+2\beta>0}^{N} \overline{a_{\alpha\beta} z^{\alpha}} w^{\beta} + o(|z|^N), \quad w = z\overline{z} + z^s + \overline{z^s} + E_{(N+1)}(z,\overline{z}). \tag{2.3}$$

Next, suppose that $N_0 = \alpha_0 + 2\beta_0$ is the smallest number such that $a_{\alpha\beta}$ is real-valued for $\alpha = 0$, and zero otherwise whenever $\alpha + 2\beta < N_0$. If $N_0 \geq N+1$ or $N_0 = \infty$, then Lemma 2.1 (iii) holds automatically. Hence, we assume that $N_0 \leq N$. For $0 < r << 1$, define $\sigma_N(\xi, r)$ to be the biholomorphic map from the unit disk $\Delta := \{\tau \in \mathbb{C}\ |\tau| < 1\}$ to the smoothly bounded simply connected domain: $\{\xi \in \mathbb{C} :\ |\xi|^2 + r^{-2}\{r^s\xi^s + r^s\overline{\xi^s} + E_{(N+1)}(r\xi, r\overline{\xi})\} < 1\}$ with $\sigma_N(\xi, r) = \xi(1 + O(r))$. (See [Lemma 2.1, Hu3]). Since the disk $\xi \longmapsto (r\sigma_N(\xi, r), r^2)$ is attached to M_{N+1} defined by $w = z\overline{z} + z^s + \overline{z^s} + E_{(N+1)}(z,\overline{z})$, it follows that

$$\sum_{\alpha+2\beta=N_0} a_{\alpha\beta} r^{N_0} \xi^{\alpha} = \sum_{\alpha+2\beta=N_0} \overline{a_{\alpha\beta}\xi^{\alpha}} r^{N_0} + o(r^{N_0}), \quad |\xi| = 1. \tag{2.4}$$

Deleting the common factor r^{N_0} of both sides and then letting $r \to 0$, we get

$$\sum_{\alpha+2\beta=N_0} a_{\alpha\beta} \xi^{\alpha} = \sum_{\alpha+2\beta=N_0} \overline{a_{\alpha\beta}\xi^{\alpha}}, \quad |\xi| = 1. \tag{2.5}$$

Hence, under the assumption that $\alpha + 2\beta = N_0$, it follows that $a_{\alpha\beta}$ is real when $\alpha = 0$, $\beta = \frac{N_0}{2} \in \mathbb{N}$ and $a_{\alpha\beta} = 0$ otherwise. This contradicts the choice of N_0 and thus completes the proof of Lemma 2.1 (iii). ■

The main purpose of this section is to prove the following uniqueness result:

Theorem 2.2. *Let n, j_0 be two integers with $n \geq 1$ and $j_0 \in [0, s-1]$. Suppose that the following formal power series*

$$\begin{cases} z' = z + f(z,w), & f(z,w) = O(|w| + |z|^2), \\ w' = w + g(w) + g_{erro}(z,w), & g(\overline{w}) = \overline{g(w)} = O(|w|^2), g_{erro}(tz, t^2 w) = o(t^{ns+j_0}) \ (as\ t \to 0), \end{cases} \tag{2.6}$$

transforms the formal Bishop surface M defined by

$$w = z\bar{z} + 2Re\left(z^s + \sum_{ks+j \leq ns+j_0; 0 \leq j \leq s-1} a_{ks+j} z^{ks+j}\right) + E_1(z, \overline{z})$$

to the formal Bishop surface defined by

$$w' = z'\bar{z'} + 2Re\left(z'^s + \sum_{ks+j \leq ns+j_0; 0 \leq j \leq s-1} b_{ks+j} z'^{ks+j}\right) + E_2(z', \overline{z'}).$$

Here for $ks + j \leq ns + j_0$, a_{ks+j}, b_{ks+j} are complex numbers with

$$a_{ks+j} = b_{ks+j} \quad for\ j = 0, 1; \tag{2.7}$$

and $E_1(z, \overline{z})$, $E_2(z, \overline{z}) = o(|z|^{ns+j_0})$. Then the following holds:
(I). $a_{ks+j} = b_{ks+j}$ for all $ks + j \leq ns + j_0$, $0 \leq j \leq s-1$.
(II). When $Ord(f(z, z\overline{z})) = 2t$ is an even number, it holds that $st + 1 > ns + j_0$ When $Ord(f(z, z\overline{z})) = 2t + 1$ is an odd number, it holds that $st + s > ns + j_0$.
(III). $Ord(g(z\overline{z})) \geq \min\{ns + j_0 + 1, Ord(f(z, z\overline{z})) + 1\}$.

One of the crucial ideas for the proof of Theorem 2.2 is to set the weight of $\overline{z}$ differently from that of z. More precisely, we set the weight of z to be 1 and that of $\bar{z}$ to be $s-1$. For a formal power series $A(z, \overline{z})$ with no constant term, we say that $wt(A(z, \overline{z})) = k$, or $wt(A(z, \overline{z})) \geq k$, if $A(tz, t^{s-1}\overline{z}) = t^k A(z, \overline{z})$, or $A(tz, t^{s-1}\overline{z}) = O(t^k)$, respectively, as $t \in \mathbb{R} \to 0$. In all that follows, we use Θ^j_l to denote a formal power series in z and $\bar{z}$ of order at least j and weight at least l. (Namely, $\Theta^j_l(tz, t\overline{z}) = O(t^j)$ and $\Theta^j_l(tz, t^{s-1}\overline{z}) = O(t^l)$ as $t \to 0$). We use $\mathbb{P}^j_l$ to denote a homogeneous polynomial in z and $\bar{z}$ with $\mathbb{P}^j_l(tz, t\overline{z}) = t^j \mathbb{P}^j_l(z, \overline{z})$ for $t \in \mathbb{R}$ and weight at least l. *We emphasize that Θ^j_l and $\mathbb{P}^j_l$ may be different in different contexts.*

In what follows, we also define the normal weight of z, w to be $1, 2$, respectively. For a formal power series $h(z, w, \overline{z}, \overline{w})$, we use $wt_{nor}(h) \geq k$ to denote the vanishing property: $h(tz, t^2 w, t\overline{z}, t^2\overline{w}) = O(t^k)$ as $t \to 0$. Let $h(z, w)$ be a formal power series in (z, w) without a constant term. Then we have the formal expansion:

$$h(z, w) = \sum_{l=1}^{\infty} h^{(l)}_{nor}(z, w),$$

where

$$h^{(l)}_{nor}(tz, t^2w) = t^l h^{(l)}_{nor}(z, w)$$

is a polynomial in (z, w). Notice that $h^{(l)}_{nor}(z, w)$ is homogeneous of degree l in the standard weighting system which assigns the weight of z and w to be 1 and 2, respectively. In this and the next sections, we write

$$h_l(z, w) = \sum_{j=l}^{\infty} h^{(j)}_{nor}(z, w) \quad \text{and} \quad h_{(l)}(z, w) = \sum_{j=1}^{l-1} h^{(j)}_{nor}(z, w). \tag{2.8}$$

Proof of Theorem 2.2: Besides proving that $a_{ks+j} = b_{ks+j}$ for $ks+j \le ns+j_0$, $0 \le j \le s-1$, we need to show that any solution (f, g) of the following equation has the vanishing property as stated in Theorem 2.2 (II)-(III), under the normalization condition for (f, g) as in the theorem:

$$\begin{aligned} w + g(w) + o(|z|^{ns+j_0}) &= (z + f(z, w))(\bar{z} + \overline{f(z, w)}) + 2Re\{(z + f(z, w))^s \\ &+ \sum_{ks+j \le ns+j_0; 0 \le j \le s-1} b_{ks+j}(z + f(z, w))^{ks+j}\} + E_2(z + f(z, w), \overline{z + f(z, w)}), \end{aligned} \tag{2.9}$$

where $w = z\bar{z} + z^s + \bar{z}^s + E(z, \bar{z})$ with $E = 2Re\left(\sum_{ks+j \le ns+j_0; 0 \le j \le s-1} a_{ks+j} z^{ks+j}\right) + E_1(z, \bar{z})$. With an immediate simplification, (2.9) takes the form:

$$\begin{aligned} g(w) &= \bar{z}f(z, w) + z\overline{f(z, w)} + |f(z, w)|^2 + 2Re\{(z + f(z, w))^s - z^s \\ &+ \sum_{ks+j \le ns+j_0; 0 \le j \le s-1} \left(b_{ks+j}(z + f(z, w))^{ks+j} - a_{ks+j} z^{ks+j}\right)\} + o(|z|^{ns+j_0}). \end{aligned} \tag{2.10}$$

In the proof of Theorem 2.2, we set the following convention. For any positive integer N, we define a_N and b_N to be as in Theorem 2.2 if $N = ks + j$ with $ks + j \le ns + j_0$, and to be 0 otherwise. For the rest of this section, we will define a positive integer N_0 as follows:

Suppose that there is a pair of positive integers (j^*, k^*) such that $(s <)k^*s + j^*(\le ns + j_0)$ is the smallest number satisfying $a_{k^*s+j^*} \ne b_{k^*s+j^*}$. We then define $N_0 = k^*s + j^*$. Otherwise, we define $N_0 = sn + j_0 + 1$. Here n, j_0 are as in Theorem 2.2.

The proof of Theorem 2.2 is carried out in two steps, according to the vanishing order of f being even or odd.

Step I of the proof of Theorem 2.2: In this step, we assume that either

$$\mathrm{Ord}(f) := \mathrm{Ord}\,(f(z, z\bar{z})) \tag{2.11}$$

is an even number denoted by $2t$ or $f \equiv 0$, where $w(z, \bar{z}) = z\bar{z} + z^s + \bar{z}^s + E(z, \bar{z})$. Write

$$g(w) = c_l w^l + o(w^l).$$

Denote by

$$\widehat{N_0} = \min\{N_0,\ \mathrm{Ord}(f), sn + j_0\}.$$

(When $f \equiv 0$, we define $\mathrm{Ord}(f) = \infty$.) Then (2.10) gives the following:

$$c_l z^l \bar{z}^l + O(|z|^{2l+1}) = 2Re[(b_{N_0} - a_{N_0})z^{N_0}] + O(|z|^{\widehat{N_0}+1}). \tag{2.12}$$

Notice that the first term on the left hand side is a mixed term, while the first term on the right hand side is a harmonic term. From this, we can easily conclude the following:

(2.I). Suppose that $2t \geq N_0$ and $c_l \neq 0$. Then $2l > \min\{N_0, sn + j_0\}$ and $b_{N_0} = a_{N_0}$. By our definition of N_0, N_0 must be $ns + j_0 + 1$. Hence, the theorem in this case readily follows. A similar argument can be used when $2t \geq N_0$ and $Ord(g) = \infty$.

(2.II). When $2t < N_0$, then $2l \geq \widehat{N_0} + 1 = \min\{2t+1, sn+j_0+1\} = 2t+1$ under the assumption that $c_l \neq 0$. Thus we either have $Ord(g) = \infty$ or we have $l > t \geq 1$ when $c_l \neq 0$.

Suppose that $N_0 = 2t + 1$ in Case (2.II). Assuming that $N_0 < ns + j_0 + 1$ and collecting terms with degree $2t + 1$ in (2.10), we obtain

$$\bar{z} f^{(2t)}_{nor}(z, z\bar{z}) + z\overline{f^{(2t)}_{nor}(z, z\bar{z})} + 2Re\left((b_{N_0} - a_{N_0})z^{N_0}\right) = 0. \tag{2.13}$$

Notice that in the above, the first two are mixed terms, while the last term is a harmonic term. This clearly forces that $a_{N_0} = b_{N_0}$. Thus, we must have $N_0 = ns + j_0 + 1$ and Theorem 2.2 also follows easily in this setting. *Hence, we will assume, in what follows:*

(2.III). $N_0 \geq 2t + 2$. (As a consequence, it also holds that $g(w) = O(|w|^l)$ with $l > t \geq 1$.)

Collecting terms with (the ordinary) degree $2t + 1$ in (2.10), we get:

$$\bar{z} f^{(2t)}_{nor}(z, z\bar{z}) + z\overline{f^{(2t)}_{nor}(z, z\bar{z})} = 0. \tag{2.14}$$

Writing $f^{(2t)}_{nor}(z, w) = \sum_{k+2l=2t} a_{kl} z^k w^l$ and substituting it back to (2.14), we then get:

$$\sum_{k+2l=2t} a_{kl} z^{k+l} \bar{z}^{l+1} + \sum_{k'+2l'=2t} \overline{a_{k'l'}} z^{l'+1} \bar{z}^{l'+k'} = 0.$$

Since $k + 2l = 2t$, $k' + 2l' = 2t$, we get $\frac{k+k'}{2} = 2t - (l + l')$. Now, for $k > 2$, we have $(k + l) - (l' + 1) = 2t - (l + l') - 1 > 0$, or $k + l > l' + 1$. Thus, we conclude that $a_{kl} = 0$ for $k > 2$. In the other cases, we get $a_{0l} + \overline{a_{2l'}} = 0$ with $l = t$ and $l' = t - 1$. Let $a = a_{0t}$. We get that

$$f^{(2t)}_{nor}(z, w) = aw^t - \bar{a} z^2 w^{t-1} \tag{2.15}$$

for $a \neq 0$. Hence

$$f(z, w) = f^{(2t)}_{nor}(z, w) + f_{2t+1}(z, w) = aw^t - \bar{a} z^2 w^{t-1} + f_{2t+1}(z, w). \tag{2.16}$$

Next, a simple computation shows that $wt(w) \geq s$, $\mathrm{Ord}(w(z,\overline{z})) \geq 2$, $wt(\overline{f_{nor}^{(2t)}(z,w)}) \geq st + 2 - s$, $wt(f_{nor}^{(2t)}(z,w)) \geq st$, $g(w) = g_{2t+2}(w)$, $wt(\overline{f_{2t+1}(z,w)}) \geq st + s - 1$. Also if $l_1 + l_2 \geq s$ with $l_2 > 1$, or $l_1 + l_2 > s$ with $l_2 \geq 1$, then $wt(z^{l_1} f_{nor}^{(2t)l_2}(z,w)) \geq l_1 + l_2(ts + 2 - s) \geq ts + 2$. Moreover, $wt(z^{l_1} f_{nor}^{(2t)l_2}(z,w) f_{2t+1}^{l_3}(z,w)) \geq s$ if $l_1 + l_2 + l_3 \geq s - 1$, $l_2^2 + l_3^2 \neq 0$. Now we can conclude that

$$wt\left(|f_{nor}^{(2t)}(z,w)|^2\right) \geq ts + 2, \; wt\left(\overline{f_{nor}^{(2t)}(z,w)} f_{2t+1}(z,w)\right) \geq ts + 2t + 1,$$
$$wt\left(f_{nor}^{(2t)}(z,w)\overline{f_{2t+1}(z,w)}\right) \geq (2 + (t-1)s) + (ts + s - 1) \geq ts + 2,$$
$$wt\left(|f_{2t+1}(z,w)|^2\right) \geq 2t + 1 + (ts + s - 1) \geq ts + 2.$$

Hence, we have the following

$$|f(z,w)|^2 = |f_{nor}^{(2t)}(z,w)|^2 + 2Re(\overline{f_{nor}^{(2t)}(z,w)} f_{2t+1}(z,w)) + |f_{2t+1}(z,w)|^2 = \Theta_{ts+2}^{2t+2}.$$

Substituting (2.16) into (2.10) and making use of the estimates we just presented, we get:

$$\begin{aligned} g_{2t+2}(w) &= 2Re\{(\overline{z} + sz^{s-1})f\} + |f(z,w)|^2 + 2Re\{\sum_{l=2}^{s} c_l z^{s-l} f^l\} \\ &+ 2Re(\sum_{s<\tau=ks+j<N_0} \sum_{l=0}^{\tau-1} b_{l\tau} z^l f^{\tau-l}) + 2Re\left((b_{N_0} - a_{N_0})z^{N_0}\right) + \Theta_{\min\{N_0+1, ns+j_0+1\}}^{\min\{N_0+1, ns+j_0+1\}} \\ &= 2Re\{(\overline{z} + sz^{s-1}) f_{nor}^{(2t)}(z,w) + (\overline{z} + sz^{s-1}) f_{2t+1}(z,w)\} \\ &+ 2Re\left((b_{N_0} - a_{N_0})z^{N_0}\right) + \Theta_s^2 f_{2t+1}(z,w) + \Theta_s^2 \overline{f_{2t+1}(z,w)} + \Theta_{N_s}^{2t+2}. \end{aligned} \tag{2.17}$$

Here $c_l, b_{l\tau}$ are complex numbers, N_0 is defined as before and

$$N_s := \min\{ts + 2, N_0 + 1, ns + j_0 + 1\}. \tag{2.18}$$

Notice that

$$\begin{aligned} &2Re\{(\bar{z} + sz^{s-1}) f_{nor}^{(2t)}(z,w)\} \\ &= 2Re\{\bar{z}(aw^t - \bar{a}z^2 w^{t-1}) + sz^{s-1}(aw^t - \bar{a}z^2 w^{t-1})\} \\ &= \overline{a}zw^t - \overline{a}\overline{z}z^2 w^{t-1} - \overline{a}sz^{s+1}w^{t-1} + \Theta_{ts+2}^{2t+2} \\ &= \overline{a}zw^{t-1}(w - |z|^2) - \overline{a}sz^{s+1}w^{t-1} + \Theta_{ts+2}^{2t+2} = (1-s)\overline{a}z^{s+1}w^{t-1} + \Theta_{ts+2}^{2t+2}. \end{aligned} \tag{2.19}$$

Hence, we obtain, over M, the following:

$$\begin{aligned} g_{2t+2}(w) &= (1-s)\bar{a}z^{s+1}(z\overline{z} + z^s)^{t-1} + (\overline{z} + sz^{s-1} + \Theta_s^2) f_{2t+1}(z,w) \\ &+ 2Re\left((b_{N_0} - a_{N_0})z^{N_0}\right) + (z + s\overline{z}^{s-1} + \Theta_s^2)\overline{f_{2t+1}(z,w)} + \Theta_{N_s}^{2t+2}. \end{aligned} \tag{2.20}$$

If $t = 1$, collecting terms of degree $s+1$ in (2.20) and noticing that

$$w = z\overline{z} + O(|z|^s), \; N_0 > s + 1$$

by the given condition, we get

$$\sum_{2j}\delta_{2j}^{s+1}g_{nor}^{(2j)}(z\bar{z}) = (1-s)\bar{a}z^{s+1} + \bar{z}f_{nor}^{(s)}(z,z\bar{z}) + z\overline{f_{nor}^{(s)}(z,z\bar{z})} + \mathbb{P}_{s+2}^{s+1}. \tag{2.21}$$

Here δ_{2j}^{s+1} takes value 1 when $2j = s+1$, and 0 otherwise.

Since $s+2 > s+1$, $\mathbb{P}_{s+2}^{s+1} = \bar{z}A(z,\bar{z})$ with $A(z,\bar{z})$ a polynomial. Thus it follows easily that $(1-s)\bar{a}z^{s+1}$ is dividable by $\bar{z}$. This is a contradiction and thus the case of $t=1$ is proved.

We next prove the following crucial lemma for the proof of Theorem 2.2:

Lemma 2.3. *Assume the hypothesis and the notation in Theorem 2.2. Let $Ord(f(z,z\bar{z})) = 2t < \infty$ and keep all the notation that we have set up so far. Suppose that $N_0 \geq 2t+2$. Assume that $2t + j(s-2) + 2 \leq m \leq 2t + (j+1)(s-2) + 1$ with $0 \leq j \leq t-1$ and $m \leq N_0$. Then, over M, we have*

$$\begin{aligned} g_m(w) =& \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z}+z^s)^{t-j-1} + (\bar{z} + sz^{s-1} + \Theta_s^2)f_{m-1}(z,w) \\ &+ (z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{m-1}(z,w)} + 2Re\left((b_{N_0} - a_{N_0})z^{N_0}\right) + \Theta_{N_s}^m. \end{aligned} \tag{2.22}$$

Proof of Lemma 2.3: When $m = ns + j_0 + 1$, we have $N_s \leq m$. Thus, (2.22) holds trivially due to the presence of the term $\Theta_{N_s}^m$. Hence, in the proof of the lemma, we always assume that $m \leq ns + j_0$ for the m in Lemma 2.3. We also recall that $N_s = \min\{ts+2, N_0+1, ns+j_0+1\}$ with n, j_0 defined as in Theorem 2.2.

The argument presented above gives the proof of the lemma with $m = 2t+2$. We complete the proof of the lemma in three steps.

Step I of the proof of Lemma 2.3: This step is not needed when $s = 3$. Denote $m_0 = 2t + j(s-2) + 2$, where j is an integer with $0 \leq j \leq t-1$. Suppose that $m_0 \leq N_0$. We also assume that there is an integer m such that $m \geq m_0$, $m+1 \leq 2t + (j+1)(s-2) + 1$ (such an m certainly does not exist if $s = 3$), $m+1 \leq N_0$ and moreover (2.22) holds for this m. Collecting terms of degree m in (2.22), we get

$$g_{nor}^{(m)}(z\bar{z}) = \bar{z}f_{nor}^{(m-1)}(z,z\bar{z}) + z\overline{f_{nor}^{(m-1)}(z,z\bar{z})} + \hat{\mathbb{P}}_{N_s}^m. \tag{2.23}$$

Since $g_{nor}^{(m)}(z\bar{z})$ is real-valued, the $\hat{\mathbb{P}}_{N_s}^m$ here is real valued. Notice also that $g_{nor}^{(m)}(z\bar{z})$ is of weight at least N_s. We can write

$$g_{nor}^{(m)}(z\bar{z}) - \hat{\mathbb{P}}_{N_s}^m = \sum_{\substack{\alpha+\beta=m \\ \alpha+\beta(s-1)\geq N_s}} a_{\alpha\beta}z^\alpha\bar{z}^\beta. \tag{2.24}$$

Write

$$f_{nor}^{(m-1)}(z,z\bar{z}) = \sum_{\widetilde{\alpha}+2\widetilde{\beta}=m-1} b_{\widetilde{\alpha}\widetilde{\beta}}z^{\widetilde{\alpha}}(z\bar{z})^{\widetilde{\beta}} = \sum_{\widetilde{\alpha}+2\widetilde{\beta}=m-1} b_{\widetilde{\alpha}\widetilde{\beta}}z^{\widetilde{\alpha}+\widetilde{\beta}}\bar{z}^{\widetilde{\beta}}. \tag{2.25}$$

Then

$$\sum_{\widetilde{\alpha}+2\widetilde{\beta}=m-1} b_{\widetilde{\alpha}\widetilde{\beta}} z^{\widetilde{\alpha}+\widetilde{\beta}} \bar{z}^{\widetilde{\beta}+1} + \sum_{\alpha^*+2\beta^*=m-1} \overline{b_{\alpha^*\beta^*}} \bar{z}^{\alpha^*+\beta^*} z^{\beta^*+1} = \sum_{\substack{\alpha+\beta=m \\ \alpha+\beta(s-1)\geq N_s}} a_{\alpha\bar{\beta}} z^{\alpha}\bar{z}^{\beta}. \tag{2.26}$$

As in the discussion for (2.15), $z^{\widetilde{\alpha}+\widetilde{\beta}}\bar{z}^{\widetilde{\beta}+1} = \bar{z}^{\alpha^*+\beta^*}z^{\beta^*+1}$ if and only if $\widetilde{a}+\alpha^*=2$. Notice also that the reality in (2.24) shows that $\beta+\alpha(s-1)\geq N_s$ for $a_{\alpha\bar{\beta}}\neq 0$.

Now, if m is even, then

$$2b_{\widetilde{\alpha}\widetilde{\beta}} = a_{\alpha\bar{\beta}} + ic \text{ with } c\in\mathbb{R} \text{ under the condition that } \alpha=\beta=\frac{m}{2},\ \widetilde{\alpha}=1,\ \widetilde{\beta}=\frac{m}{2}-1. \tag{2.27}$$

The other relations are as follows:

$$\begin{gathered} b_{\widetilde{\alpha}\widetilde{\beta}} = a_{\alpha\bar{\beta}},\ \text{if } \widetilde{\alpha}+\widetilde{\beta}=\alpha,\ \widetilde{\alpha}+2\widetilde{\beta}=m-1,\ \widetilde{\beta}+1=\beta, \widetilde{\alpha}\neq 1, \alpha+\beta=m. \\ \text{Here } \alpha+(s-1)\beta\geq N_s, \beta+(s-1)\alpha\geq N_s. \end{gathered} \tag{2.28}$$

If m is odd, we still have the same relation as in (2.28) except when $\widetilde{\alpha}=0,\ \widetilde{\beta}=\frac{m-1}{2}$ or when $\widetilde{\alpha}=2,\ \widetilde{\beta}=\frac{m-3}{2}$.

Next, for m even and $\widetilde{\alpha}=1,\ \widetilde{\beta}=\frac{m}{2}-1$, letting $\alpha=\beta=\frac{m}{2}$, we also have $\alpha=\widetilde{\alpha}+\widetilde{\beta},\ \beta=\widetilde{\beta}+1$, $\alpha+\beta=m$, $\alpha+(s-1)\beta=\beta+(s-1)\alpha=\frac{m}{2}s\geq(t+1)s>N_s$.

Assume that m is old. (Thus $m\geq 2t+3$). When $\widetilde{\alpha}=0,\ \widetilde{\beta}=\frac{m-1}{2}$, let $\alpha=\frac{m-1}{2}, \beta=\frac{m+1}{2}$. When $\widetilde{\alpha}=2,\ \widetilde{\beta}=\frac{m-3}{2}$, let $\alpha=\frac{m+1}{2}$ and $\beta=\frac{m-1}{2}$. We similarly have the relation as in (2.28): $\alpha=\widetilde{\alpha}+\widetilde{\beta},\ \beta=\widetilde{\beta}+1$, $\alpha+\beta=m$, $\alpha+(s-1)\beta\geq N_s,\ \beta+(s-1)\alpha\geq N_s$. Thus for all the α,β uniquely determined by $\widetilde{\alpha}$ and $\widetilde{\beta}$ as just discussed above, we always have:

$$\widetilde{\alpha}+\widetilde{\beta}+(s-1)\widetilde{\beta} = \alpha+(s-1)(\beta-1) = \alpha+(s-1)\beta-(s-1)\geq N_s-(s-1). \tag{2.29}$$

From this, one easily sees that

$$wt(f_{nor}^{(m-1)}(z,z\overline{z})) \geq \min_{\widetilde{\alpha}\geq 0}\{\widetilde{\alpha}+\widetilde{\beta}+(s-1)\widetilde{\beta}\} = \min_{\alpha}\{\alpha+(s-1)\beta-s+1\}\geq N_s-s+1, \tag{2.30}$$

$$wt(f_{nor}^{(m-1)}(z,w))\geq N_s-s+1,\ wt(g_{nor}^{(m)}(z,z\overline{z})), wt(g_{nor}^{(m)}(z,w))\geq N_s, \tag{2.31}$$

$$wt\{f_{nor}^{(m-1)}(z,z\overline{z})-f_{nor}^{(m-1)}(z,w)\}\geq N_s-s+1, \tag{2.32}$$

$$wt(\overline{f_{nor}^{(m-1)}(z,z\overline{z})}) \geq \min_{\widetilde{\alpha}\geq 0}\{(s-1)\widetilde{\alpha}+s\widetilde{\beta}\} = \min_{\alpha}\{(s-1)(\alpha-\beta+1)+s(\beta-1)\}\geq N_s-1, \tag{2.33}$$

$$wt\{\overline{f_{nor}^{(m-1)}(z,z\overline{z})-f_{nor}^{(m-1)}(z,w)}\}\geq N_s-1. \tag{2.34}$$

Substituting $f_{m-1}(z,w)=f_{nor}^{(m-1)}(z,w)+f_m(z,w)$ into (2.22) and making use of (2.23), (2.30)-(2.34), we get

$$\begin{aligned} g_{m+1}(w) =& (1-s)^{j+1}\bar{a}z^{(j+1)s+1}(z\bar{z}+z^s)^{t-j-1} + (\bar{z}+sz^{s-1}+\Theta_s^2)f_m(z,w) \\ &+ (z+s\bar{z}^{s-1}+\Theta_s^2)\overline{f_m(z,w)} + \Theta_{N_s}^{m+1} + (sz^{s-1}+\Theta_s^2)f_{nor}^{(m-1)}(z,z\overline{z}) \\ &+ 2Re\left((b_{N_0}-a_{N_0})z^{N_0}\right) + (s\bar{z}^{s-1}+\Theta_s^2)\overline{f_{nor}^{(m-1)}(z,z\overline{z})}. \end{aligned} \tag{2.35}$$

By (2.30) and (2.33), we get

$$(sz^{s-1}+\Theta_s^2)f_{nor}^{(m-1)}(z,w)+(s\bar{z}^{s-1}+\Theta_s^2)\overline{f_{nor}^{(m-1)}(z,w)}=\Theta_{N_s}^{m+1}. \tag{2.36}$$

Hence

$$\begin{aligned} g_{m+1}(w)=&(1-s)^{(j+1)}\bar{a}z^{(j+1)s+1}(z\bar{z}+z^s)^{t-j-1}+(\bar{z}+sz^{s-1}+\Theta_s^2)f_m(z,w)\\ &+(z+s\bar{z}^{s-1}+\Theta_s^2)\overline{f_m(z,w)}+2Re\left((b_{N_0}-a_{N_0})z^{N_0}\right)+\Theta_{N_s}^{m+1}. \end{aligned} \tag{2.37}$$

By induction, we showed that if the lemma holds for m_0 defined above, then it holds for any m with $m_0\le m\le 2t+(j+1)(s-2)+1$ and $m\le N_0$.

Step II of the proof of Lemma 2.3: In this step, suppose that we know that the lemma holds for $m\in[2t+j(s-2)+2, 2t+(j+1)(s-2)+1]$ with $m\le N_0$, where j is a certain non-negative integer bounded by $t-2$. We then proceed to prove that the lemma holds also for $m\in[2t+(j+1)(s-2)+2, 2t+(j+2)(s-2)+1]$, whenever $m\le N_0$.

Suppose that $2t+(j+1)(s-2)+1<N_0$. By the assumption, we have over M

$$\begin{aligned} g_{2t+(j+1)(s-2)+1}(w)=&\bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z}+z^s)^{t-j-1}+(\bar{z}+sz^{s-1}+\Theta_s^2)f_{2t+(j+1)(s-2)}(z,w)\\ &+(z+s\bar{z}^{s-1}+\Theta_s^2)\overline{f_{2t+(j+1)(s-2)}(z,w)}+2Re\left((b_{N_0}-a_{N_0})z^{N_0}\right)+\Theta_{N_s}^{2t+(j+1)(s-2)+1}. \end{aligned} \tag{2.38}$$

Collecting terms of degree $2t+(j+1)(s-2)+1$ in (0.1), we get

$$\begin{aligned} g_{nor}^{(2t+(j+1)(s-2)+1)}(z\bar{z})=&\bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1}+\hat{\mathbb{P}}_{N_s}^{2t+(j+1)(s-2)+1}\\ &+\bar{z}f_{nor}^{(2t+(j+1)(s-2))}(z,z\bar{z})+z\overline{f_{nor}^{(2t+(j+1)(s-2))}(z,z\bar{z})}. \end{aligned} \tag{2.39}$$

Here $\hat{\mathbb{P}}_{N_s}^{2t+(j+1)(s-2)+1}$ is a certain homogeneous polynomial of degree $2t+(j+1)(s-2)+1$ with weight at least N_s.

Now, we solve (0.2) as follows. Denote by $\Lambda=2t+(j+1)(s-2)$. Notice that

$$I:=-\hat{\mathbb{P}}_{N_s}^{\Lambda+1}+a(1-s)^{j+1}\bar{z}^{(j+1)s+1}(z\bar{z})^{t-j-1}+g_{nor}^{(\Lambda+1)}(z\bar{z})$$

is real valued and $I=\mathbb{P}_{N_s}^{\Lambda+1}$. Then (0.2) can be rewritten as

$$\begin{aligned} I=&\bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1}+a(1-s)^{j+1}\bar{z}^{(j+1)s+1}(z\bar{z})^{t-j-1}\\ &+\bar{z}f_{nor}^{(2t+(j+1)(s-2))}(z,z\bar{z})+z\overline{f_{nor}^{(2t+(j+1)(s-2))}(z,z\bar{z})}. \end{aligned} \tag{2.40}$$

Write

$$I=\sum_{\substack{\alpha+\beta=\Lambda+1\\ \alpha+(s-1)\beta\ge N_s}}a_{\alpha\bar{\beta}}z^{\alpha}\bar{z}^{\beta}.$$

Since $a_{\alpha\bar{\beta}} = \overline{a_{\beta\bar{\alpha}}}$, we also require that $\beta + (s-1)\alpha \geq N_s$. We next have the following general solution of (0.3):

$$\begin{array}{l} f_{nor}^{(2t+(j+1)(s-2))}(z,w) = f_1^{(\Lambda)}(z,w) + f_2^{(\Lambda)}(z,w) \quad \text{with} \\ f_1^{(\Lambda)}(z,w) = -\bar{a}(1-s)^{j+1} z^{(j+1)s+2} w^{t-j-2}, \\ f_2^{(\Lambda)}(z,w) = \sum_{\widetilde{\alpha}+2\widetilde{\beta}=\Lambda} h_{\widetilde{\alpha}\widetilde{\beta}} z^{\widetilde{\alpha}} w^{\widetilde{\beta}}, \end{array} \tag{2.41}$$

where $h_{\widetilde{\alpha}\widetilde{\beta}}$ are determined by the following:

$$\sum_{\widetilde{\alpha}+2\widetilde{\beta}=\Lambda} h_{\widetilde{\alpha}\widetilde{\beta}} z^{\widetilde{\alpha}+\widetilde{\beta}} \bar{z}^{\widetilde{\beta}+1} + \sum_{\widetilde{\alpha}+2\widetilde{\beta}=\Lambda} \overline{h_{\widetilde{\alpha}\widetilde{\beta}}} z^{\widetilde{\beta}+1} \bar{z}^{\widetilde{\alpha}+\widetilde{\beta}} = \sum_{\substack{\alpha+\beta=\Lambda+1 \\ \alpha+\beta(s-1)\geq N_s}} a_{\alpha\bar{\beta}} z^{\alpha} \bar{z}^{\beta}. \tag{2.42}$$

Now, (0.5) can be handled exactly in the same way as for (2.26). (The only difference is that the role of $m-1$ is now played by Λ.) For convenience of the reader, we repeat some details as follows:

First, we have similar relations as those in (2.27), (2.28), (2.29), etc. Next, we can conclude the following:

$$wt(f_2^{(\Lambda)}(z,z\bar{z})) \geq \min_{\widetilde{\alpha}\geq 0}\{\widetilde{\alpha} + \widetilde{\beta} + (s-1)\widetilde{\beta}\} = \min_{\alpha}\{\alpha + (s-1)\beta - s + 1\} \geq N_s - s + 1, \tag{2.43}$$

$$wt\{f_2^{(\Lambda)}(z,w)\} \geq N_s - s + 1,\ wt\{g^{(\Lambda+1)}(z,w)\}, wt\{g^{(\Lambda+1)}(z,z\bar{z})\} \geq N_s, \tag{2.44}$$

$$wt\{f_2^{(\Lambda)}(z,w) - f_2^{(\Lambda)}(z,z\bar{z})\} \geq N_s - s + 1, \tag{2.45}$$

$$wt\{\overline{f_2^{(\Lambda)}(z,w)}\},\ wt\{\overline{f_2^{(\Lambda)}(z,z\bar{z})}\} \geq N_s - 1, \tag{2.46}$$

$$(sz^{s-1} + \Theta_s^2) f_2^{(\Lambda)}(z,z\bar{z}) + (s\bar{z}^{s-1} + \Theta_s^2)\overline{f_2^{(\Lambda)}(z,z\bar{z})} = \Theta_{N_s}^{\Lambda+2}, \tag{2.47}$$

$$(sz^{s-1} + \Theta_s^2) f_2^{(\Lambda)}(z,w) + (s\bar{z}^{s-1} + \Theta_s^2)\overline{f_2^{(\Lambda)}(z,w)} = \Theta_{N_s}^{\Lambda+2}, \tag{2.48}$$

$$wt\{f_1^{(\Lambda)}(z,z\bar{z})\} \geq st - s + 2, \quad wt\{\overline{f_1^{(\Lambda)}(z,w)}\},\ wt\{\overline{f_1^{(\Lambda)}(z,z\bar{z})}\} \geq N_s. \tag{2.49}$$

For instance, to see (0.11), it suffices to notice that by (0.6)- (0.9), we have

$$wt\{(sz^{s-1} + \Theta_s^2) f_2^{(\Lambda)}(z,w) + (s\bar{z}^{s-1} + \Theta_s^2)\overline{f_2^{(\Lambda)}(z,w)}\} \geq s - 1 + N_s - s + 1 = N_s. \tag{2.50}$$

Hence, from (0.1)-(0.12), we get

$$\begin{aligned} g_{\Lambda+2}(w) + g_{nor}^{(\Lambda+1)}(w) =& (\bar{z} + sz^{s-1} + \Theta_s^2) f_{\Lambda+1}(z,w) + (z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{\Lambda+1}(z,w)} \\ &+ \Theta_{N_s}^{\Lambda+2} + \hat{\mathbb{P}}_{N_s}^{\Lambda+1} + \bar{a}(1-s)^{j+1} z^{(j+1)s+1}(z\bar{z} + z^s)^{t-j-1} \\ &+ (\bar{z} + sz^{s-1} + \Theta_s^2) f_{nor}^{(\Lambda)}(z,w) + (z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{nor}^{(\Lambda)}(z,w)} \\ &+ 2Re\left((b_{N_0} - a_{N_0}) z^{N_0}\right). \end{aligned} \tag{2.51}$$

Notice that

$$g_{nor}^{(\Lambda+1)}(z\bar{z}) = \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1} + \bar{z}f_{nor}^{(\Lambda)}(z,z\bar{z}) + z\overline{f_{nor}^{(\Lambda)}(z,z\bar{z})} + \hat{\mathbb{P}}_{N_s}^{\Lambda+1}. \tag{2.52}$$

Also,

$$wt\{g_{nor}^{(\Lambda+1)}(w)\}, wt\{g_{nor}^{(\Lambda+1)}(z\bar{z})\} \geq \frac{s(\Lambda+1)}{2} = ts + \frac{1}{2}s(j+1)(s-2) + \frac{s}{2} \geq ts+2.$$

Hence

$$g_{nor}^{(\Lambda+1)}(w) - g_{nor}^{(\Lambda+1)}(z\bar{z}) \in \Theta_{N_s}^{\Lambda+2}. \tag{2.53}$$

Subtracting (0.15) from (0.14) and then making use of (0.16), we obtain

$$\begin{aligned} g_{\Lambda+2}(w) =&(\bar{z} + sz^{s-1} + \Theta_s^2)f_{\Lambda+1}(z,w) + 2Re\left((b_{N_0} - a_{N_0})z^{N_0}\right) \\ &+ (z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{\Lambda+1}(z,w)} + \Theta_{N_s}^{\Lambda+2} + J, \end{aligned} \tag{2.54}$$

where

$$\begin{aligned} J =&(\bar{z} + sz^{s-1} + \Theta_s^2)f_{nor}^{(\Lambda)}(z,w) + (z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{nor}^{(\Lambda)}(z,w)} \\ &+ \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z}+z^s)^{t-j-1} - \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1} \\ &- (\bar{z}f_{nor}^{(\Lambda)}(z,z\bar{z}) + z\overline{f_{nor}^{(\Lambda)}(z,z\bar{z})}). \end{aligned} \tag{2.55}$$

Here, by (0.10)-(0.12)and the formula in (0.4) for $f_1^{(\Lambda)}$, we notice that

$$\begin{aligned} &\bar{z}f_{nor}^{(\Lambda)}(z,w) + z\overline{f_{nor}^{(\Lambda)}(z,w)} - (\bar{z}f_{nor}^{(\Lambda)}(z,z\bar{z}) + z\overline{f_{nor}^{(\Lambda)}(z,z\bar{z})}) \\ &\quad +\bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z}+z^s)^{t-j-1} - \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1} \\ &= -\bar{a}(1-s)^{j+1}z^{(j+1)s+1}z\bar{z}(z\bar{z}+z^s)^{t-j-2} + \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1} \\ &\quad +\bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z}+z^s)^{t-j-1} - \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1} + \Theta_{N_s}^{\Lambda+2} \\ &= -\bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z}+z^s)^{t-j-2}\left(z\bar{z} - (z\bar{z}+z^s)\right) + \Theta_{N_s}^{\Lambda+2} \\ &= \bar{a}(1-s)^{j+1}z^{(j+2)s+1}(z\bar{z}+z^s)^{t-j-2} + \Theta_{N_s}^{\Lambda+2}. \end{aligned}$$

Hence by the formula in (0.4) for $f_1^{(\Lambda)}$ and by (0.12) (0.13), we have

$$\begin{aligned} J =&(sz^{s-1} + \Theta_s^2)f_1^{(\Lambda)}(z,w) + (s\bar{z}^{s-1} + \Theta_s^2)\overline{f_1^{(\Lambda)}(z,w)} \\ &+ \bar{a}(1-s)^{j+1}z^{(j+2)s+1}(z\bar{z}+z^s)^{t-j-2} + \Theta_{N_s}^{\Lambda+2} \\ =&\bar{a}(1-s)^{j+2}z^{(j+2)s+1}(z\bar{z}+z^s)^{t-j-2} + \Theta_{N_s}^{\Lambda+2}. \end{aligned} \tag{2.56}$$

This proves the lemma when $m = 2t + (j+1)(s-2) + 2$. Now, the result obtained in the previous step completes the proof of the claim in this step.

Step III of the proof of Lemma 2.3: We now can complete the proof of the lemma by inductively using results obtained in Steps I-II. Indeed, since we know that the Lemma holds for $m = 2t+2$, we see, by Step I, that the lemma holds for any $m \le N_0$ with $m \in [2t+2, 2t+(s-2)+1]$. First applying Step II and then applying Step I again, we see the lemma holds for any $m \le N_0$ with $m \in [2t + j(s-2)+2, 2t+(j+1)(s-2)+1]$ and with $j = 1$. Now, by an induction argument on j, we see the proof of the lemma. ■

Now we are ready to complete the proof of Theorem 2.2 in case $\mathrm{Ord}(f) = 2t$. Since by (2.III), we need only to consider the situation when $N_0 \ge 2t+2$, it suffices for us to study the following two subcases:

Case I: If $N_0 > m = ts+1$, then $m \in [2t + j(s-2)+2, 2t+(j+1)(s-2)+1]$ with $j = t-1$. Also, notice in this setting that $N_s = ts+2$. Applying Lemma 2.3 with $m = ts+1$ and $j = t-1$, we have:

$$\begin{aligned} g_{ts+1}(w) =& \bar{a}(1-s)^t z^{ts+1} + \Theta_{ts+2}^{ts+1} + (\bar{z} + sz^{s-1} + \Theta_s^2) f_{ts}(z,w) \\ &+ (z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{ts}(z,w)}. \end{aligned}$$

Collecting terms of degree $ts+1$ in the above equation, we obtain:

$$g_{nor}^{(ts+1)}(z\bar{z}) = \bar{a}(1-s)^t z^{ts+1} + \mathbb{P}_{ts+2}^{ts+1} + \bar{z} f_{nor}^{(ts)}(z, z\bar{z}) + z\overline{f_{nor}^{(ts)}(z, z\bar{z})}. \tag{2.57}$$

Since $ts+2 > ts+1$, we can write $\mathbb{P}_{ts+2}^{ts+1} = \bar{z}A(z,\bar{z})$ for some polynomial function A. Hence, the equation is solvable only if $a = 0$, which is a contradiction.

Case II: Suppose $(2t+1 <) N_0 \le ts+1$.

Assume that $N_0 \le ns + j_0$. By the assumption that $a_{ks+1} = b_{ks+1}$ for $ks+1 \le ns+j_0$ and by the definition of N_0, we notice that $N_0 \ne ts+1$. Hence, we must have $2t+1 < N_0 < ts+1$. Notice that $N_s = N_0 + 1$ now. Assume that j is the integer such that $2t + j(s-2)+2 \le N_0 \le 2t+(j+1)(s-2)+1$. By Lemma 2.3 and collecting terms of degree N_0 in (2.22), we have

$$\begin{aligned} g_{nor}^{(N_0)}(z\bar{z}) =& 2Re\{(b_{N_0} - a_{N_0}) z^{N_0}\} + \delta(1-s)^{j+1}\bar{a} z^{(j+1)s+1}(z\bar{z})^{t-j-1} \\ &+ \bar{z} f_{nor}^{(N_0-1)}(z, z\bar{z}) + z\overline{f_{nor}^{(N_0-1)}(z, z\bar{z})} + \Theta_{N_0+1}^{N_0}. \end{aligned}$$

Here $\delta = 0$ if $N_0 < 2t+(j+1)(s-2)+1$ and $\delta = 1$ if $N_0 = 2t+(j+1)(s-2)+1$. Notice that when $j = t-1$, $2t+(j+1)(s-2)+1 = ts+1 > N_0$. Hence, when $\delta = 1$, we have $t-j-1 > 0$. Now, since $2Re\{(b_{N_0} - a_{N_0})z^{N_0}\}$ is the only non-mixed term, by the same argument as above, we can see a contradiction too. Hence, to reach no contradiction, we must have $b_N = a_N$ for any $N \le ns + j_0$, namely, $N_0 = ns + j_0 + 1$. Back to the hypothesis in Case II, we obtain $ts+1 \ge ns + j_0 + 1$. This finally completes the proof.

Step II of the proof of Theorem 2.2: In this step, we show that we also have the result stated in Theorem 2.2 when $\mathrm{Ord}(f)$ is a finite odd number by applying the same argument as that in Step I. (See (2.11) for the definition of $\mathrm{Ord}(f)$). Since the argument is completely parallel to that in Step I, we will be very brief.

Suppose that $\mathrm{Ord}(f) = 2t+1 < \infty$, $g(w) = c_l w^l + o(w^l)$. And let $\widehat{N_0} = \min\{N_0, \mathrm{Ord}(f), sn + j_0\}$ as in Step I. We then also have (2.12) and the proof of the theorem in the case of $2t+1 \geq N_0$ can be similarly achieved.

Assume that $2t+1 < N_0 (\leq ns + j_0 + 1)$. As before, we have $2l \geq \widehat{N_0} + 1 = \min\{2t+2, sn + j_0 + 1\} = 2t+2$ under the assumption that $c_l \neq 0$. Thus we have $l \geq t+1$ when $c_l \neq 0$.

Suppose that $N_0 = 2t+2$. Assuming that $N_0 < ns + j_0 + 1$ and collecting terms with degree $2t+2$ in (2.10), we obtain

$$-g_{nor}^{(2t+2)}(z\bar{z}) + \bar{z} f_{nor}^{(2t+1)}(z, z\bar{z}) + z\overline{f_{nor}^{(2t+1)}(z, z\bar{z})} + 2Re\left((b_{N_0} - a_{N_0})z^{N_0}\right) = 0. \tag{2.58}$$

Since the last term is harmonic and the others are divisible by $z\bar{z}$, we see that $a_{N_0} = b_{N_0}$. This is a contradiction. We thus have $N_0 = ns + j_0 + 1$ and Theorem 2.2 also follows easily as before. Hence, it suffices to consider the following case:

$$N_0 \geq 2t+3. \text{ (Then } g(w) = O(|w|^l) \text{ with } l \geq t+1.)$$

Collecting terms of degree 2t+2 in (2.10), we get

$$g_{nor}^{(2t+2)}(z\bar{z}) = \bar{z} f_{nor}^{(2t+1)}(z, z\bar{z}) + z\overline{f_{nor}^{(2t+1)}(z, z\bar{z})}. \tag{2.59}$$

Its solution is given by

$$f_{nor}^{(2t+1)}(z, w) = bzw^t, \;\; g_{nor}^{(2t+2)}(w) = (b + \bar{b})w^{t+1}, \; b \neq 0. \tag{2.60}$$

Similar to the definition of N_s, we set

$$N_s' := \min\{ts + s + 1, N_0 + 1, ns + j_0 + 1\}.$$

Then substituting (2.60) into (2.10) and letting $A = (s-1)b - \bar{b}$, we get the following dual version of (2.20):

$$\begin{aligned} g_{2t+3}(w) &= Az^s(z\bar{z} + z^s)^t + (\bar{z} + sz^{s-1} + \Theta_s^2) f_{2t+2}(z, w) \\ &+ (z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{2t+2}(z, w)} + 2Re\left((b_{N_0} - a_{N_0})z^{N_0}\right) + \Theta_{N_s'}^{2t+3}. \end{aligned} \tag{2.61}$$

Exactly the same argument as that in Lemma 2.3 (except a few obvious and trivial changes to fit into the current situation that $\mathrm{Ord}(f)$ is odd) can be applied to prove the following corresponding lemma:

Lemma 2.4. *Assume the hypothesis and the notation in Theorem 2.2. Let $Ord(f(z, z\bar{z})) = 2t+1 < \infty$ and keep all the notation that we have set up so far. Suppose that $N_0 \geq 2t+3$. Assume that $2t + j(s-2) + 3 \leq m \leq 2t + (j+1)(s-2) + 2$ with $0 \leq j \leq t$ and $m \leq N_0$. Then we have on M the following:*

$$\begin{aligned} g_m(w) = &A(1-s)^j z^{(j+1)s}(z\bar{z} + z^s)^{t-j} + (\bar{z} + sz^{s-1} + \Theta_s^2) f_{m-1}(z, w) \\ &+ (z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{m-1}(z, w)} + 2Re\left((b_{N_0} - a_{N_0})z^{N_0}\right) + \Theta_{N_s'}^m. \end{aligned} \tag{2.62}$$

We next proceed in the same way as before.

Case I: Assume $N_0 > m = ts + s$. Let $j = t$ and $m = ts + s$ in (2.62). Then we get $N'_s = ts + s + 1$ and

$$
\begin{aligned}
g_{ts+s}(w) =& A(1-s)^t z^{ts+s} + (\bar{z} + s z^{s-1} + \Theta_s^2) f_{ts+s-1}(z, w) \\
&+ (z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_{ts+s-1}(z, w)} + \Theta_{ts+s+1}^{ts+s}.
\end{aligned} \tag{2.63}
$$

Collecting terms of degree $ts + s$ in (2.63), we obtain:

$$
g_{nor}^{(ts+s)}(z\bar{z}) = A(1-s)^t z^{ts+s} + \mathbb{P}_{ts+s+1}^{ts+s} + \bar{z} f_{nor}^{(ts+s-1)}(z, z\bar{z}) + z\overline{f_{nor}^{(ts+s-1)}(z, z\bar{z})}. \tag{2.64}
$$

As in Step I, it is solvable if and only if $A = 0$, and thus $b = 0$. This gives a contradiction.

Case II: Suppose $(2t + 3 \leq) N_0 \leq ts + s$. Assume that $N_0 \leq ns + j_0$. By the assumption that $a_{ks} = b_{ks}$ for $k \leq n$ and by the definition of N_0, we must have $N_0 \neq ts + s$. This gives that $2t + 3 \leq N_0 < ts + s$. Suppose that j is the integer satisfying $2t + j(s-2) + 3 \leq N_0 = k_0 s + j_0 \leq 2t + (j+1)(s-2) + 2$. Collecting terms of degree N_0 in (2.62) and making use of Lemma 2.4, we get

$$
\begin{aligned}
g_{nor}^{(N_0)}(z\bar{z}) =& 2Re\{(b_{N_0} - a_{N_0})z^{N_0}\} + \delta A(1-s)^j z^{(j+1)s}(z\bar{z})^{t-j} \\
&+ \bar{z} f_{nor}^{(N_0-1)}(z, z\bar{z}) + z\overline{f_{nor}^{(N_0-1)}(z, z\bar{z})} + \Theta_{N_0+1}^{N_0}.
\end{aligned}
$$

Here δ is 0 when $N_0 < 2t+(j+1)(s-2)+2$, and $\delta = 1$ when $N_0 = 2t+(j+1)(s-2)+2$.Notice that when $j = t$, $(j+1)s + 2(t-j) = ts + s > N_0$. Hence, when $\delta = 1$, we have $t - j > 0$. As before, we can easily reach a contradiction by considering the divisibility by $\bar{z}$. Hence, we have $b_N = a_N$ for any $N \leq ns + j_0$, that is, $N_0 = ns + j_0 + 1$. This is a contradiction. Hence $ts + s \geq ns + j_0 + 1$ by the assumption in Case (II), which gives immediately Theorem 2.2(II). This also completes the proof of Theorem 2.2 when $\text{ord}(f) = 2t + 1$. The proof of Theorem 2.2 is finally complete. ■

The following is a combination of Theorem 2.2 and Lemma 2.1 (ii) (iii):

Corollary 2.5. *Suppose that the origin preserving formal equivalence map*

$$
(z', w') = (F(z, w), G(z, w))
$$

transforms the formal Bishop surface M defined by

$$
w = z\bar{z} + 2Re\left(z^s + \sum_{j=2,\cdots,s-1; ks+j \leq N} a_{ks+j} z^{ks+j}\right) + o(|z|^N)
$$

to the formal Bishop surface defined by

$$
w' = z'\bar{z}' + 2Re\left(z'^s + \sum_{j=2,\cdots,s-1; ks+j \leq N} b_{ks+j} z'^{ks+j}\right) + o(|z'|^N),
$$

where $N(> s) = ns + j_0$ with a certain $j_0 \in [2, s-1]$, a_{ks+j}, b_{ks+j} are complex numbers. Then there is a constant θ with $e^{\sqrt{-1}s\theta} = 1$ such that $(F, G) = (e^{\sqrt{-1}\theta}z + f(z,w), w + g(z,w))$. Moreover, we have the following conclusions stated in (I), (II) and (III), respectively:
(I). When $Ord(f) = 2t$, it holds that $st + 1 > N$; and when $Ord(f) = 2t+1$, it holds that $st + s > N$.
(II). $g(z,w) = g(w) + g_{erro}(z,w)$ with $\overline{g(w)} = g(\overline{w})$, $wt_{nor}(g_{erro}(z,w)) > N$ and

$$wt_{nor}(g(w)) \geq \min\{N, wt_{nor}(f(z,w)) + 1\}.$$

(III). $a_{ks+j} = e^{j\sqrt{-1}\theta} b_{ks+j}$ for $ks + j \leq N$.

3 A complete set of formal invariants, proofs of Theorem 1.1, Theorem 1.3 and Corollary 1.4

In this section, we will establish a formal normal form for the formal surface defined in (2.2), by applying a formal transformation preserving the origin. This will give a complete classification of germs of formal surfaces $(M, 0)$ with $\lambda = 0$, $s < \infty$ in the formal setting, which, in particular, can be used to answer an open question raised by J. Moser in 1985 ([pp 399, Mos]).

As another application of our complete set of formal invariants, we show that a generic Bishop surface with the Bishop invariant vanishing is not equivalent to an algebraic surface, by applying a Baire category argument similar to the study in the CR setting. (See the nice paper of Forstneric [For].) Notice that this phenomenon is strikingly different from the theory for elliptic Bishop surfaces with non-vanishing Bishop invariants, where Moser-Webster proved their celebrated theorem, that states that any elliptic Bishop surface with a non-vanishing Bishop invariant has an algebraic normal form.

Let M be a formal Bishop surface in $\mathbb{C}^2$ defined by

$$w = H(z, \overline{z}) = z\overline{z} + 2Re\{\sum_{j=s}^{N} a_j z^j\} + E_{N+1}(z, \bar{z}), \tag{3.1}$$

where $s \geq 3$ is a positive integer and E_{N+1} is a formal power series in $(z, \overline{z})$ with $\mathrm{Ord}(E_{N+1}) \geq N+1$. Moreover, $a_s = 1$ and for $m > s$, $m \leq N$,

$$a_m = 0 \quad \text{if} \quad m = 0, 1 \text{ mod s}.$$

Our first result of this section is the following normalization theorem:

Theorem 3.1. *With the above notation, there is a polynomial map*

$$\begin{cases} z' = z + f(z,w), & f(z,w) = O(|w| + |z|^2), \\ w' = w + g(z,w), & g(z,w) = O(|w|^2 + |z|^3 + |zw|), \end{cases} \tag{3.2}$$

that transforms the formal Bishop surface M defined in (3.1) to the formal Bishop surface defined by

$$w' = H^*(z', \overline{z'}) = z'\bar{z}' + 2Re\{\sum_{j=s}^{N+1} b_j z'^j\} + E^*_{N+2}(z', \bar{z}'). \tag{3.3}$$

Here $E^*_{N+2} = O(|z|^{N+2})$, $a_j = b_j$ *for* $s \le j \le N$ *and*

$$b_{N+1} = 0 \quad if \quad N+1 = 0, 1 \ mod\ s.$$

Moreover, we have the following conclusions:
(I). When $N+1 \neq 0, 1 \mod(s)$, *then* $wt_{nor}(f) \ge N$ *and* $wt_{nor}(g) \ge N+1$.
(II). When $N = ts$, *then* $wt_{nor}(f) \ge 2t$ *and* $wt_{nor}(g) \ge 2t+1$.
(III). When $N = ts-1$, *then* $wt_{nor}(f) \ge 2t-1$ *and* $wt_{nor}(g) \ge 2t$.

Before proceeding to the proof, we recall a result of Moser, which will be used for our consideration here. For any $m \ge 4$ and holomorphic polynomials

$$f^{(m-1)}_{nor}(z, w), \ g^{(m)}_{nor}(z, w), \ \phi^{(m)}(z),$$

we define an operator, which we call the Moser operator $\mathcal{L}$, as follows:

$$\mathcal{L}(f^{(m-1)}_{nor}(z, w), g^{(m)}_{nor}(z, w), \phi^{(m)}(z)) := g^{(m)}_{nor}(z, z\overline{z}) - 2Re\{\overline{z} f^{(m-1)}_{nor}(z, z\overline{z}) + \phi^{(m)}(z)\}.$$

The following lemma is an immediate consequence of [Proposition 2.1, Mos] and [(2.10), pp401, Mos]:

Lemma 3.2. *Let* $G(z, \overline{z})$ *be a homogeneous polynomial of degree* m. *Then*

$$\mathcal{L}(f^{(m-1)}_{nor}(z, w), g^{(m)}_{nor}(z, w), \phi^{(m)}(z)) = G(z, \overline{z})$$

has a unique solution $\{f^{(m-1)}_{nor}(z, w), g^{(m)}_{nor}(z, w), \phi^{(m)}(z)\}$ *under the normalization condition:* $f^{(m-1)}_{nor} = z^2 f^*$ *with* f^* *a holomorphic polynomial. In case* $G(z, \overline{z})$ *is real-valued, then we have the reality property for* $g^{(m)}_{nor}$: $g^{(m)}_{nor}(z, w) = g^{(m)}_{nor}(w)$, $\overline{g^{(m)}_{nor}(w)} = g^{(m)}_{nor}(\overline{w})$. *Moreover, when* G *has no harmonic terms, then* $\mathcal{L}\left(f^{(m-1)}_{nor}(z, w), g^{(m)}_{nor}(z, w), 0\right) = G(z, \overline{z})$ *has a unique solution* $\{f^{(m-1)}_{nor}(z, w), g^{(m)}_{nor}(z, w)\}$ *under the same normalization condition just mentioned. (When* G *is further assumed to be real-valued, we then also have the same reality property for* $g(w)$*.)*

The proof of Theorem 3.1 follows from a similar induction argument that we used in the previous section.

Proof of Theorem 3.1: We complete the proof in three steps.

Step 1: We first show that there is a polynomial map: $z' = z + f_{nor}^{(N)}(z,w)$, $w' = w + g_{nor}^{(N+1)}(z,w)$, which maps M to a surface defined by the following equation:

$$w = z\bar{z} + 2Re\{\sum_{j=s}^{N+1} b_j z^j\} + \widetilde{E}_{N+2}(z,\bar{z}) \tag{3.4}$$

with $b_j = a_j$ for $s \le j \le N$ and b_{N+1} to be determined. Substituting the map into (3.4) and collecting terms of degree $N+1$, we see that the existence of the map is equivalent to the existence of the solution of the following functional equation:

$$\mathcal{L}(f_{nor}^{(N)}(z,w), g_{nor}^{(N+1)}(z,w), b_{N+1}z^{N+1}) = -E_{N+1}^{(N+1)}(z,\overline{z}). \tag{3.5}$$

By Lemma 3.2, we know that (3.5) is indeed solvable and is uniquely solvable under the normalization condition as in Lemma 3.2.

For the rest of the proof of the theorem, we can assume that $E_{N+1} = 2Re\{b_{N+1}z^{N+1}\} + o(|z|^{N+1})$.

Step 2: We now assume that M is defined by (3.4). In this step, we assume that $N+1 = 1$ mod s. Write $N = ts$. We then show that there is a polynomial map of the form:

$$\begin{aligned} z' &= z + \textstyle\sum_{l=0}^{N-2t}\{f_{nor}^{(2t+l)}(z,w)\},\\ w' &= w + \textstyle\sum_{\tau=0}^{N+1-2t-2}\{g_{nor}^{(2t+2+\tau)}(w)\}, \end{aligned} \tag{3.6}$$

such that under this transformation, M is mapped to a formal surface M' defined by (3.3) with $b_{N+1} = 0$, where $\overline{g_{nor}^{(j)}(u)} = g_{nor}^{(j)}(u)$ for $u \in \mathbb{R}$, $j \le N+1$. The map is also uniquely determined by imposing the normalization condition as in Lemma 3.2 for $f_{nor}^{(j)}(z,w)$ with $2t < j \le N$.

As in Step I, this amounts to studying a series of normally weighted homogeneous functional equations with the normally weighted degree running from $2t+1$ to $N+1$. Substituting (3.6) into (3.3), over $w = z\bar{z} + 2Re\{\sum_{j=s}^{N+1} b_j z^j\} + \widetilde{E}_{N+2}(z,\bar{z})$, we get the following:

$$\begin{aligned} w + \sum_{\tau=0}^{N+1-2t-2} g_{nor}^{(2t+2+\tau)}(w) &= \big(z + \sum_{l=0}^{N-2t} f_{nor}^{(2t+l)}(z,w)\big)\big(\overline{z} + \sum_{l=0}^{N-2t} \overline{f_{nor}^{(2t+l)}(z,w)}\big)\\ &\quad + 2Re\Big\{\sum_{j=s}^{N} b_j\big(z + \sum_{l=0}^{N-2t} f_{nor}^{(2t+l)}(z,w)\big)^j\Big\} + E_{N+2}^*(z',\overline{z}'). \end{aligned} \tag{3.7}$$

With an immediate simplification, (3.7) takes the following form:

$$
\begin{aligned}
\sum_{\tau=0}^{N+1-2t-2} g_{nor}^{(2t+2+\tau)}(w) = \bar{z}\Big(\sum_{l=0}^{N-2t} f_{nor}^{(2t+l)}(z,w)\Big) + z\Big(\sum_{l=0}^{N-2t} \overline{f_{nor}^{(2t+l)}(z,w)}\Big) \\
+ \Big(\sum_{l=0}^{N-2t} f_{nor}^{(2t+l)}(z,w)\Big) \cdot \Big(\sum_{l=0}^{N-2t} \overline{f_{nor}^{(2t+l)}(z,w)}\Big) - 2Re\big(b_{N+1}z^{N+1}\big) \\
+ 2Re\Big\{ \sum_{j=s}^{N} b_j\Big(\big(z + \sum_{l=0}^{N-2t} f_{nor}^{(2t+l)}(z,w)\big)^j - z^j\Big)\Big\} + O(|z|^{N+2}), \\
w = z\bar{z} + 2Re\{\sum_{j=s}^{N+1} b_j z^j\} + \widetilde{E}_{N+2}(z,\bar{z}).
\end{aligned}
\tag{3.8}
$$

We need to inductively solve the above equation up to order $N+1$. Collecting terms of degree $2t+1$ in $(z,\bar{z})$, we obtain the equation (2.14), which can be solved as:

$$
f_{nor}^{(2t)}(z,w) = aw^t - \bar{a}z^2w^{t-1}
$$

with a to be (uniquely) determined later.

Now, suppose we are able to solve $f_{nor}^{(2t+l)}$, $g_{nor}^{(2t+1+l)}$ for $2t+l = 2t, \cdots, m-1 \le st-1$, by making use of the equation (3.8) up to the level of degree $m \le st$. Also, suppose that $g_{nor}^{(j)}(z\bar{z})$ is real-valued for $j \le m$. By arguing exactly in the same way as in the proof of Lemma 2.3, we obtain from (3.8) the following equation in our setting:

$$
\begin{aligned}
g_{m+1}(w) =& \bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z}+z^s)^{t-j-1} + (\bar{z} + sz^{s-1} + \Theta_s^2)f_m(z,w) \\
&+ (z + s\bar{z}^{s-1} + \Theta_s^2)\overline{f_m(z,w)} - 2Re\left(b_{ts+1}z^{ts+1}\right) + \Theta_{ts+2}^{m+1}, \text{ for } m \le ts = N,
\end{aligned}
\tag{3.9}
$$

where $2t + j(s-2) + 2 \le m+1 \le 2t + (j+1)(s-2) + 1$ with $0 \le j \le t-1$. Suppose that $m-1 < st-1$. Collecting terms of degree $m+1$ in (3.9), we get

$$
g_{nor}^{(m+1)}(z\bar{z}) = \bar{z}f_{nor}^{(m)}(z,z\bar{z}) + z\overline{f_{nor}^{(m)}(z,z\bar{z})} + \hat{\mathbb{P}}_{ts+2}^{m+1} + \delta_{2t+(j+1)(s-2)+1}^{m+1}\bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1}. \tag{3.10}
$$

Here $\delta_{2t+(j+1)(s-2)+1}^{m+1}$ takes value 1 when $m+1 = 2t+(j+1)(s-2)+1$ for some integer $j \in [0, t-2]$, and 0 otherwise. Notice that $g_{nor}^{(j)}(z\bar{z})$ is real-valued for $j \le m$. Since $w(z,\bar{z})$ and the right hand side of (3.8) are also real valued at each homogeneous level of degree up to $N+1$, we easily see that the sum of the last two terms in (3.10) must be real valued. Here $\hat{\mathbb{P}}_{ts+2}^{m+1}$ is uniquely determined by the known data such as M and $f_{nor}^{(2t+l)}$, $g_{nor}^{(2t+1+l)}$ for $2t+l = 2t, \cdots, m-1 \le st-1$. Since $m+1 < ts+1$, this equation, in terms of the Moser operator, can be rewritten as:

$$
\mathcal{L}\left(f_{nor}^{(m)}(z,z\bar{z}), g_{nor}^{(m+1)}(z\bar{z}), 0\right) = \hat{\mathbb{P}}_{ts+2}^{m+1} + \delta_{2t+(j+1)(s-2)+1}^{m+1}\bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1}. \tag{3.11}
$$

Here $\delta^{m+1}_{2t+(j+1)(s-2)+1}$ is defined as in (3.10). Since $\hat{\mathbb{P}}^{m+1}_{ts+2}+\delta^{m+1}_{2t+(j+1)(s-2)+1}\bar{a}(1-s)^{j+1}z^{(j+1)s+1}(z\bar{z})^{t-j-1}$ is real-valued and divisible by $\overline{z}$, it does not contain any harmonic terms. By Lemma 3.2, it can be solved, and can be uniquely solved under the normalization condition as in Lemma 3.2. Also $g^{(m+1)}_{nor}(z\overline{z})$ is real-valued. By induction, we can uniquely obtain $f^{(m)}_{nor}$, $g^{(m+1)}_{nor}$ for $m \leq ts-1$ with the reality property for $g^{(m+1)}_{nor}$. Collecting terms of degree $m+1 = ts+1$ in (3.9), we obtain an equation similar to (2.57), which can be rewritten as:

$$\begin{aligned}\mathcal{L}(g^{(ts+1)}_{nor}(z\bar{z}), f^{(ts)}_{nor}(z, z\overline{z}), 0) = 2Re\{\bar{a}(1-s)^t z^{ts+1}\}\\ +\hat{\mathbb{P}}^{ts+1}_{ts+2} - a(1-s)^t\overline{z}^{ts+1} - 2Re(b_{ts+1}z^{ts+1}).\end{aligned} \tag{3.12}$$

As argued above and as in the proof of Theorem 2.2, the real-valued homogeneous polynomial $\hat{\mathbb{P}}^{ts+1}_{ts+2} - a(1-s)^t\overline{z}^{ts+1}$ has a $\overline{z}$ factor and thus has no harmonic terms. Hence, if we choose $a = \overline{b_{ts+1}}/(1-s)^t$, then (3.12) is uniquely solvable, under the normalization condition in Lemma 3.2, with $g^{(ts+1)}_{nor}(z\overline{z})$ real-valued. This completes the proof of the claim in this step.

Step 3: In this step, we assume that $N+1 = 0$ mod s. Write $N = (t+1)s-1$. We then show that there is a unique polynomial map of the form:

$$\begin{aligned}z' &= z + \textstyle\sum_{l=0}^{N-1-2t}\{f^{(2t+l+1)}_{nor}(z, w)\},\\ w' &= w + \textstyle\sum_{\tau=0}^{N+1-2t-2}\{g^{(2t+2+\tau)}_{nor}(w)\},\end{aligned} \tag{3.13}$$

such that under this transformation, M is mapped to a formal surface M' defined by (3.3) with $\underline{b_{N+1}} = 0$. Here $f^{(m)}_{nor}$ satisfies the normalization condition in Lemma 3.2 for $m \neq 2t+1$, and $\overline{g^{(j)}_{nor}(u)} = g^{(j)}_{nor}(u)$ for u real and $j \leq N+1$.

The argument for this step is the same as that for Step 2. We first have to choose

$$f^{(2t+1)}_{nor}(z, w) = bzw^t, \quad g^{(2t+2)}_{nor}(w) = (b+\bar{b})w^{t+1}$$

with b to be uniquely determined later. Arguing exactly in the same way as in Step 2, we can inductively find the unique solution (under the normalization condition) for $f^{(2t+l)}_{nor}$, $g^{(2t+1+l)}_{nor}$ with $2t+l = 2t+2, \cdots, < st+s-1$ with the reality property for $g^{(2t+1+l)}_{nor}$. At the level with degree $ts+s$, we have the following equation:

$$\begin{aligned}2Re(b_{N+1}z^{N+1}) + g^{(ts+s)}_{nor}(z\bar{z}) = ((s-1)b - \bar{b})(1-s)^t z^{ts+s}\\ +\hat{\mathbb{P}}^{ts+s}_{ts+s+1} + \bar{z}f^{(ts+s-1)}_{nor}(z, z\bar{z}) + \overline{zf^{(ts+s-1)}_{nor}(z, z\bar{z})}.\end{aligned} \tag{3.14}$$

Now, arguing in the same way as in Step 2, (3.14) is uniquely solvable by choosing b such that $((s-1)b - \overline{b})(1-s)^t = b_{N+1}$ and by imposing the normalization condition as in Lemma 3.2 to $f^{(ts+s-1)}_{nor}$. The reality for $g^{(ts+s)}_{nor}$ follows in the same way.

Now, the map in Theorem 3.1 can be chosen as the map in Step 1 if $N+1 \neq 0, 1$ mod(s). When $N+1 = 0$, or 1 mod(s), the map in Theorem 3.1 can be defined by composing the map in Step 2 or that in Step 3, respectively, with the map in Step 1. We see the proof of

Theorem 3.1. Moreover, with such fixed procedures and normalizations described in the above steps, for $k+2l \le N$ and $j+2\tau \le N+1$ there are polynomials $\{P_{kl}(a_{\alpha\beta}, \overline{a_{\alpha\beta}})_{1\le\alpha+\beta\le N+1}\}$ and $\{Q_{j\tau}(a_{\alpha\beta}, \overline{a_{\alpha\beta}})_{1\le\alpha+\beta\le N+1}\}$ (depending only on s and N) such that the coefficients of the map $(z', w') = (z, w) + (f, g) = (z, w) + (\sum_{k+2l\ge 2} b_{kl} z^k w^l, \sum_{j+2\tau\ge 3} c_{j\tau} z^j w^\tau)$ in Theorem 3.1 are determined by

$$b_{kl} = P_{kl}(a_{\alpha\beta}, \overline{a_{\alpha\beta}}),\ \ c_{j\tau} = Q_{j\tau}(a_{\alpha\beta}, \overline{a_{\alpha\beta}}) \text{ with } 1 \le \alpha+\beta \le N+1, \tag{3.15}$$

where $k+2l \le N,\ j+2\tau \le N+1$ and $H = \sum_{\alpha+\beta\ge 2} a_{\alpha\beta} z^\alpha \overline{z}^\beta$.

The rest of the proof of Theorem 3.1 follows from the procedures that we used to prove the existence part. ■

We next choose the map $z' = z + f,\ w' = w + g$ in Theorem 3.1 such that its coefficients are determined by (3.15). Let $z = z' + f^*(z', w')$ and $w = w' + g^*(z', w')$ be its inverse transformation. Notice that the coefficients of (f^*, g^*) in its Taylor expansion up to degree, say m, are universal polynomial functions of the coefficients of (f, g) up to degree m for any m. Hence we have the defining equation of M^*, the image of M, as follows:

$$w' + g^*(z', w') = H(z' + f^*(z', w'), \overline{z' + f^*(z', w')}).$$

Applying an implicit function theorem to solve for w' and making use of the uniqueness of the graph function, we see that the coefficients in the Taylor expansion of H^* up to degree m must also be (possibly non-holomorphic) polynomial functions of the coefficients of H of degree not exceeding m in its Taylor expansion. Repeating such a normalization procedure that we did for M to M^* and by an induction argument, we get the following theorem: (The uniqueness part follows from Lemma 2.1 and Theorem 2.2.)

Theorem 3.3. *Let M be a formal Bishop surface defined by*

$$w = H(z, \overline{z}) = z\overline{z} + z^s + \overline{z}^s + E(z, \bar{z}), \tag{3.16}$$

where $s \ge 3$ is a positive integer and $E(z, \overline{z}) = \sum_{\alpha+\beta\ge s+1} a_{\alpha\beta} z^\alpha \overline{z^\beta}$. Then there is a unique formal transformation of the form:

$$\begin{cases} z' = z + f(z, w), & f(z, w) = O(|w| + |z|^2), \\ w' = w + g(z, w), & g(z, w) = O(|w|^2 + |z|^3 + |zw|), \end{cases} \tag{3.17}$$

that transforms M to the formal Bishop surface defined by

$$w' = H^*(z', \overline{z'}) = z'\bar{z}' + z'^s + \overline{z'}^s + 2Re\{\sum_{j=2,\cdots,s-1;\ k\ge 1}^{\infty} \lambda_{ks+j} z'^{ks+j}\}. \tag{3.18}$$

The normal form in (3.18), up to a transformation of the form $z'' = e^{i\theta} z'$, $w'' = w$ with $e^{is\theta} = 1$, uniquely determines the formal equivalence class of M. Moreover, there are a set of universal polynomial functions

$$\{\Lambda_{ks+j}(Z_{\alpha\beta}, \overline{Z_{\alpha\beta}})_{s+1\le\alpha+\beta\le ks+j}\}_{\ j=2,\cdots,s-1;\ k\ge 1}$$

depending only on s, such that:

$$\lambda_{ks+j} = \Lambda_{ks+j}(a_{\alpha\beta}, \overline{a_{\alpha\beta}})_{s+1\le\alpha+\beta\le ks+j;\ j=2,\cdots,s-1;\ k\ge 1}. \tag{3.19}$$

Proofs of Theorem 1.1 and Corollary 1.4: Theorem 1.1 follows immediately from Theorem 3.3 and Lemma 2.1 (ii) (iii).

The proof of Corollary 1.4 (a), (b), (d) also follows easily from Theorem 3.3. To see Corollary 1.4 (c), we let $\mathcal{G}$ be a proper subgroup of $\mathcal{Z}_s$. Define $J_G := \{j :\ 2 \le j \le s-1,\ e^{i\theta j} = 1,\ \text{for any } (e^{i\theta}z, w) \in \mathcal{G}\}$. Let M_G be defined by

$$w = z\overline{z} + z^s + \overline{z}^s + 2Re\{\sum_{j\in J_G} a_{s+j} z^{s+j}\},$$

with $a_{s+j} \ne 0$. Then we will verify that $aut_0(M_G) = \mathcal{G}$. To this aim, write $\mathcal{G}^*$ to be the collection of $\xi' s$ with $(z, w) \to (\xi z, w)$ belonging to $\mathcal{G}$. By Corollary 1.3 (a), we need only to show that if $\xi^{*s} = 1$ and $\xi^{*j} = 1$ for any $j \in J_G$, then $\xi^* \in \mathcal{G}^*$. Write $k = |\mathcal{G}^*|$. Then $s = km$ with $m(\in \mathbf{N}) > 1$. For any $\xi(\in \mathcal{G}^*) \ne 1$, since the order of ξ must be divisible by k, we see that $\xi^k = 1$. Therefore, $\mathcal{G}^*$ forms a complete set of the solutions of $z^k = 1$. Now, it is clear that $J_G = \{k, \cdots, (m-1)k\}$. Hence, we see that $\xi^{*k} = 1$. Thus, $\xi^* \in \mathcal{G}^*$. This completes the proof of Corollary 1.4 (c).

Now, by Corollary 1.4(a), we see that for M as in Corollary 1.4(e), M must be formally equivalent to M_s. Assuming Theorem 1.5, which we will prove in the next section, we also conclude that M is biholomorphically equivalent to M_s. Corollary 1.4 (f) is a simple consequence of the results in (a) and (e). ∎

Corollary 3.4. *Let M be a real analytic Bishop surface defined by an equation of the form:*

$$w = H(z, \overline{z}) = z\overline{z} + 2Re\{z^s + \sum_{k\ge 1,\ j=2,\cdots,s-1} a_{ks+j} z^{ks+j}\} \ \text{ with infinitely many } \ a_{ks+j} \ne 0.$$

Then for any $N > s$, M is not equivalent to the Bishop surface M_N defined by

$$w = H_{(N+1)}(z, \overline{z}) = z\overline{z} + 2Re\{z^s + \sum_{k\ge 1,\ j=2,\cdots,s-1}^{ks+j\le N} a_{ks+j} z^{ks+j}\}.$$

Here $H_{(N+1)}$ is the N^{th}-truncation from the Taylor expansion of H at 0. In fact, $M_{(N+1)}$ is equivalent to $M_{(N'+1)}$ with $N' > N$ if and only if $a_{ks+j} = 0$ for any $N < ks + j \le N'$.

Corollary 3.4 answers, in the negative, the second problem that J. Moser asked in his paper ([pp 399, Mos]).

As a less obvious application of Theorem 3.3, we next show that a generic Bishop surface with the Bishop invariant vanishing at 0 and with $s < \infty$ is not even formally equivalent to any algebraic surface in $\mathbb{C}^2$. For this purpose, we borrow the idea used in the CR setting based on the Baire category argument. For the consideration in the CR setting by using the Baire category theorem, the reader is referred to the paper of Forstneric [For].

Write $\mathcal{M}_s$ for the collection of all formal Bishop surfaces defined as in (3.16):

$$w = H(z,\bar{z}) = z\bar{z} + 2Re(z^s) + \sum_{\alpha+\beta \geq s+1} a_{\alpha\beta} z^{\alpha} \overline{z^{\beta}}. \tag{3.20}$$

Write $\mathcal{F} := \{\vec{a} = (a_1, \cdots, a_n, \cdots) : \ a_j \in \mathbb{C}\}$, equipped with the usual distance function:

$$dist(\vec{a}, \vec{b}) = \sum_{j=1}^{\infty} \frac{|a_j - b_j|}{2^j(1 + |a_j - b_j|)}.$$

We know that $\mathcal{F}$ is a Fréchet space. There is a one-to-one correspondence between $\mathcal{M}_s$ and $\mathcal{F}$, which assigns each $M \in \mathcal{M}_s$ to an element: $\vec{M} = (a_{\alpha\beta}) \in \mathcal{F}$ labeled in the lexicographical order. Therefore, we can, in what follows, identify $\mathcal{M}_s$ as a Fréchet space. We define the operator $\mathcal{J}$ such that it sends any $M \in \mathcal{M}_s$ to $(\lambda_{ks+j})_{j\neq 0,1; k\geq 1}$, where (λ_{sk+j}) is described as in Theorem 3.3. By (3.19), we easily see that $\mathcal{J}$ is a continuous map from $\mathcal{M}_s$ to $\mathcal{F}$.

(M, p) in $\mathbb{C}^2$ is called the germ of an algebraic surface if M near p possesses a real polynomial defining equation. If $p \in M$ is a point with an elliptic complex tangent, whose Bishop invariant is 0 and whose Moser invariant is $s < \infty$, then there is a change of coordinates (see [Hu1], for instance) such that $p = 0$ and M near 0 is defined by an equation of the form:

$$w = z\overline{z} + B(z,\overline{z},w,\overline{w}), \quad B(z,\overline{z},w,\overline{w}) = \sum_{3 \leq \alpha+\beta+2\gamma+2\tau} c_{\alpha\beta\gamma\tau} z^{\alpha} \overline{z}^{\beta} w^{\gamma} \overline{w}^{\tau}, \tag{3.21}$$

where B is a polynomial in its variables. By using the implicit function theorem and using the argument in the step 1 of the proof of Theorem 3.1, it is not hard to see that there is a fixed procedure to transform (3.21) into a surface defined by an equation as in (3.20), in which $a_{\alpha\beta}$ are presented by polynomials of $c_{\alpha\beta\gamma\tau}$ and $H(z,\overline{z})$ becomes what we call a Nash algebraic function to be defined as follows:

We call a real analytic function $h(z,\overline{z})$ near 0 a Nash algebraic function if either $h \equiv 0$ or there is an irreducible polynomial $P(z,\overline{z}; X)$ in X with polynomial coefficients in $(z,\overline{z})$ such that $P(z,\overline{z}; h(z,\overline{z})) \equiv 0$. Certainly, we can always assume that the coefficients of (z,ξ,X) (in $P(z,\xi,X)$) of terms with highest power in X have maximum value 1. The degree of h is defined as the total degree of P in $(z,\overline{z},X)$.

For $d,\ n,\ m \geq 1$, we define $\mathcal{A}^d_B(n,m) \subset \mathcal{M}_s$ to be the subset of Bishop surfaces defined in (3.20), where $H(z,\overline{z})'s$ are Nash algebraic functions derived from the $B's$ in (3.21) by the procedure described above with the degree of $B's$ bounded by d, that further satisfy the following properties:

Cond (1): $H(z,\xi)'s$ are holomorphic over $|z|^2+|\xi|^2<1/m^2$;

Cond(2): $\max_{(|z|^2+|\xi|^2)<1/m^2}|H(z,\xi)| \leq n$ and $|c_{\alpha\beta\gamma\tau}| \leq n$.

Write $\mathcal{A}^d_B = \cup^\infty_{n,m=1}\mathcal{A}^d_B(n,m)$ and $\mathcal{A}_B = \cup^\infty_{d=1}\mathcal{A}^d_B$. It is a consequence of Theorem 3.3 that M, defined in (3.16), is formally equivalent to an algebraic surface if and only if $\mathcal{J}(M) \in \mathcal{J}(\mathcal{A}_B)$. (Therefore, M defined in (3.16) is not formally equivalent to an algebraic surface if and only if $\mathcal{J}(M) \notin \mathcal{J}(\mathcal{A}_B)$.)

Now, for any sequence $\{M_j\} \subset \mathcal{A}^d_B(n,m)$ with $M_j : w = H_j(z,\overline{z}) = z\overline{z} + z^s + \overline{z}^s + o(|z|^s)$, by a normal family argument and by passing to a subsequence, we can assume that $H_j(z,\xi) \to H_0(z,\xi)$ over any compact subset of $\{|z|^2+|\xi|^2<1/m^2\}$. If follows easily that M_0 defined by $w = H_0$ is also in $\mathcal{A}^d_B(n,m)$. Moreover, $D^\alpha_z D^\beta_\xi H_j(0) \to D^\alpha_z D^\beta_\xi H_0(0)$ for any (α,β). By (3.19), $\mathcal{J}(M_j) \to \mathcal{J}(M_0)$ in the topology of $\mathcal{F}$. Therefore, we easily see that $\mathcal{J}(\mathcal{A}_B)$ is a subset of $\mathcal{F}$ of the first category.

Next, for any $R > 0$, we let

$$\mathcal{S}_R := \{\vec{\lambda} = (\lambda_{sk+j})_{k\geq 1; j=2,\cdots,s-1}\} : \ \|\vec{\lambda}\|_R := \sum_{ks+j} |\lambda_{ks+j}| R^{ks+j} < \infty\}.$$

It can be verified that $\mathcal{S}_R$ is a Banach space under the above defined $\|\cdot\|_R$-norm. (In fact, it reduces to the standard l^1-space when $R = 1$.) We now claim that $\mathcal{K}^d_B$, defined as the closure of $\mathcal{J}\left(\mathcal{A}^d_B(n,m)\right) \cap \mathcal{S}_R$ in $\mathcal{S}_R$ in its Banach norm, has no interior point.

Suppose, to the contrary, that a certain ϵ-ball $\mathcal{B}$ of $\vec{a_0} = (\lambda^0_{sk+j})_{k\geq 1; j=2,\cdots,s-1}$ in $\mathcal{S}_R$ is contained in $\mathcal{K}^d_B$. We must then have $\mathcal{B} \subset \mathcal{J}\left(\mathcal{A}^d_B(n,m)\right) \cap \mathcal{S}_R$. Indeed, for any $\vec{a} \in \mathcal{B}$, let $\mathcal{J}(M_j) \to \vec{a}$ with $M_j \in \mathcal{A}^d_B(n,m)$. By the argument in the above paragraph, we can assume, without loss of generality, that $M_j \to M_0 \in \mathcal{A}^d_B(n,m)$ in the $\mathcal{F}$-norm. By (3.19), we see that $\mathcal{J}(M_0) = \vec{a}$. Choose $\vec{a} = \{\lambda_{ks+j}\}$ such that $|\lambda_{ks+j} - \lambda^0_{ks+j}| \cdot (2R)^{ks+j} < \epsilon$ for any $ks + j$. For any $N \geq 1$, then we see that there is a certain $H = z\overline{z} + z^s + \overline{z^s} + \sum_{s+1\leq\alpha+\beta} a_{\alpha\beta} z^\alpha \overline{z^\beta}$ Nash algebraic near 0 such that

$$\lambda_{ks+j} = \Lambda_{ks+j}(a_{\alpha\beta}, \overline{a_{\alpha\beta}}),\quad N \geq ks+j \geq s+1,\ \alpha+\beta \leq ks+j,\quad \Lambda = (\Lambda_{ks+j})_{s+1\leq ks+j\leq N}. \tag{3.22}$$

Here H is obtained from B in (3.21) with degree of B bounded by d. Since $a_{\alpha\beta}$ are polynomial functions of $c_{\alpha\beta\gamma\tau}$, we can conclude a contradiction from (3.22). Indeed, since the variables on the right hand side of (3.22) are polynomially parametrized by less than $(2d)^8$ free variables $(c_{\alpha\beta\gamma\tau})$, the image of (3.22) can not fill in an open subset of $\mathbb{R}^{N-s}$ as $N \gg 1$.

Therefore, we proved that $\mathcal{A}_B = \cup^\infty_{d,n,m=1}\mathcal{A}^d_B(n,m)$ is a set of the first category in $\mathcal{S}_R$. By the Baire category theorem, we conclude that most elements in $\mathcal{S}_R$ are not from $\mathcal{J}(\mathcal{A}_B \cap \mathcal{S}_R)$. For any $\vec{a} = (\lambda_{sk+j})(\in \mathcal{S}_R) \notin \mathcal{J}(\mathcal{A}_B \cap \mathcal{S}_R)$, the Bishop surface defined by: $w = z\overline{z} + z^s + \overline{z}^s +$

$2\mathrm{Re}(\sum_{k\geq 1;2\leq j\leq s-1}\lambda_{ks+j}z^{ks+j})$ is not equivalent to any algebraic surface in $\mathbb{C}^2$. When R varies, we complete a proof of Theorem 1.3. ∎

A real analytic surface in $\mathbb{C}^2$ is called a Nash algebraic surface if it can be defined by a Nash algebraic function. By the same token, we can similarly prove the following:

Theorem 3.5. *Most real analytic elliptic Bishop surfaces with the Bishop invariant $\lambda = 0$ and the Moser invariant $s < \infty$ at 0 are not equivalent to Nash algebraic surfaces in $\mathbb{C}^2$.*

Proof of Theorem 3.5: To prove Theorem 3.5, we define $\mathcal{A}^d_B(n,m)$ in the same way as before except that we now only require that $H(z,\overline{z}) = z\overline{z} + z^s + \overline{z}^s + \sum_{\alpha+\beta\geq s+1} a_{\alpha\beta}z^\alpha\overline{z}^\beta$ is a general Nash algebraic function with total degree bounded by d and with the same conditions described as in Cond (1) and the first part of Cond (2). The last part of Cond (2) is replaced by the condition that $|b_{\alpha\beta\gamma}| \leq n$, where $P(z,\overline{z},X) = \sum b_j(z,\overline{z})X^j = \sum_{\alpha\beta\gamma} b_{\alpha\beta\gamma}z^\alpha\overline{z}^\beta X^\gamma$ is a minimal polynomial of H with the same coefficient restriction as imposed before.

We fix an H_0 and its minimal polynomial $P_0(z,\overline{z};X)$. (We will fix a certain coefficient of P in the top degree terms of X to be 1 to make the minimal polynomial P_0 unique). Let $\mathcal{A}^d_B(n,m;H_0,\delta)$ be a subset of $\mathcal{A}^d_B(n,m)$, where $M = \{w = H(z,\overline{z})\} \in \mathcal{A}^d_B(n,m;H_0,\delta)$ if and only if $|b_{\alpha\beta\gamma} - b^0_{\alpha\beta\gamma}| \leq \delta$. Here $P = \sum b_{\alpha\beta\gamma}z^\alpha\overline{z}^\beta X^\gamma$ and $P_0 = \sum b^0_{\alpha\beta\gamma}z^\alpha\overline{z}^\beta X^\gamma$ are the minimal polynomials of H and H_0, respectively. We assume that P is normalized in the same manner as for P_0. (Certainly, we can always do this if $\delta \ll 1$.)

Consider an H and its minimal polynomial P associated with an element from $\mathcal{A}^d_B(n,m;H_0,\delta)$. Let R be the resultant of P and P'_X with respect to X. We know that R is a non-zero polynomial of $(z,\overline{z})$ of degree bounded by $C_1(d)$, a constant depending only on d. Write $H = H^*_{(N)} + H^{**}_N$ with $H^*_{(N)}$ the Taylor polynomial of H up to order $N-1$ and H^{**}_N the remainder. Then from $P(z,\overline{z},H^*_{(N)} + H^{**}_N) = 0$, we obtain

$$P^{**}(z,\overline{z},X^{**}) = 0 \quad \text{with} \quad X^{**} = H^{**}_N. \tag{3.23}$$

Here P^{**} is a polynomial of total degree bounded by $C_2(d,N)$, a constant depending only on d and N, and its coefficients are determined polynomially by the coefficients of P and $H^*_{(N)}$. Notice that $D_{X^{**}}\left(P^{**}(z,\overline{z},X^{**})\right)|_{X^{**}=0} = D_X\left(P(z,\overline{z},X)\right)_{X=H^*_{(N)}}$. Since there are polynomials G_1 and G_2 such that $G_1P + G_2P'_X = R$ and since $P(z,\overline{z},H^*_{(N)}) = o(|z|^N)$, we conclude that the degree k_0 of the lowest non-vanishing order term of $P'_X(z,\overline{z},H^*_{(N)})$ is bounded by $C_1(d)$, depending only on d.

Choose an $N > C_1(d)$ and a sufficiently small positive number δ. We can apply a comparing coefficient method to (3.23) to conclude that each $a_{\alpha_0\beta_0}$ with $\alpha_0+\beta_0 \geq N$ is determined by $b_{\alpha\beta\gamma}$ and $a_{\alpha\beta}$ with $\alpha+\beta \leq N-1$ through rational functions in $a_{\alpha\beta}$ $(\alpha+\beta \leq N-1)$ and $b_{\alpha\beta\gamma}$ $(\alpha+\beta+\gamma \leq d)$ with at most $C(k_0,d,N)$ variables, here $C(k_0,d,N)$ depends only on k_0,d,N. Now, (3.22) can be used in the same manner to show that the interior of the closure of $\mathcal{J}(\mathcal{A}^d_B(n,m;H_0,\delta)) \cap \mathcal{S}_R$ in $\mathcal{S}_R$ is empty. It is easy to see that $\mathcal{J}(\mathcal{A}_B)$ can be written as a countable union of these sets. We see that $\mathcal{J}(\mathcal{A}_B)$ is a set of the first category in $\mathcal{S}_R$. This completes the proof of Theorem 3.5. ■

Remark 3.6. (A). The crucial point for Theorem 3.5 to hold is that the modular space of surfaces with a vanishing Bishop invariant and $s < \infty$ is parameterized by an infinitely dimensional space. Hence, any subclass of $\mathcal{M}_s$, that is represented by a countable union of finite dimensional subspaces of $\mathcal{M}_s$, is a thin set of $\mathcal{M}_s$ under the equivalence relation. This idea, that the infinite dimensionality of the modular space would generally have the consequence of the generic non-algebraicity for its elements, dates back to the early work of Poincaré [Po]. In the CR setting, Forstneric in [For] has used the infinitely dimensional modular space of CR manifolds and the Baire category argument to give a short and quick proof that a generic CR submanifold in a complex space is not holomorphically equivalent to any algebraic manifold. Some earlier studies related to non-algebraicity for CR manifolds can be found, for instance, in [BER] [Hu2] [Ji]. However, by a result of the first author with Krantz [HK] and a result of the first author in [Hu3], a Bishop surface with an elliptic complex tangent can always be holomoirphically transformed into the algebraic Levi-flat hypersurface $\mathbb{C} \times \mathbb{R}$ and also into the Heisenberg hypersurface in $\mathbb{C}^2$.

(B). In the normal form (3.18), the condition that $\lambda_{ks+j} = 0$ for $j = 0, 1,\ k = 1, 2, \cdots$ can be compared with the Cartan-Chern-Moser chain condition in the case of strongly pseudoconvex hypersurfaces (see [CM]). In the hypersurface case, the chain condition is also described by a finite system of differential equations. It would be very interesting to know if, in our setting here, there also exist similar equations describing our chain condition.

4 Surface hyperbolic geometry and a convergence argument

In this section, we study the convergence problem for the formal consideration in the previous section. Our starting point is the flattening theorem of Huang-Krantz [HK], which says that an elliptic Bishop surface with a vanishing Bishop invariant can be holomorphically mapped into $\mathbb{C} \times \mathbb{R}$.

Hence, to study the convergence problem, we can restrict ourselves to a real analytic Bishop surface M defined by

$$w = z\bar{z} + z^s + \bar{z}^s + E(z, \bar{z}),\ E(z, \bar{z}) = \overline{E(z, \bar{z})} = o(|z|^s), \quad z \approx 0,\ 3 \le s < \infty. \tag{4.1}$$

Here E is real analytic.

For the rest of this section, we assume that *all Bishop surfaces (which we will denote by $M, M', M_{nor}, M'_{nor}, \cdots$) are real analytic and are holomorphically flattened.* (Namely, they are defined by real analytic equations of the form as in (4.1)). Thus the second-component (denoted by $(w + g(z,w)))$ of any map (formal or holomorphic) between such surfaces has the reality property: (See Lemma (2.1)(iii))

$$g(z,w) = g(w),\ \overline{g(w)} = g(\overline{w}). \tag{4.2}$$

Recall that the Moser-Webster complexification $\mathfrak{M}$ of M is the complex surface near $0 \in \mathbb{C}^4$ defined by:

$$\begin{cases} w = z\zeta + z^s + \zeta^s + E(z,\zeta), \\ \eta = z\zeta + z^s + \zeta^s + E(z,\zeta). \end{cases} \tag{4.3}$$

We define the projection $\pi : \mathfrak{M} \longrightarrow \mathbb{C}^2$ by sending $(z,\zeta,w,\eta) \in \mathfrak{M}$ to (z,w). Then π is generically s to 1. Write B for the branching locus of π near the origin. Namely, $(z,w) \in B$ if and only if $\exists(\zeta_0,\eta_0)$ such that $(z,\zeta_0,w,\eta_0) \in \mathfrak{M}$ and π is not biholomorphic near (z,ζ_0,w,η_0). Write $\mathfrak{B} = \pi^{-1}(B)$. Then

$$\begin{aligned} &(z,w) \in B \\ &\Longleftrightarrow \exists\zeta \ \text{ such that } \ w = z\zeta + z^s + \zeta^s + E(z,\zeta) \ \text{ and } \ z + s\zeta^{s-1} + E_\zeta(z,\zeta) = 0, \\ &\Longleftrightarrow \sharp\{\pi^{-1}(z,w)\} < s. \end{aligned}$$

It is easy to see that near 0, B is a holomorphic curve passing through the origin.

Now, suppose M' is defined by

$$w' = z'\bar{z}' + z'^s + \bar{z}'^s + E^*(z',\bar{z}'), \quad E^*(z',\bar{z}') = \overline{E^*(z',\bar{z}')} = o(|z'|^s) \text{ for } z' \approx 0. \tag{4.4}$$

Write $\mathfrak{M}'$ for the complexification of M'. Suppose that $F : (M,0) \longrightarrow (M',0)$ is a biholomorphic map. Then F induces a biholomorphic map $\mathcal{F}$ from $(\mathfrak{M},0)$ to $(\mathfrak{M}',0)$ such that $\pi' \circ \mathcal{F} = F \circ \pi$. From this, it follows that $F(B) = B'$, where B' is the branching locus of π' near the origin.

We next give the precise defining equation of B near 0. From the equation $z + s\zeta^{s-1} + E_\zeta(z,\zeta) = 0$, we can solve, by the implicit function theorem, that

$$z = h_1(\zeta) = -s\zeta^{s-1} + o(\zeta^{s-1}), \tag{4.5}$$

where $h_1(\zeta)$ is holomorphic near 0. Substituting (4.5) into (4.3), we get

$$w = h_2(\zeta) = (1-s)\zeta^s + o(\zeta^s). \tag{4.6}$$

From (4.6), we get

$$-\frac{w}{s-1} = (h_3(\zeta))^s \ \text{ with } \ h_3(\zeta) = \zeta + o(\zeta). \tag{4.7}$$

Hence, we get

$$\begin{aligned} &\zeta = h_3^{-1}((-\frac{w}{s-1})^{\frac{1}{s}}) = (-1)^{\frac{1}{s}}(\frac{1}{s-1})^{\frac{1}{s}}w^{\frac{1}{s}} + o(w^{\frac{1}{s}}), \\ &z = h_1 \circ h_3^{-1}((-\frac{w}{s-1})^{\frac{1}{s}}) = h_1((-1)^{\frac{1}{s}}(\frac{w}{s-1})^{\frac{1}{s}} + o(w^{\frac{1}{s}})) = s(-1)^{-\frac{1}{s}}w^{\frac{s-1}{s}} \cdot (s-1)^{\frac{1-s}{s}} + o(w^{\frac{s-1}{s}}). \end{aligned} \tag{4.8}$$

Here, $h_j's$ are holomorphic functions near 0. B is now defined by the second (multiple-valued) function in (4.8).

Next, let $w = u \geq 0$ and we define

$$\begin{aligned} A_j(u) &= h_1 \circ h_3^{-1}\left(e^{-\frac{(2j+1)\pi\sqrt{-1}}{s}}\left(\frac{u}{s-1}\right)^{1/s}\right) \\ &= se^{\frac{(1+2j)\pi\sqrt{-1}}{s}} u^{\frac{s-1}{s}} \cdot (s-1)^{\frac{1-s}{s}} + o(u^{\frac{s-1}{s}}), \quad j = 0, 1, \cdots, s-1. \end{aligned} \tag{4.9}$$

Then $A_j(u)$ is a well-defined function for $0 \leq u \ll 1$ and has a convergent power series expansion in $u^{1/s}$.

The following immediate fact will be crucial for this section:

Lemma 4.1. *(a). For any u with $0 < u \ll 1$, $A_j(u) \in D(u)$. Here*

$$D(u) = \{z \in \mathbb{C}^1 : w = z\bar{z} + z^s + \bar{z}^s + E(z, \bar{z}) < u\}. \tag{4.10}$$

(b).$\{(A_j(u), u)\}_{j=0}^{s-1} = B \cap \{w = u\}$ and $A_j(u)$ has a convergent power series expansion in $u^{1/s}$ for each fixed $j \in [0, s-1]$.

Proof of Lemma 4.1: The proof of Lemma 4.1 (a) follows clearly from the following estimate:

$$|A_j(u)|^2 + Re\{2A_j^s(u) + E(A_j(u), \overline{A_j(u)})\} = O(u^{\frac{2(s-1)}{s}}) \ll u$$

as far as $0 < u \ll 1$ and $s \geq 3$.

Lemma 4.1 (b) follows from the way $A_j(u)'s$ were defined and the result in Lemma 4.1(a). ∎

We remark that (4.1)-(4.9) also hold in the formal sense, when M is just assumed to be a formal Bishop surface with a vanishing Bishop invariant.

Consider a surface (M, p) in $\mathbb{C}^2$. We say that M near p is defined by a complex-valued function ρ, if M near p is precisely the zero set of ρ and $\{Re(\rho), Im(\rho)\}$ has constant rank two near p as functions in (x, y, u, v). For a surface (M, p) defined by ρ and a biholomorphic map F from a neighborhood of p to a neighborhood of p', we say that $F(M)$ approximates (M^*, p') defined by $\rho^* = 0$ to the order m at p' if there are smooth functions h_1 and h_2 with $|h_1|^2 - |h_2|^2 \neq 0$ at p such that $\rho^* \circ F(Z) = h_1 \cdot \rho(Z) + h_2 \cdot \overline{\rho(Z)} + o(|Z - p|^m)$. It is easy to check that this notion is independent of the choices of ρ and ρ^*.

Lemma 4.2. *Let M, M' be Bishop surfaces near 0 defined by (4.1) and (4.4), respectively. Suppose that $F(M)$ approximates M' to the order $\widetilde{N} = Ns + s - 1$ at 0 with $N > 1$. Then*

$$|\widetilde{f}(A_j(u), u) - A_j^*(u')| \lesssim |u|^{N-1}, \text{ for } j = 0, \cdots, s-1, \ 0 < u \ll 1,$$

where $A_j^*(u)$ is the function associated with M' defined as in (4.9). Here $F = (\widetilde{f}, \widetilde{g}) = (z + f, w + g)$ is assumed to be a holomorphic map with $f = O(|w| + |z|^2)$, $g(z, w) = g(w) = O(w^2)$, $\overline{g(w)} = g(\overline{w})$ and $u' = u + g(u)$.

Remark 4.3. In Lemma 4.2, since it is not assumed that $F(M) \subset M'$, the reality of g is not automatic from the property that $F(M)$ approximates M' to a high order.

Proof of Lemma 4.2: Let Φ_1 be a biholomorphic map, which maps M into M^N_{nor} defined by

$$w = z\bar{z} + 2Re\{z^s + \sum_{k=1}^{N}\sum_{j=2}^{s-1} a_{ks+j} z^{ks+j}\} + R(z, \bar{z}), \tag{4.11}$$

and let Φ_2 be a biholomorphic map from M' to ${M'}^N_{nor}$ with ${M'}^N_{nor}$ defined by

$$w' = z'\bar{z}' + 2Re\{{z'}^s + \sum_{k=1}^{N}\sum_{j=2}^{s-1} {a'}_{ks+j} {z'}^{ks+j}\} + R'(z', \bar{z}'). \tag{4.12}$$

Here $R(z, \bar{z}) = \overline{R(z, \bar{z})} = o(|z|^{sN+s-1})$ and $R'(z, \bar{z}) = \overline{R'(z, \bar{z})} = o(|z|^{sN+s-1})$. Define $\Phi^\sharp = \Phi_2 \circ F \circ \Phi_1^{-1}$. Here we assume $\Phi_1,\ \Phi_2$ satisfy the normalization as in Theorem 3.1 at the origin. (Notice that the second components of Φ_1, Φ_2 have the reality property as mentioned before). Then $\Phi^\sharp(M^N_{nor})$ approximates ${M'}^N_{nor}$ up to order $\widetilde{N}$.

By Theorem 2.2(I) (II), we conclude that

$$a_{ks+j} = a'_{ks+j} \text{ for } ks + j \le \widetilde{N} \text{ and } \Phi^\sharp = \mathrm{Id} + O(|(z, w)|^N), \text{ with } \widetilde{N} = Ns + s - 1. \tag{4.13}$$

In what follows, we write $A_j(u),\ A^*_j(u),\ A^{nor}_j(u),\ A^{*nor}_j(u)$ $(j = 0, \cdots, s-1)$ for those functions in u for $0 < u \ll 1$, defined as in (4.9), corresponding to $M,\ M',\ M^N_{nor},\ {M'}^N_{nor}$, respectively. Notice that they have convergent power series expansions in $u^{1/s}$ with the same first nonzero term $C_{s-2,j}u^{\frac{s-1}{s}}$, where

$$C_{s-2,j} = se^{\frac{(1+2j)\pi\sqrt{-1}}{s}} \cdot (s-1)^{\frac{1-s}{s}}. \tag{4.14}$$

Write $h^{nor}_j(\zeta)$ and $h^{*nor}_j(\zeta)$ $(j = 1, 2, 3)$ for those holomorphic functions, defined as in (4.5), (4.6) and (4.7), corresponding to M^N_{nor} and ${M'}^N_{nor}$, respectively. Then from the way these functions were constructed, we have

$$h^{nor}_j(\zeta) = h^{*nor}_j(\zeta) + O(|\zeta|^{\widetilde{N}-s}) \text{ for } j = 1, 2, 3.$$

Thus,

$$(h^{nor}_3)^{-1}(\zeta) = (h^{*nor}_3)^{-1}(\zeta) + O(|\zeta|^{\widetilde{N}-s}), \text{ and } h^{nor}_1 \circ (h^{nor}_3)^{-1}(\zeta) = h^{*nor}_1 \circ (h^{*nor}_3)^{-1}(\zeta) + O(|\zeta|^{\widetilde{N}-s}).$$

Hence, from the way A_j and A^*_j were defined, we have

$$A^{nor}_j(u) = A^{*nor}_j(u) + O(u^{N-1}). \tag{4.15}$$

Write $\Phi_j(z, w) = (\phi_j(z, w), \psi_j(w))$ with $\overline{\psi_j(u)} = \psi_j(u)$ for $j = 1, 2$.

By the invariant property that we mentioned above, we have, for $0 < u \ll 1$ and $0 \le j \le s-1$,

$$\Phi_1(A_j(u),u) = \left(A_j^{nor}(\psi_1(u)), \psi_1(u)\right), \ \Phi_2(A_j^*(u),u) = \left(A_j^{*nor}(\psi_2(u)), \psi_2(u)\right).$$

Since $F = \Phi_2^{-1} \circ \Phi^\sharp \circ \Phi_1$ and $\Phi^\sharp = \mathrm{Id} + O(|(z,w)|^N)$, we see that $u + g(u) = \psi_2^{-1}(\psi_1(u)) + O(u^N)$.

This immediately gives the following:

$$\begin{aligned} &F(A_j(u),u) = \Phi_2^{-1} \circ \Phi^\sharp \circ \Phi_1(A_j(u),u) = \Phi_2^{-1} \circ \Phi^\sharp \left(A_j^{nor}(\psi_1(u)), \psi_1(u)\right) \\ &= \Phi_2^{-1} \circ \Phi^\sharp \left(A_j^{*nor}(\psi_1(u)), \psi_1(u)\right) + O(u^{N-1}) = \Phi_2^{-1} \left(A_j^{*nor}(\psi_1(u)), \psi_1(u)\right) + O(u^{N-1}) \\ &= \left(A_j^*(\psi_2^{-1}(\psi_1(u))), \psi_2^{-1}(\psi_1(u))\right) + O(u^{N-1}) \\ &= \left(A_j^*(u+g(u)), u+g(u)\right) + O(u^{N-1}). \end{aligned} \tag{4.16}$$

The proof of Lemma 4.2 follows. ∎

We now state the following proposition, whose first part is the content of Lemma 4.2.

Proposition 4.4. *(1). Suppose that there is a holomorphic map F from $(\mathbb{C}^2,0)$ to $(\mathbb{C}^2,0)$ such that $F(M)$ approximates M' up to order $\widetilde{N} = Ns+s-1 > 2s-1$ at 0. Then*

$$A_j^*(u+g(u)) = \widetilde{f}(A_j(u),u) + O(u^{N-1}), \quad j = 0,1,\cdots,s-1, \ \textit{for } 0 < u \ll 1. \tag{4.17}$$

Here we assume that $F = (\widetilde{f}(z,w), \widetilde{g}(z,w)) = (z+f(z,w), w+g(w))$ with $f(z,w) = O(|w|+|z|^2)$, $g(w) = O(w^2)$ and $\overline{g(w)} = g(\overline{w})$.
(2). Suppose that there is a formal holomorphic map $F: M \longrightarrow M'$, where we write $F = (\widetilde{f}(z,w), \widetilde{g}(z,w)) = (z+f(z,w), w+g(w))$ with $f(z,w) = O(|w|+|z|^2)$ and $g(w) = O(w^2)$. For an $N > 1$, write, for the rest of this paper, $\widetilde{f}_{(\widetilde{N}+1)}(z,w)$, $\widetilde{g}_{(\widetilde{N}+1)}(z,w)$ for the (Taylor) polynomials consisting of terms of degree $\le \widetilde{N}$ in the Taylor expansions at the origin of $\widetilde{f}$ and $\widetilde{g}$, respectively, with $\widetilde{N} = Ns+s-1$. Then

$$A_j^*(u+g_{(\widetilde{N}+1)}(u)) - \widetilde{f}_{(\widetilde{N}+1)}(A_j(u),u) = O(u^{N-1}), \ \textit{as } u \to 0^+. \tag{4.18}$$

Remark 4.5. Proposition 4.4 (2) is the first place in this section, in which we use the truncation to deal with formal power series. We give a little more detailed explanation in this remark.

(**A**) We emphasize that A_j^* is a function in its variable over the domain $(0,\epsilon_0)$ with $0 < \epsilon_0 \ll 1$. Hence, for any other function $h(u) > 0$ with $0 < u < u_0$, if $\lim_{u\to 0} h(u) = 0$, then $A_j^* \circ h := A_j^*(h(u))$ is also a well defined function for $0 < u \ll 1$. In Proposition 4.4 (2), since $h(u) = u + g_{(\widetilde{N}+1)}(u) > 0$ for $0 < u \ll 1$ and $\lim_{u\to 0^+} h(u) = 0$, $A_j^* \circ h(u) = A_j^*(u+g_{(\widetilde{N}+1)}(u))$ is a well-defined function for $0 < u \ll 1$. As a point in the complex plane, $A_j^*(h(u)) \in$

$D^*(h(u))$ for each $0 < u \ll 1$. (See, for example, (4.20) for the notation of $D^*(u)$.) Of course, since $\widetilde{f}_{(\tilde{N}+1)}(z,w)$ is a polynomial in (z,w), $\widetilde{f}_{(\tilde{N}+1)}(A_j(u),u)$ is a well defined function in u for $0 < u \ll 1$. The precise meaning of (4.18) is that

$$\Big|\frac{A_j^*(u+g_{(\tilde{N}+1)}(u)) - \widetilde{f}_{(\tilde{N}+1)}(A_j(u),u)}{u^{N-1}}\Big| \le C$$

for a certain constant C when $0 < u < \epsilon_1$ with ϵ_1 a sufficiently small positive number. In what follows, the same explanation applies in the similar situations.

(B). Let m be a positive integer and let n be an integer. Let $h_1(u) = \sum_{k=n}^{\infty} a_k u^{\frac{k}{m}}$ and $h_2(u) = \sum_{k=n}^{\infty} b_k u^{\frac{k}{m}}$ be formal Laurent series in $u^{1/m}$ with at most finitely many negative power terms in $u^{1/s}$. In what follows, we say that $h_1(u) = h_2(u)$ in the formal sense if $a_k = b_k$ for any $k \ge n$. Now, in Proposition 4.4 (2), since $u + g(u)$ is a formal power series without constant term and $A_j^*(u)$ admits a power series expansion in $u^{1/s}$, $A_j^*(u+g(u)) = C_{s-2,j}u^{(s-1)/s} + \cdots$ also has a formal power series expansion in $u^{1/s}$. Similarly, $\widetilde{f}(A_j(u),u)$ admits a formal power series expansion in $u^{1/s}$. Then it follows from (4.18) that

$$A_j^*(u+g(u)) = \widetilde{f}(A_j(u),u) \quad \text{in the formal sense,} \tag{4.19}$$

which is all we need for our later application of Proposition 4.4 (2). Namely, the precise estimate for the error term $O(|z|^{N-1})$ in (4.18) is not crucial for our application. All we need is that there is an N', depending only on N with $N' \to \infty$ as $N \to \infty$, such that the right hand side of (4.18) is $O(|z|^{N'})$. (There are many similar situations in the later discussions where what is important is the error term of order $O(|z|^{N'})$ with $N' \to \infty$ as $N \to \infty$.) Indeed, to see (4.19), write

$$A_j^*(u+g(u)) = \sum_{k=s-1}^{\infty} a_k u^{k/s}, \quad \text{and} \quad \widetilde{f}(A_j(u),u) = \sum_{k=s-1}^{\infty} b_k u^{k/s}.$$

Then by (4.18), we have

$$\sum_{k=s-1}^{s(N-1)-1} b_k u^{k/s} = \sum_{k=s-1}^{s(N-1)-1} a_k u^{k/s} + O(u^{N-1}).$$

Hence, we have $a_k = b_k$ for any $s-1 \le k \le s(N-1)-1$. Since N is arbitrary, we see $a_k = b_k$ for $k \ge s-1$.

(C): In what follows, we often use the following simple fact without mentioning specifically: Let $B(u)$ be a formal power series in $u^{1/m}$ and $\chi_j(u) = c_j u^n + \cdots$ be a formal power series in u without constant term for $j = 1, 2$. If $\chi_1(u) = \chi_2(u) + O(u^N)$, then $B(\chi_1(u)) = B(\chi_2(u)) + O(u^{N-n+n/m}) = B(\chi_2(u)) + O(u^{N-n})$.

(D). We emphasize again that since the real analytic surfaces M, M' are assumed to be holomorphically flattened, the formal reality for g in Proposition 4.4(2) follows from Lemma

2.1(iii), as mentioned before. However, the reality for g in Proposition 4.4(1) has to be taken as part of the hypothesis there. The same remark applies in the other similar situations.

(E). Fix an M. Notice that for any u with $0 < u \ll 1$, $\{(z,u) : z \in D(u)\}$, with $D(u)$ being defined in (4.10), is a simply connected Riemann surface attached to M, whose Euclidean diameter $d(u)$ is of the quantity $2\sqrt{u} + o(u^{1/2})$. We notice that the Euclidean distance from $A_j(u)$ to the boundary of $D(u)$ divided by the diameter of $D(u)$ approaches to 1/2 as $u \to 0^+$. This roughly says that $A_j(u)'s$ are close to the the center of $D(u)$. More precisely, when we scale both $D(u)$ and $A_j(u)$, for each $0 < u \ll 1$, by the factor $\frac{1}{\sqrt{u}}$, $\frac{1}{\sqrt{u}}D(u)$ uniformly approaches to the unit disk in the sense that for any $0 < \delta \ll 1$, when $u > 0$ is sufficiently small, $\Delta_{1-\delta} \subset \frac{1}{\sqrt{u}}D(u) \subset \Delta_{1+\delta}$; while $\frac{1}{\sqrt{u}}A_j(u) \to 0$, the center of Δ. Here for any $R > 0$, $\Delta_R := \{\xi \in \mathbb{C} : \ |\xi| < R\}$.

Proof of Proposition 4.4: We need only to prove the second part of the proposition. We fix $0 \le j \le s-1$. Let the polynomial map $F_{(\widetilde{N}+1)} = (\widetilde{f}_{(\widetilde{N}+1)}, \widetilde{g}_{(\widetilde{N}+1)})$ be the Taylor polynomial of F of order $\widetilde{N}$ at the origin, namely, polynomial consisting of terms in the Taylor expansion of F at the origin of degree $\le \widetilde{N}$. Then $F_{(\widetilde{N}+1)}(M)$ approximates M' up to order $\widetilde{N}$. By the first part of the proposition, (see Remark 4.5(D) for the explanation for the formal reality of g), we have

$$A_j^*(u + g_{(\widetilde{N}+1)}(u)) = \widetilde{f}_{(\widetilde{N}+1)}(A_j(u), u) + O(u^{N-1}), \quad 0 < u \ll 1,$$

which is precisely (4.18). ∎

Let $z = r\sigma(\tau, r)$ with $u = r^2$ and $r > 0$ be the uniquely determined conformal map from the unit disk $\Delta := \{\tau \in \mathbb{C} : \ |\tau| < 1\}$ to $D(u)$ with $\sigma(0,r) = 0,\ \sigma'_\tau(0,r) > 0$. Here, as defined before,

$$D(u) = \{z \in \mathbb{C}^1 : z\bar{z} + z^s + \bar{z}^s + E(z,\bar{z}) < u = r^2\}.$$

Then $\phi(\tau) = (r\sigma(\tau,r), r^2)$ is a holomorphic disk attached to M.

Similarly, let $z = r\sigma^*(\tau^*, r)$ with $u = r^2$ and $r > 0$ be the conformal map from the disk Δ to $D^*(u)$ with $\sigma^*(0,r) = 0,\ {\sigma^*}'_{\tau^*}(0,r) > 0$. Here,

$$D^*(u) = \{z \in \mathbb{C}^1 : z\bar{z} + z^s + \bar{z}^s + E^*(z,\bar{z}) < u = r^2\}, \tag{4.20}$$

with M' being defined by $w = z\bar{z} + z^s + \bar{z}^s + E^*(z,\bar{z})$ as before. Then we know that

$$\sigma(\tau,r) = \tau(1 + O(r)) \text{ and } \sigma \text{ extends to a real analytic function in } (\tau,r) \text{ over } \Delta_{1+\varepsilon} \times (-\varepsilon,\varepsilon) \tag{4.21}$$

with $0 < \varepsilon \ll 1$. (See [Lemma 2.1, Hu3]). Similar properties also hold for σ^*.

For each $j \in [0, s-1]$, we will write, in what follows, $\tau_j(u) \in \Delta$ for the point such that $r\sigma(\tau_j(u), r) = A_j(u)$. Then

$$\tau_j(u) = \sigma^{-1}\left(\frac{A_j(u)}{u^{\frac{1}{2}}}, \sqrt{u}\right) = \frac{A_j(u)}{u^{\frac{1}{2}}}(1 + O(\sqrt{u})) = C_{s-2,j}u^{\frac{s-2}{2s}} + o(u^{\frac{s-2}{2s}}),\ 0 \le j \le s-1. \tag{4.22}$$

Here $\sigma^{-1}(\cdot, r)$ denotes the inverse of $\sigma(\cdot, r)$. In particular, as a function of u with $0 < u \ll 1$, we have the following property for $\tau_j(u)$ for each $j \in [0, s-1]$:

Lemma 4.6. *When $u \to 0^+$, $\tau_j(u)$ approaches to the origin.*

Remark 4.7. By (4.22) and (4.14), for each $0 < u \ll 1$, $\{\tau_0(u), \cdots, \tau_{s-1}(u)\}$, as points in Δ, are approximately equally distributed on the circle with radius equal to $s \cdot (s-1)^{\frac{1-s}{s}} u^{\frac{1}{2}-\frac{1}{s}}$. $\{\tau_0, \cdots, \tau_{s-1}\}$ are labeled counter-clock-wisely along the circle starting with $\tau_0(u) = se^{\frac{\pi\sqrt{-1}}{s}} \cdot (s-1)^{\frac{1-s}{s}} u^{\frac{1}{2}-\frac{1}{s}} + o(u^{1/2-1/s})$. For $0 < u \ll 1$, $\{A_0(u), \cdots, A_{s-1}(u)\}$, as points in $D(u)$, are approximately equally distributed counter-clock-wisely on the circle centered at the origin with radius equal to $s \cdot (s-1)^{\frac{1-s}{s}} u^{\frac{s-1}{s}}$, while $D(u)$ is approximately a disk centered at the origin with radius approximately equal to $\sqrt{u} \gg u^{\frac{s-1}{s}}$. Notice that the ratio of the Euclidean distance from $A_j(u)$ to $\partial D(u)$ with the Euclidean distance from $A_j(u)$ to the origin is approximately of the quantity $C_0 u^{\frac{2-s}{2s}} (\to \infty, \text{ as } u \to 0)$ with the constant $C_0 \neq 0$.

Notice that $\tau_j(u)$ has a convergent power series expansion in $u^{1/(2s)}$: (Or, we will simply say that $\tau_j(u)$ is analytic in $u^{\frac{1}{2s}}$)

$$\tau_j(u) = \sum_{l=s-2}^{\infty} C_{l,j} u^{\frac{l}{2s}}.$$

Here, as before,

$$C_{s-2,j} = s(s-1)^{\frac{1-s}{s}} e^{\frac{\pi\sqrt{-1}(1+2j)}{s}}, \quad 0 \leq j \leq s-1. \tag{4.23}$$

Recall that for any $0 \leq j, l \leq s-1$ and any u with $0 < u \ll 1$, $A_j(u), A_l(u) \in D(u)$. We define the hyperbolic distance between $A_j(u)$ and $A_l(u)$, as points in $D(u)$, to be the distance determined through the metric pulled back, by a conformal map, the classical Poincaré metric $d^2s = \frac{4dzd\bar{z}}{(1-|z|^2)^2}$ over the unit disk Δ. Now, the hyperbolic distance between $A_j(u)$ and $A_l(u)$ as points in $D(u)$ is the same as the classical hyperbolic distance between $\tau_j(u)$ and $\tau_l(u)$ as points in Δ with respect to the Poincaré metric $d^2s = \frac{4dzd\bar{z}}{(1-|z|^2)^2}$. Write $d_{hyp}(\tau_j(u), \tau_l(u))$ for the classical hyperbolic distance between $\tau_j(u)$ and $\tau_l(u)$ as points in Δ.

Write $L_{1(j+1)}(u) = e^{d_{hyp}(\tau_0, \tau_j)} - 1$, which is a function in u for $0 < u \ll 1$ and for each $j \in [1, s-1]$. In particular, $L_{12}(u) = e^{d_{hyp}(\tau_0, \tau_1)} - 1$. Since

$$d_{hyp}(\tau_0, \tau_1) = \ln \left(\frac{1 + |\frac{\tau_0 - \tau_1}{1 - \bar{\tau}_0 \tau_1}|}{1 - |\frac{\tau_0 - \tau_1}{1 - \bar{\tau}_0 \tau_1}|} \right), \text{ we have}$$

$$L_{12}(u) = 2s(s-1)^{\frac{1-s}{s}} |e^{\frac{\sqrt{-1}\pi}{s}} - e^{\frac{3\sqrt{-1}\pi}{s}}| u^{\frac{s-2}{2s}} + o(u^{\frac{s-2}{2s}}).$$

Also, $L_{12}(u)$ has a convergent power series expansion in $u^{\frac{1}{2s}}$.

Next, suppose that $F : M \longrightarrow M'$ is a biholomorphic map with $F = (\widetilde{f}, \widetilde{g}) = (z, w) + (O(|w| + |z|^2), O(w^2))$. Then $\widetilde{f}(z, u) = z + f(z, u)$ is a conformal map from $D(u)$ to $D^*(u')$ with $u' = u + g(u)$ for each u with $0 < u \ll 1$. Hence the hyperbolic distance between $A_0(u)$ to $A_1(u)$ is the same as the hyperbolic distance from $A_0^*(u')$ to $A_1^*(u')$ with respect to the hyperbolic metric in $D^*(u')$; for $\widetilde{f}(A_j(u), u) = A_j^*(u + g(u))$ as mentioned at the beginning of this section. Similarly, we can define functions $L_{1(j+1)}^*$ associated with M'. We have the following:

Lemma 4.8. *Suppose that F is a biholomorphic map with $F = (\widetilde{f}(z, w), \widetilde{g}(w)) = (z + f(z, w), w + g(w)) = (z, w) + (O(|w| + |z|^2), O(w^2))$ such that $F(M)$ approximates M' at 0 up to order $\widetilde{N} = Ns + s - 1 > 2s - 1$. Assume that $\overline{g(w)} = g(\overline{w})$. Then, we have*

$$L_{12}^*(u + g(u)) - L_{12}(u) = O(u^{N-2}) \text{ as } u \to 0^+.$$

Proof of Lemma 4.8: We first assume that M, M' are already normalized up to order $\widetilde{N}$. Then, by Theorem 2.2, we see that $F = \mathrm{Id} + O(|(z, w)|^N)$, M is defined by $w = z\bar{z} + 2Re\{\varphi_0(z)\} + o(|z|^{\widetilde{N}})$, M is defined by $w = z\bar{z} + 2Re\{\varphi_0(z)\} + o(|z|^{\widetilde{N}})$, where $\varphi_0(z) = z^s + o(z^s)$, $u' = u + g(u) = u + O(|u|^N)$ and $\varphi_0^{(sk+j)}(0) = 0$ for $j = 0, 1 \mod (s)$.

Since $u' = u + g(u) = u + O(u^N)$ and $u = r^2, u' = r'^2$, we have $r' = r + O(u^{N-1})$. From the way σ and σ^* were constructed, we claim that there is a constant C independent of τ and u such that for $0 < u \ll 1$, we have the following:

$$|\sigma^*(\tau, r') - \sigma(\tau, r)| \le C|\tau|u^{N-1} \text{ for } \tau \in \overline{\Delta}. \tag{4.24}$$

Indeed, by the way $\sigma^*(\cdot, r)$ was constructed, we can write $\sigma^*(\tau, r) = \tau(1 + \chi(\tau, r))$, where $\chi(\tau, r)$ extends to a real analytic function over $\overline{\Delta} \times (-\epsilon_0, \epsilon_0)$. (See [Lemma 2.1, Hu3] or the following Lemma.) We see that

$$\sigma^*(\tau, r') - \sigma^*(\tau, r) = \tau O(u^{N-1}). \tag{4.25}$$

Hence, (4.24) follows from (4.25) and the following more general result:

Lemma 4.9. *Let $\sigma(\xi, r) = \xi \cdot (1 + O(r))$ and $\sigma^*(\xi, r) = \xi \cdot (1 + O(r))$ be the biholomorphic map from the unit disk Δ to*

$$\begin{aligned} D(r) &:= \{\xi \in \mathbb{C}(\approx \overline{\Delta}) : \ |\xi|^2 + rF_1(r, \xi, \overline{\xi}) < 1\}, \\ D^*(r) &:= \{\xi \in \mathbb{C}(\approx \overline{\Delta}) : \ |\xi|^2 + rF_1(r, \xi, \overline{\xi}) + r^m F_2(r, \xi, \overline{\xi}) < 1\}, \end{aligned} \tag{4.26}$$

respectively. Here $F_j(r, \xi, \overline{\xi})$ are real-valued real analytic functions in a neighborhood of $\{0\} \times \overline{\Delta} \times \overline{\Delta}$. Then there is a constant C, depending only on F_1, F_2, such that

$$|\sigma^*(\xi, r) - \sigma(\xi, r)| \le C|\xi|r^m, \quad \xi \in \overline{\Delta}.$$

Proof of Lemma 4.9: From the way σ and σ^* were constructed (see [Lemma 2.1, Hu3]), there are $U, U^* \in C^\omega(\partial\Delta \times (-\epsilon_0, \epsilon_0))$ with $0 < \epsilon_0 \ll 1$ such that

$$\sigma(\xi, r) = \xi\left(1 + U(\xi, r) + \mathcal{H}(U(\cdot, r))\right),\ \sigma^*(\xi, r) = \xi\left(1 + U^*(\xi, r) + \mathcal{H}(U^*(\cdot, r))\right),\ \xi \in \partial\Delta.$$

Here $\mathcal{H}$ is the standard Hilbert transform and $U,\ U^*$ satisfy the following equations:

$$U = G_1(r, \xi, U, \mathcal{H}(U)),\quad U^* = G_1(r, \xi, U^*, \mathcal{H}(U^*)) + r^m G_2(r, \xi, U^*, \mathcal{H}(U^*)),$$

where $G_j(r, \xi, x, y)$ are real analytic in (r, ξ, x, y) with $|G_j| \lesssim |r| + |x|^2 + |y|^2$. Notice by the implicit function (see [Lemma 2.1, Hu3]), $\|U\|_{1/2},\ \|U^*\|_{1/2} \le C_1|r|$ with $\|\cdot\|_{1/2}$ the Hölder-$\frac{1}{2}$ norm in $\xi \in \partial\Delta$. Next, we have

$$\begin{aligned} &U^* - U = \int_0^1 \tfrac{\partial G_1}{\partial x}\big(r, \xi, \tau U^* + (1-\tau)U, \tau\mathcal{H}(U^*) + (1-\tau)\mathcal{H}(U)\big)(U^* - U)\mathrm{d}\tau \\ &+ \int_0^1 \tfrac{\partial G_1}{\partial y}\big(r, \xi, \tau U^* + (1-\tau)U, \tau\mathcal{H}(U^*) + (1-\tau)\mathcal{H}(U)\big)(\mathcal{H}(U^*) - \mathcal{H}(U))\mathrm{d}\tau \\ &+ r^m G_2\big(r, \xi, U^*, \mathcal{H}(U^*)\big). \end{aligned} \tag{4.27}$$

By noticing that the Hilbert transform is bounded acting on the Hölder space, we easily conclude that when $0 < u \ll 1$, it holds that $\|U^* - U\|_{1/2} \le C r^m$ for a certain constant C. The result in the lemma follows accordingly for $0 < r \ll 1$. ∎

Now, recall that $u' = u + g(u) = u + O(u^N)$, $r = \sqrt{u}, r' = \sqrt{u'}$ and $r' = r + O(u^{N-1})$. Notice that $A_j^*(u') = A_j(u) + O(u^{N-1})$ as a function of u with $u \to 0^+$, by Proposition 4.4 (1). Hence

$$\frac{A_j^*(u')}{r'} - \frac{A_j(u)}{r} = \frac{A_j^*(u')}{r} - \frac{A_j(u)}{r} + O(u^{N-2}) = O(u^{N-2}). \tag{4.28}$$

By the definition of $\tau_j(u)$ and $\tau_j^*(u')$, we have

$$A_j(u) = r\sigma(\tau_j(u), r),\quad \text{and } A_j^*(u') = r'\sigma^*(\tau_j^*(u'), r'). \tag{4.29}$$

Recall that, by Lemma 4.6, we have

$$\tau_j(u),\ \tau_j^*(u') \text{ are inside } \Delta, \text{ and approach to 0 as } u \to 0^+. \tag{4.30}$$

Now, from (4.28) (4.29), we get the following

$$\sigma^*(\tau_j^*(u'), r') - \sigma(\tau_j(u), r) = O(u^{N-2}). \tag{4.31}$$

On the other hand,

$$\begin{aligned} &\sigma^*(\tau_j^*(u'), r') - \sigma(\tau_j(u), r) \\ &= \big(\sigma^*(\tau_j^*(u'), r') - \sigma(\tau_j^*(u'), r)\big) + \big(\sigma(\tau_j^*(u'), r) - \sigma(\tau_j(u), r)\big). \end{aligned} \tag{4.32}$$

Notice that $\sigma(\xi, r) = \xi(1 + O(r))$. Also, notice that $|\frac{\partial\{\sigma(\xi,r)-\xi\}}{\partial\xi}| \le r \cdot C$ for $|\xi| < 1/2$ with C a constant independent of r. (This can be seen immediately from the Cauchy estimate, for instance; or it can be easily derived by the property of $\sigma(\tau, r)$ itself.) Hence for $0 < u \ll 1$, by (4.30) and the estimate just mentioned, we have

$$\sigma(\tau_j^*(u'), r) - \sigma(\tau_j(u), r) = (\tau_j^*(u') - \tau_j(u)) \cdot (1 + o(1)). \tag{4.33}$$

Now, it follows from (4.32), (4.31), (4.24) and (4.33) that

$$O(u^{N-2}) = \sigma^*(\tau_j^*(u'), r') - \sigma(\tau_j(u), r) = O(u^{N-2}) + (\tau_j^*(u') - \tau_j(u)) \cdot (1 + o(1)).$$

This immediately gives

$$\tau_j^*(u') - \tau_j(u) = O(u^{N-2}) \text{ as } u \to 0^+. \tag{4.34}$$

Here, we mention again that, as in Remark 4.5 (A), $\tau_j^*(u')$ is understood as a well defined composition function of τ_j^* with $u' = u + g(u)$. Hence, we have

$$\begin{aligned}
&|L_{12}^*(u') - L_{12}(u)| = |e^{d_{hyp}(\tau_0^*(u'), \tau_1^*(u'))} - e^{d_{hyp}(\tau_0(u), \tau_1(u))}| \\
&= 2\big||\tau_0^*(u') - \tau_1^*(u')| - |\tau_0(u) - \tau_1(u)|\big| \cdot (1 + o(1)) \\
&\leq 2\left(\big|(\tau_0^*(u') - \tau_0(u)) + (\tau_1(u) - \tau_1^*(u'))\big|\right) \cdot (1 + o(1)) \\
&= O(u^{N-2}).
\end{aligned}$$

We thus obtain

$$L_{12}^*(u + g(u)) = L_{12}(u) + O(u^{N-2}), \text{ as } u \to 0^+.$$

This completes the proof of Lemma 4.8.

For the general M and M', using the invariant property for the hyperbolic distance function under a conformal transformation, we can proceed in exactly the same way as in the proof of Lemma 4.2 to reduce the proof of Lemma 4.9 to the case when M and M' are already normalized up to order $\widetilde{N} = Ns + s - 1$. For convenience of the reader, we say a few words as follows:

Let M_{nor}^N, $M_{nor}^{\prime N}$, $\Phi_1 = (\phi_1, \psi_1)$, $\Phi_2 = (\phi_2, \psi_2), \Phi^\sharp$ be defined as in the proof of Lemma 4.2. For $0 < u \ll 1$, define $L_{12}^{nor}(u)$ and $L_{12}^{*nor}(u)$ in a similar way as for L_{12}. Since $\Phi^\sharp(M_{nor}^N)$ approximates $M_{nor}^{\prime N}$ up to order $\widetilde{N}$ and since $\Phi^\sharp(z, w) = (z, w) + O(|(z, w)|^N)$, by what we have obtained and Remark 4.5(C), we have $L_{12}^{*nor}(u) = L_{12}^{nor}(u) + O(u^{N-2})$ for $0 < u \ll 1$. Recall that $u + g(u) = \psi_2^{-1} \circ \psi_1(u) + O(u^N)$. Also by the invariant property of hyperbolic distances, we have $L_{12}^{nor}(\psi_1(u)) = L_{12}(u)$ and $L_{12}^{*nor}(\psi_2(u)) = L_{12}^*(u)$. Therefore, we obtain the following:

$$\begin{aligned}
&L_{12}^*(u + g(u)) = L_{12}^*(\psi_2^{-1} \circ \psi_1(u)) + O(u^{N-1}) \\
&= L_{12}^{*nor}(\psi_1(u)) + O(u^{N-1}) = L_{12}^{nor}(\psi_1(u)) + O(u^{N-2}) \\
&= L_{12}(u) + O(u^{N-2}).
\end{aligned} \tag{4.35}$$

∎

Now, let $F : M \longrightarrow M'$ be a formal equivalence map with $F := (\widetilde{f}, \widetilde{g}) = (z, w) + (O(|w| + |z|^2), O(w^2))$ and let the polynomial map $F_{(\widetilde{N}+1)}$ be the Taylor polynomial of F of order $\widetilde{N}$, as before. Here $\widetilde{N} = Ns + s - 1$. Then $F_{(\widetilde{N}+1)}(M)$ approximates M' up to order $\widetilde{N}$. Applying Lemma 4.8, we get

$$L^*_{12}(\widetilde{g}_{(\widetilde{N}+1)}(u)) = L_{12}(u) + O(u^{N-2}). \tag{4.36}$$

Here, as before, the polynomial $\widetilde{g}_{(\widetilde{N}+1)}(u)$ is the Taylor polynomial of $\widetilde{g}(u)$ at the origin of order $\widetilde{N}$. We mention again that if ϕ is a formal power series in $u^{\frac{1}{2s}}$ and $h(u)$ is a formal power series in u without constant term, then $\phi \circ h$ gives a formal power series in $u^{\frac{1}{2s}}$. Now, since N is arbitrary, we get from (4.36) the following:

$$L^*_{12}(\widetilde{g}(u)) = L_{12}(u) \text{ in the formal sense.} \tag{4.37}$$

Namely, the right hand side and left hand side of (4.37) have the same formal power series expansion in $u^{1/(2s)}$. (See Remark 4.5(B) for the related notion.)

Since $L_{12}(u)$ is a well-defined function of u for $0 < u \ll 1$, (4.37) shows that $L^*_{12}(\widetilde{g}(u))$ also gives a function in u even though we do not know yet the convergence of $\widetilde{g}(u)$. This fact will be one of the crucial points for our convergence argument.

Making use of (4.37), we next prove the following:

Lemma 4.10. *Let $F : M \longrightarrow M'$ be a formal equivalence map such that*

$$F(z, w) := (\widetilde{f}(z, w), \widetilde{g}(w)) = (z + f(z, w), w + g(w))$$

with $f(z, w) = O(|w| + |z|^2)$ and $g(w) = O(w^2)$. Then $\widetilde{g}$ is convergent.

Proof of Lemma 4.10: We remark again that the reality property of g follows from Lemma 2.1(iii).

Notice that we already proved (see (4.37)) that

$$L^*_{12}(\widetilde{g}(u)) = L_{12}(u) \text{ in the formal sense.}$$

Write $u = V^{2s}$. Define $U = (\widetilde{g}(u))^{1/(2s)} = u^{1/(2s)} + \cdots$, which has a formal power series expansion in $u^{\frac{1}{2s}}$ and thus can be regarded as a formal power series in V.

Then

$$L^*_{12}(U^{2s}(V)) = L_{12}(V^{2s})$$

in the formal sense. Notice that $L^*_{12}(t^{*2s})$ and $L_{12}(t^{2s})$ have convergent power series expansions in t^* and t, respectively. Moreover,

$$L^*_{12}(t^{*2s}) = (\psi^*(t^*))^{s-2}, \; L_{12}(t^{2s}) = (\psi(t))^{s-2}$$

with ψ, ψ^* invertible holomorphic map of $(\mathbb{C}, 0)$ to itself, and with $\psi'(0) = \psi^{*\prime}(0)(= |2(C_{s-2,0} - C_{s-2,1})|^{\frac{1}{s-2}})$. Hence, we get

$$U = \psi^{*-1} \circ \psi(u^{\frac{1}{2s}}) \text{ and } \widetilde{g}(u) = U^{2s} = \left(\psi^{*-1} \circ \psi(u^{\frac{1}{2s}})\right)^{2s}.$$

The above are regarded as equalities as formal power series in $u^{\frac{1}{2s}}$. Notice that $(\psi^{*-1}\circ\psi(z^{\frac{1}{2s}}))^{2s}$ defines a multiple valued holomorphic function near the origin. By the Puiseux expansion, we get

$$(\psi^{*-1}\circ\psi(u^{\frac{1}{2s}}))^{2s}=\sum_{j=2s}^{\infty}c_j u^{\frac{j}{2s}}.$$

Here $|c_j|\lesssim R^j$ for some $R\gg 1$. However, $(\psi^{*-1}\circ\psi(u^{\frac{1}{2s}}))^{2s}=\widetilde{g}$ in the formal sense and the latter has a formal power series expansion in u. We conclude that $c_j=0$ if $2s$ does not divide j. This proves the convergence of $\widetilde{g}(u)$. ∎

We next prove the following theorem:

Theorem 4.11. *Let M and M' be real analytic Bishop surfaces near 0 defined by (4.1) and (4.4), respectively. Suppose that $F=(\widetilde{f},\widetilde{g}):(M,0)\longrightarrow(M',0)$ is a formal equivalence map. Then F is biholomorphic near 0.*

Proof of Theorem 4.11: We can assume that $\widetilde{f}=z+f$ with $wt_{nor}(f)\geq 2$ and $\widetilde{g}=w+g(w)$ with $wt_{nor}(g)\geq 4$. We can also assume that M and M' have been normalized to a certain high order, say, to the order of $2s^2$, such that $F=(z,w)+O(|(z,w)|^{s+1})$. Then $F_0(M')$ is still defined by an equation of the form as in (4.4), where $F_0(z,w)=(z,(\widetilde{g})^{-1}(w))$. By Lemma 4.10 and by considering $F_0\circ F$, $F_0(M')$ instead of F and M', we can assume, without loss of generality, that $\widetilde{g}=w$. We will prove the convergence of $\widetilde{f}$ by the hyperbolic geometry associated to the surface discussed above.

By Proposition 4.4(2), we first notice that $\widetilde{f}_{(\widetilde{N}+1)}(A_j(u),u)=A_j^*(u)+O(u^{N-1})$ for $\widetilde{N}=Ns+s-1>2s-1$. Here, $\widetilde{f}_{(\widetilde{N}+1)}$ is the Taylor polynomial of order $\widetilde{N}^{th}$ in the Taylor expansion of f at 0, as defined before.

Write $\widetilde{M}$ and $\widetilde{M'}$ for the local holomorphic hull of M and M' near the origin, respectively. We next construct a holomorphic map from $\widetilde{M}\setminus M$ to $\widetilde{M'}\setminus M'$ as follows:

Let $\Psi(\cdot,r)$ be the biholomorphic map from Δ to itself such that $\Psi(\tau_j(u),r)=\tau_j^*(u)$ for $j=0,1$. Since $\tau_j(u),\tau_j^*(u)\in\Delta$, to see the existence and uniqueness of $\Psi(\cdot,r)$, it suffices for us to explain that $d_{hyp}(\tau_0(u),\tau_1(u))=d_{hyp}(\tau_0^*(u),\tau_1^*(u))$. But, this readily follows from (4.37) with $\widetilde{g}(u)=u$; for once we know that (4.37) holds in the formal sense and when both sides are well defined analytic functions in $u^{1/(2s)}$, then (4.37) holds for $0<u\ll 1$ as functions in u.

For a non-zero complex number z, its principal argument $arg(z)$ is set such that $0\leq arg(z)<2\pi$. Now, for $\tau\in\Delta$, $0<r=\sqrt{u}\ll 1$ and $j\in[1,s-1]$, write

$$\theta_j(r)=arg\{\tfrac{\tau_j(u)-\tau_0(u)}{1-\overline{\tau_0}(u)\tau_j(u)}\tfrac{1}{u^{\frac{s-2}{2s}}}\},\ \theta_j^*(r)=arg\{\tfrac{\tau_j^*(u)-\tau_0^*(u)}{1-\overline{\tau_0^*}(u)\tau_j^*(u)}\tfrac{1}{u^{\frac{s-2}{2s}}}\}.$$

Then, by (4.22) and (4.23), we get $\theta_j(r)=(\frac{\pi}{2}+\frac{(1+j)\pi}{s})+O(u^{1/(2s)})$, $\theta_j^*(r)=(\frac{\pi}{2}+\frac{(1+j)\pi}{s})+O(u^{1/(2s)})$, for $0<u\ll 1$, which also have convergent power series expansion in $u^{1/(2s)}$. Write

$$\Psi_1(\tau,r)=\frac{\tau-\tau_0(u)}{1-\overline{\tau_0}(u)\tau},\ \Psi_1^*(\tau,r)=\frac{\tau-\tau_0^*(u)}{1-\overline{\tau_0^*}(u)\tau},\ \mathcal{R}(\tau,r)=e^{-i\theta_1(r)+i\theta_1^*(r)}\tau.$$

Then

$$\Psi(\cdot, r) = \Psi_1^{*-1}(\cdot, r) \circ \mathcal{R}(\cdot, r) \circ \Psi_1(\cdot, r). \tag{4.38}$$

It is clear that there is a real analytic function $\Psi^{ext}(\tau, \nu)$ in $(\tau, \nu) \in \Delta_{1+\varepsilon_0} \times (-\varepsilon_0, \varepsilon_0)$ with $0 < \epsilon_0 \ll 1$ such that $\Psi(\tau, r) = \Psi^{ext}(\tau, u^{\frac{1}{2s}})$ for $0 < u = r^2 \ll 1$. For simplicity of notation, we shall simply say, in what follows, that $\Psi(\tau, r)$ has a real analytic extension in $(\tau, u^{1/(2s)})$ to $\Delta_{1+\varepsilon_0} \times (-\varepsilon_0, \varepsilon_0)$.

We notice that when f is a priori known to be convergent, we then have, by the uniqueness property of the conformal transformation, that

$$\widetilde{f}(r\sigma(\xi, r), r^2) = r\sigma^*(\Psi(\xi, r), r), \tag{4.39}$$

The idea for the proof of the theorem is actually to find a way to make sense of (4.39) even in the formal case.

Write, for each $0 < u \ll 1$, $\Theta_j(u)$ ($j = 2, \cdots, s-1$) for the (counter-clockwise) angle from the hyperbolic geodesic (in Δ) connecting $\tau_0(u)$ to $\tau_1(u)$ to the hyperbolic geodesic (in Δ) connecting τ_0 to τ_j at their intersection $\tau_0(u)$. As a function of u (or $r = \sqrt{u}$) for $0 < u \ll 1$, we have the following, which can also be taken as the definition of $\Theta_j(u)$, $j = 2, \cdots, s-1$:

$$\begin{aligned} \Theta_j(u) &= arg\{\frac{\tau_j(u) - \tau_0(u)}{\tau_1(u) - \tau_0(u)} \cdot \frac{1 - \overline{\tau_0(u)}\tau_1(u)}{1 - \overline{\tau_0(u)}\tau_j(u)}\} \\ &= arg\{\frac{C_{s-2,j} - C_{s-2,0}}{C_{s-2,1} - C_{s-2,0}}\} + O(u^{1/(2s)}). \end{aligned} \tag{4.40}$$

Remark 4.12. We remark that $\Theta_j(u) = \theta_j(u) - \theta_1(u) = \frac{j-1}{s}\pi + O(u^{\frac{1}{2s}})$ for $j \in [2, s-1]$, as far as $0 < u \ll 1$. A geometric way to see $\Theta_j(u)$ is as follows: Find an automorphism χ of Δ to transform $\tau_0(u)$ to the origin and $\tau_1(u)$ to the positive real axis. Then the principal argument of $\chi(\tau_j(u))$ is $\Theta_j(u)$.

We can similarly define Θ_j^* for M'. Then the same argument, which we used to show that $L_{12}(u) = L_{12}^*(u)$, can be used to prove that

$$\Theta_j(u) = \Theta_j^*(u) \ \text{ and } \ L_{1(j+1)}(u) = L_{1(j+1)}^*(u), \ 2 \le j \le s-1 \tag{4.41}$$

first in the formal sense and thus also hold as functions of u.

Now, we can use an automorphism of Δ to map τ_0 to the origin and τ_1 to a point in the positive real line. Then we easily see that Θ_j and $L_{1(j+1)}$ uniquely determine $\tau_j(u)$.

Recall $\Psi(\cdot, \tau)$ is an automorphism of Δ and thus is an isometry with respect to the Poincaré metric, that maps $\tau_j(u)$ to $\tau_j^*(u)$ for $j = 0, 1$. Write $\widetilde{\tau}_j(u) = \Psi(\tau_j(u), r) \in \Delta$ for each $j \in [2, s-1]$. Then the hyperbolic distance between $\widetilde{\tau}_j(u)$ and $\tau_0^*(u) = \Psi(\tau_0(u), r)$ equals to that between $\tau_j(u)$ and $\tau_0(u)$, that is $L_{1(j+1)}(u)$ and thus is also the same as $L^*_{1(j+1)}(u)$. Moreover, the angle between the hyperbolic geodesic (in Δ) connecting $\tau^*_0(u)$ to $\tau^*_1(u)$ and the hyperbolic geodesic (in Δ) connecting τ^*_0 to $\widetilde{\tau}_j$ at their intersection $\tau^*_0(u)$ equals, first, to $\Theta_j(u)$ and thus also equals to $\Theta^*_j(u)$. Hence, we see that $\widetilde{\tau}_j(u) = \tau^*_j(u)$. Namely, we proved the following:

Lemma 4.13. $\Psi(\tau_j(u), r) = \tau^*_j(u)$ *for* $j = 0, \cdots, s-1$.

Now, for $(z, u) \in \widetilde{M} \setminus M$ close to the origin, we define

$$f^*(z,u) = \sqrt{u}\sigma^*(\Psi(\sigma^{-1}(\frac{z}{\sqrt{u}}, \sqrt{u}), \sqrt{u}), \sqrt{u}). \tag{4.42}$$

Here, we recall that $\sigma^{-1}(\cdot, r)$ denotes the inverse of $\sigma(\cdot, r)$. Then $f^*(z,u)$ is analytic in $\widetilde{M} \setminus M$. Our crucial point is to show that $f^*(z,u)$ is actually the same as $\widetilde{f}(z,u)$ in a certain sense. For this purpose, we next prove the following lemma:

Lemma 4.14. *Let α be a non-negative integer. Let $\widetilde{N} = Ns + s - 1 \gg 1$. Still write $\widetilde{f}_{(\widetilde{N}+1)}$ for the polynomial consisting of terms of degree $\leq \widetilde{N}$ in the Taylor expansion of $\widetilde{f}$ at 0. Then we have*

$$\Big|\frac{\partial^\alpha f^*}{\partial z^\alpha}(0,u) - \frac{\partial^\alpha \widetilde{f}_{(\widetilde{N}+1)}}{\partial z^\alpha}(0,u)\Big| \leq Cu^{N'}, \text{ for } 0 < u \ll 1. \tag{4.43}$$

Here C is a constant independent of u, N' is an integer depending only on N and α such that $N' \to \infty$ when $N \to \infty$. (Indeed, we can take $N' = [\frac{2}{3}N] - \alpha - 3$.)

Proof of Lemma 4.14: Let $S(u)$ be the hyperbolic polygon in $D(u)$ with vertices $A_j(u)(j = 0, 1, \cdots, s-1)$, whose boundary consists of the geodesic segment connecting $A_j(u)$ to $A_{j+1}(u)$ for $j = 0, \cdots, s-2$ and the geodesic segment connecting $A_{s-1}(u)$ to $A_0(u)$. Let $S^*(u)$ be the one corresponding to M'. For any points $P, Q \in \Delta$, we define the following curve, whose image is precisely the geodesic segment connecting P to Q:

$$\gamma^\Delta_{P,Q}(t) = \frac{t\frac{Q-P}{1-Q\overline{P}} + P}{1 + t\overline{P} \cdot \frac{Q-P}{1-Q\overline{P}}}, \quad 0 \leq t \leq 1. \tag{4.44}$$

For a more general bounded simply connected domain D and $P, Q \in D$, let σ_D be a conformal map from D to Δ with $\sigma_D(P) = 0$. We then define $\gamma^D_{P,Q}(t)$ to be $\sigma_D^{-1}(t\sigma_D(Q))$ for $0 \leq t \leq 1$.

$\gamma^D_{P,Q}(t)$ is independent of the choice of σ_D, by the fact that $\gamma^D_{P,Q}(t)$ is sitting on the hyperbolic geodesic (with respect to the hyperbolic metric in D) connecting P to Q with the hyperbolic distance from P to $\gamma^D_{P,Q}(t)$ being

$$\ln \frac{(1-t) + (1+t)e^l}{(1+t) + (1-t)e^l},$$

where l is the hyperbolic distance from P to Q (with respect to the Poincaré metric over D). $\gamma^D_{P,Q}(t)$ coincides with (4.44) when $D = \Delta$.

Next, we have

Lemma 4.15. *For $P \in \partial S(u)$ and $0 < u \ll 1$, it holds that*

$$f^*(P,u) = \widetilde{f}_{(\widetilde{N}+1)}(P,u) + Error(P,u), \tag{4.45}$$

where $|Error(P,u)| \leq Cu^{\frac{2}{3}N-2}$ with C a constant independent of $P \in \partial S(u)$ and u.

Proof of Lemma 4.15: This can be done by the same argument used in the proof of Lemma 4.2 and by making use of the property that $\widetilde{f}(A_j(u),u) = A^*_j(u)$ (in the formal sense) as a formal power series in $u^{1/s}$. In detail, we argue as follows:

Let $u > 0$ be sufficiently small. Without loss of generality, we just explain how to obtain (4.45) for points sitting on the hyperbolic geodesic segment in $D(u)$ connecting $A_0(u)$ to $A_1(u)$. Write $P(t,u) := \gamma^{D(u)}_{A_0(u),A_1(u)}(t)$ and $P^*(t,u) := \gamma^{D^*(u)}_{A^*_0(u),A^*_1(u)}(t)$ for $t \in [0,1]$. Here, as before, $D^*(u), A^*_0(u), A^*_1(u)$ denote, respectively, the similarly defined objects (but associated with M') as $D(u), A_0(u), A_1(u)$.

Notice that $F_{(\widetilde{N}+1)}(M)$ approximates M' up to order $\widetilde{N} = Ns + s - 1$, where $F_{(\widetilde{N}+1)} = (\widetilde{f}_{(\widetilde{N}+1)}(z,w), w)$ is defined as before. As in the proof of Lemma 4.2, we have biholomorphic maps Φ_1 and Φ_2 satisfying the normalization in Theorem 3.1, such that $\Phi_1(M) = M^N_{nor}$, $\Phi_2(M') = {M'}^N_{nor}$. Moreover, M^N_{nor} and ${M'}^N_{nor}$ are defined by equations of the form as in (4.11) and (4.12), respectively. Write $(z_{nor}, w_{nor}) = \Phi_1(z,w)$ and write $(z^*_{nor}, w^*_{nor}) = \Phi_2(z', w')$. As in Lemma 4.2, we have

$$\begin{aligned}\Phi^\sharp = (\widetilde{\phi}^\sharp, \widetilde{\psi}^\sharp) = (z^*_{nor}(z_{nor}, w_{nor}), w^*_{nor}(w_{nor})) := \Phi_2 \circ F_{(\widetilde{N}+1)} \circ \Phi_1^{-1}(z_{nor}, w_{nor}) \\ = (z_{nor}, w_{nor}) + O(|(z_{nor}, w_{nor})|^N).\end{aligned} \tag{4.46}$$

Define $D^{nor}(u)$ and $D^{*nor}(u)$, associated with M^N_{nor} and M'^N_{nor}, respectively, in a similar way as for $D(u)$. Let $r_{nor} \cdot \sigma_{nor}(\cdot, r_{nor})$ be the conformal map from Δ to $D^{nor}(u^{nor})$, where $\sigma_{nor}(\cdot, r_{nor})$ has the same normalization at the origin as that for $\sigma(\tau, r)$. (Notice that $u_{nor} = r^2_{nor}$.) Then $\tau^{nor}_j(u_{nor})$ is defined such that $A^{nor}_j(u_{nor}) = r_{nor} \cdot \sigma_{nor}(\tau^{nor}_j(u_{nor}), r_{nor})$. Similarly, we can define $\sigma^*_{nor}(\tau, r^*_{nor})$, τ^{*nor}_j.

Notice that $\Phi^\sharp(M^{nor})$ approximates M^{*nor} to the order $\widetilde{N} = Ns + s - 1$ and the defining equation of M^{*nor} given in the form of (4.12) coincides with that of M^{nor} given in the form of (4.11) up to order $\widetilde{N}$. As in (4.13), (4.15) and (4.34), we obtain

$$\begin{aligned}&A^{*nor}_j(u^*_{nor}) = A^{nor}_j(u_{nor}) + O(u^{N-1}_{nor}) \text{ and} \\ &\tau^{*nor}_j(u^*_{nor}) - \tau^{nor}_j(u_{nor}) = O((u_{nor})^{N-2}), \quad \text{as } u_{nor} \to 0^+.\end{aligned} \tag{4.47}$$

Write $P_{nor}(t,u_{nor}) = \gamma^{D^{nor}(u)}_{A_0^{nor}(u_{nor}),A_1^{nor}(u_{nor})}(t)$ and $P^*_{nor}(t,u^*_{nor}) = \gamma^{D^{*nor}(u)}_{A_0^{*nor}(u^*_{nor}),A_1^{*nor}(u^*_{nor})}(t)$ for $t\in[0,1]$.

Define, for $|X|,|Y|<1$,

$$\Xi(t,X,Y) := \frac{t\frac{Y-X}{1-\overline{X}Y}+X}{1+t\overline{X}\cdot\frac{Y-X}{1-Y\overline{X}}}. \tag{4.48}$$

And define for $0<u\ll 1$,

$$\beta^*_{nor}(t,u) := \Xi\left(t,\tau_0^{*nor}(u),\tau_1^{*nor}(u)\right),\ \beta_{nor}(t,u) := \Xi\left(t,\tau_0^{nor}(u),\tau_1^{nor}(u)\right).$$

We then have, for a certain constant C, the following

$$|\beta^*_{nor}(t,u^*_{nor})|,\ |\beta^*_{nor}(t,u_{nor})|\ |\beta_{nor}(t,u_{nor})| \le C|u_{nor}|^{\frac{s-2}{2s}}(\to 0,\ \text{as } u_{nor}\to 0^+).$$

Notice that

$$|\frac{\partial\Xi}{\partial X}(t,X,Y)|,\ |\frac{\partial\Xi}{\partial Y}(t,X,Y)|$$

are uniformly bounded when $|X|,|Y|<1/2$. Together with (4.47), we thus obtain the following estimate:

$$\begin{aligned}
&\beta^*_{nor}(t,u^*_{nor}) = (\beta^*_{nor}(t,u^*_{nor})-\beta_{nor}(t,u_{nor}))+\beta_{nor}(t,u_{nor})\\
&=\beta_{nor}(t,u_{nor})+\int_0^1\Big(\frac{\partial\Xi}{\partial X}\big(t,\zeta\tau_0^{*nor}(u^*_{nor})\\
&+(1-\zeta)\tau_0^{nor}(u_{nor}),\zeta\tau_1^{*nor}(u^*_{nor})+(1-\zeta)\tau_1^{nor}(u_{nor})\big)\big(\tau_0^{*nor}(u^*_{nor})-\tau_0^{nor}(u_{nor})\big)\Big)d\zeta\\
&+\int_0^1\Big(\frac{\partial\Xi}{\partial Y}\big(t,\tau_0^{*nor}(u^*_{nor})\zeta+(1-\zeta)\tau_0^{nor}(u_{nor}),\zeta\tau_1^{*nor}(u^*_{nor})\\
&+(1-\zeta)\tau_1^{nor}(u_{nor})\big)\big(\tau_1^{*nor}(u^*_{nor})-\tau_1^{nor}(u_{nor})\big)\Big)d\zeta\\
&=\beta_{nor}(t,u_{nor})+O(u^{N-2})
\end{aligned} \tag{4.49}$$

Let $\chi(\tau,\tau_0(u)) = \frac{\tau-\tau_0(u)}{1-\overline{\tau_0(u)}\tau}$. Then $\chi^{-1}(\tau,\tau_0(u)) = \frac{\tau+\tau_0(u)}{1+\overline{\tau_0(u)}\tau}$. Write

$$\sigma_{D^{nor}(u_{nor})}(z) := \chi\left((\sigma_{nor})^{-1}(\frac{z}{\sqrt{u_{nor}}},\sqrt{u_{nor}}),\tau_0^{nor}(u_{nor})\right),$$

which is a conformal map from $D_{nor}(u)$ to Δ, mapping $A_0^{nor}(u_{nor})$ to the origin. By the definition,

$$\begin{aligned}
P_{nor}(t,u_{nor}) &= (\sigma_{D^{nor}(u_{nor})})^{-1}\left(t\sigma_{D^{nor}(u_{nor})}(A_1^{nor}(u_{nor}))\right)\\
&=\sqrt{u_{nor}}\sigma_{nor}\left(\beta_{nor}(t,u_{nor}),\sqrt{u_{nor}}\right).
\end{aligned} \tag{4.50}$$

Similarly, we have

$$P^*_{nor}(t,u^*_{nor}) = \sqrt{u^*_{nor}}\sigma^*_{nor}(\beta^*_{nor}(t,u^*_{nor}),\sqrt{u^*_{nor}}).$$

Applying Lemma 4.9 and (4.49), arguing as before, we arrive at the following estimate:

$$\begin{aligned}
&|P_{nor}(t,u_{nor}) - P^*_{nor}(t,u^*_{nor})| \\
&\le \sqrt{u_{nor}}|\sigma_{nor}(\beta_{nor}(t,u_{nor}),\sqrt{u_{nor}}) - \sigma^*_{nor}(\beta^*_{nor}(t,u^*_{nor}),\sqrt{u^*_{nor}})| + |O(u_{nor}^{N-1})| \\
&\le |O(u_{nor}^{N-2}) + \sigma_{nor}(\beta_{nor}(t,u_{nor}),\sqrt{u_{nor}}) - \sigma^*_{nor}(\beta_{nor}(t,u_{nor}),\sqrt{u^*_{nor}})| \\
&\le |O(u_{nor}^{N-2}) + \sigma_{nor}(\beta_{nor}(t,u_{nor}),\sqrt{u_{nor}}) - \sigma^*_{nor}(\beta_{nor}(t,u_{nor}),\sqrt{u_{nor}})| \\
&\le C u_{nor}^{N-2},
\end{aligned} \tag{4.51}$$

for a certain constant C independent of t and for $0 < u_{nor} \ll 1$.

By (4.46), we have

$$F_{(\widetilde{N}+1)} \circ \Phi_1^{-1}(z_{nor}, w_{nor}) = \Phi_2^{-1}\big((z_{nor}, w_{nor}) + O(|(z_{nor}, w_{nor})|^N)\big) \tag{4.52}$$

Letting $(z_{nor}, w_{nor}) = (P_{nor}(t,u_{nor}), u_{nor})$ in (4.52) and making use of (4.51), we have

$$F_{(\widetilde{N}+1)} \circ \Phi_1^{-1}(P_{nor}(t,u_{nor}), u_{nor}) = \Phi_2^{-1}\big(P^*_{nor}(t,u^*_{nor}), u^*_{nor}\big) + O(|(P_{nor}(t,u_{nor}), u_{nor})|^N + u_{nor}^{N-2}).$$

Since we clearly have $|P_{nor}(t,u_{nor})| \lesssim (u_{nor})^{\frac{s-1}{s}}$ (see (4.56), for instance) and since $P(t,u) = \Phi_1^{-1}(P_{nor}(t,u_{nor}), u_{nor})$, $P^*(t,u^*) = \Phi_2^{-1}(P^*_{nor}(t,u^*_{nor}), u^*_{nor})$, we get

$$\widetilde{f}_{(\widetilde{N}+1)}(P(t,u),u) = P^*(t,u) + O(u^{\frac{s-1}{s}N}) + O(u^{N-2}) = f^*(P(t,u),u) + O(u^{\frac{2}{3}N-2}),$$

uniformly on t. This completes the proof of Lemma 4.15. ∎

We next claim that for a certain constant $C \gg 1$, it holds that

$$\text{if } \ z \in \partial S(u), \text{ then } \ C^{-1}u^{\frac{s-1}{s}} \le |z| \le C u^{\frac{s-1}{s}}, \text{ and thus } |\frac{1}{z}| \lesssim u^{-1} \text{ for } 0 < u \ll 1. \tag{4.53}$$

Assume the claim for the moment.

First, we mention that by the observation in Remark 4.7, one can easily see that $0 \in S(u)$. (Indeed, this is equivalent to the fact that the origin is inside the hyperbolic polygon $\widetilde{S}(u)$ with vertices $\tau_0(u), \cdots, \tau_{s-1}(u)$ in Δ. To see this, using the asymptotic expansion for $\tau_j(u)$ in (4.22) and using the geodesic segment formula in (4.44), one concludes easily that the boundary of $\widetilde{S}(u)$ can be deformed, in $\Delta\setminus\{0\}$, to the circle centered at the origin with radius $s\cdot(s-1)^{\frac{1-s}{s}}u^{\frac{s-1}{2s}}$. Hence, 0 is an interior point of the hyperbolic polygon $\widetilde{S}(u)$.)

Now, by the Cauchy formula, it holds that

$$\frac{\partial^\alpha f^*}{\partial z^\alpha}(0,u) = \frac{\alpha!}{2\pi\sqrt{-1}} \oint_{\partial S(u)} \frac{f^*(\zeta,u)}{\zeta^{\alpha+1}} d\zeta$$

and

$$\frac{\partial^\alpha \widetilde{f}_{(\widetilde{N}+1)}}{\partial z^\alpha}(0,u) = \frac{\alpha!}{2\pi\sqrt{-1}} \oint_{\partial S(u)} \frac{\widetilde{f}_{(\widetilde{N}+1)}}{\zeta^{\alpha+1}} d\zeta.$$

Hence, it follows that

$$\Big|\frac{\partial^\alpha f^*}{\partial z^\alpha}(0,u) - \frac{\partial^\alpha \widetilde{f}_{(\widetilde{N}+1)}}{\partial z^\alpha}(0,u)\Big| \le Cu^{\frac{2}{3}N-\alpha-3}. \tag{4.54}$$

Here, we used the obvious fact that the Euclidean length of $\partial S(u)$ is bounded by a constant. Hence, to complete the proof of Lemma 4.14, we need only to explain (4.53). Assume that z is on the hyperbolic geodesic segment in $D(u)$ connecting $A_j(u)$ to $A_{j+1}(u)$ for a certain $j \in [0, s-1]$. (Here, we write $A_s(u) = A_0(u)$ and $\tau_s(u) = \tau_0(u)$.)

Then, as in (4.50), it holds that

$$z = z(u,t) = \sqrt{u}\sigma\left(\Xi(t, \tau_j(u), \tau_{j+1}(u)), \sqrt{u}\right) = \sqrt{u}\sigma\left(\frac{\frac{\tau_{j+1}(u)-\tau_j(u)}{1-\overline{\tau_j(u)}\tau_{j+1}(u)}t + \tau_j(u)}{1+\overline{\tau_j(u)}\frac{\tau_{j+1}(u)-\tau_j(u)}{1-\overline{\tau_j(u)}\tau_{j+1}(u)}t}, \sqrt{u}\right) \tag{4.55}$$

for a certain $t \in [0,1]$. By (4.21), (4.22), we get

$$|z(u,t)| = s\cdot(s-1)^{\frac{1-s}{s}}|1+t(e^{\frac{2\pi\sqrt{-1}}{s}}-1)|u^{\frac{s-1}{s}} + o(u^{\frac{s-1}{s}}).$$

Since

$$\min_{0\le t\le 1}|1+t(e^{\frac{2\pi\sqrt{-1}}{s}}-1)| \ge \sqrt{\frac{1}{2}\left(1+\cos(\frac{2\pi}{s})\right)} > 0,$$

we get that

$$8u^{\frac{s-1}{s}}s\cdot(s-1)^{\frac{1-s}{s}} \ge |z(u,t)| \ge \sqrt{\frac{1}{4}\left(1+\cos(\frac{2\pi}{s})\right)}u^{\frac{s-1}{s}}s\cdot(s-1)^{\frac{1-s}{s}} \tag{4.56}$$

for $0 < u \ll 1$. This completes the proof of the claim and thus also the proof of Lemma 4.14. ∎

We continue our proof of Theorem 4.11 as follows. We notice that
(i) : $\sigma^*(\zeta, \sqrt{u})$ has a convergent power series expansion in $(\zeta, \sqrt{u})$ near $(0,0)$,
(ii) : $\Psi(\tau, \sqrt{u})$ has a convergent power series expansion in τ and $u^{\frac{1}{2s}}$ and,
(iii) : $\sigma^{-1}(\frac{z}{\sqrt{u}}, \sqrt{u})$ has a convergent power series expansion in $(\frac{z}{\sqrt{u}}, \sqrt{u})$, too.
Write

$$\Psi(\tau, \sqrt{u}) = \sum_{\alpha,\beta=0}^{\infty} a_{\alpha\beta}\tau^\alpha u^{\frac{\beta}{2s}} \text{ and } \widetilde{\Psi}(\tau, Y_1) = \sum_{\alpha,\beta=0}^{\infty} a_{\alpha\beta}\tau^\alpha Y_1^\beta.$$

Then

$$H(X, Y_1, Y_2) = Y_2\sigma^*(\widetilde{\Psi}(\sigma^{-1}(X, Y_2), Y_1), Y_2) \tag{4.57}$$

is analytic in X, Y_1, Y_2 near 0. Write

$$H(X, Y_1, Y_2) = \sum_{\alpha,\beta,\gamma=0}^{\infty} b_{\alpha\beta\gamma}X^\alpha Y_1^\beta Y_2^\gamma. \tag{4.58}$$

Then there is an ϵ_0 with $0 < \epsilon_0 \ll 1$ such that when $|X|, |Y_1|, |Y_2| < \epsilon_0$, (4.58) and the following power series in (4.59) converge uniformly for $|X|, |Y_1|, |Y_2| < \epsilon_0$:

$$\frac{\partial^\alpha H}{\partial X^\alpha}(0, Y_1, Y_2) = \sum_{\beta,\gamma=0}^{\infty} b_{\alpha\beta\gamma}\alpha! Y_1^\beta Y_2^\gamma. \tag{4.59}$$

Hence, we have

$$|b_{\alpha\beta\gamma}| \le C_0 \cdot R^{\alpha+\beta+\gamma} \text{ for a certain positive number } C_0 \text{ and a certain } R \gg 1. \tag{4.60}$$

Next, for the above ϵ_0, choose (z, u) such that $|\frac{z}{\sqrt{u}}| < \epsilon_0$ and $0 < u^{1/(2s)} < \epsilon_0$, we get from (4.42), (4.57), (4.58) the following:

$$\begin{aligned} &f^*(z,u) = H(\frac{z}{\sqrt{u}}, u^{\frac{1}{2s}}, \sqrt{u}) = \sum_{\alpha,\beta,\gamma=0}^{\infty} b_{\alpha\beta\gamma} z^\alpha u^{\frac{\gamma-\alpha}{2}+\frac{\beta}{2s}} \quad \text{and from } (4.59), \quad \text{we get} \\ &u^{\frac{\alpha}{2}} \frac{\partial^\alpha f^*}{\partial z^\alpha}(0,u) = \frac{\partial^\alpha H}{\partial X^\alpha}(0, u^{\frac{1}{2s}}, \sqrt{u}) = \sum_{\beta,\gamma=0}^{\infty} b_{\alpha\beta\gamma}\alpha! u^{\frac{\gamma}{2}+\frac{\beta}{2s}}. \end{aligned} \tag{4.61}$$

Here, making use of the Cauchy estimates for $b_{\alpha\beta\gamma}$ (4.60), the second series in (4.61) can be easily shown to be uniformly convergent (in its variable u) over $[0, b]$ for $b \ll 1$. (Indeed, let R be as in (4.60). We can then simply take $b = (\frac{1}{2R})^{2s}$.) We thus see that for any $m > 1$

$$\sum_{\frac{\beta}{2s}+\frac{\gamma}{2}\ge m}^{\infty} b_{\alpha\beta\gamma}\alpha! u^{\frac{\gamma}{2}+\frac{\beta}{2s}} = O(u^m) \quad \text{as } u \to 0^+.$$

On the other hand, for each fixed $\alpha \ge 0$, $m \gg 1$, $\frac{\partial^\alpha \widetilde{f}}{\partial z^\alpha}(0,u)$ also has a formal power series expansion in u and thus in $u^{1/(2s)}$. By Lemma 4.14, for each fixed integer $\alpha \ge 0$ and $\widetilde{N} = sN + s - 1$, we have

$$\left|\frac{\partial^\alpha \widetilde{f}_{(\widetilde{N}+1)}}{\partial z^\alpha}(0,u) - \sum_{0 \le \frac{\beta}{2s}+\frac{\gamma}{2} \le m} b_{\alpha\beta\gamma}\alpha! u^{\frac{\gamma-\alpha}{2}+\frac{\beta}{2s}}\right| \le C(u^{N'} + u^{m-\frac{\alpha}{2}}), \ 0 < u \ll 1$$

where C, N' are independent of u and $N' \to \infty$ as $N \to \infty$. We thus have, for each fixed $\alpha \ge 0$, that

$$\frac{\partial^\alpha \widetilde{f}}{\partial z^\alpha}(0,u) = \sum_{\beta,\gamma=0}^{\infty} b_{\alpha\beta\gamma}\alpha! u^{\frac{\gamma-\alpha}{2}+\frac{\beta}{2s}} \tag{4.62}$$

in the formal sense as formal Laurent series in $u^{\frac{1}{2s}}$ with only finitely many negative power terms. (See Remark 4.5(B) for the definition.) It thus follows that if $\beta' = \frac{\gamma-\alpha}{2} + \frac{\beta}{2s}$ is not a non-negative integer, then the finite sum $\sum_{\beta,\gamma;\frac{\gamma-\alpha}{2}+\frac{\beta}{2s}=\beta'} b_{\alpha\beta\gamma} = 0$. Hence for $|\frac{z}{\sqrt{u}}| < \epsilon_0$ and $0 < u^{1/(2s)} < \epsilon_0$, we have

$$f^*(z,u) = \sum_{\alpha,\beta,\gamma=0}^{\infty} b_{\alpha\beta\gamma} z^\alpha u^{\beta'} = \sum_{\alpha,\beta'=0}^{\infty} b'_{\alpha\beta'} z^\alpha u^{\beta'}, \tag{4.63}$$

where $\beta' = \frac{\gamma-\alpha}{2}+\frac{\beta}{2s} \in \{0,1,2,\cdots\}$ and $b'_{\alpha\beta'} = \sum_{\beta,\gamma;\beta'=\frac{\gamma-\alpha}{2}+\frac{\beta}{2s}} b_{\alpha\beta\gamma}$. Now, by (4.60), we conclude that, for each fixed α and for any β,γ with $\beta' = \frac{\gamma-\alpha}{2}+\frac{\beta}{2s}$, it holds that $|b_{\alpha\beta\gamma}| \le C_0 \cdot R^{2s\alpha+2s\beta'} \le C_0 \cdot (R^{2s})^{\alpha+\beta'}$. Thus, $|b'_{\alpha\beta'}| \le C_0 \cdot (2s(\alpha+\beta')+1)\, R^{2s\alpha+2s\beta'} \le C_0(1+R)^{4s(\alpha+\beta')}$. Since $f^*(z,u)$ is real analytic over $\widetilde{M}$, we conclude that $f^*(z,u)$ extends to a analytic function in (z,u) near 0 through the power series in the right hand side of (4.63). Since (4.62) holds for each $\alpha \ge 0$, we see that $\widetilde{f}(z,u) = f^*(z,u)$ in the formal sense. Hence, $\widetilde{f}(z,u)$ is also given by a convergent power series. The proof of Theorem 4.11 is finally complete. ∎

Proofs of Theorem 1.5 and Theorem 1.2: Theorem 1.5 and Theorem 4.11 have the same content. Theorem 1.2 follows from Theorem 1.1 and Theorem 1.5. ∎

We finish off the paper by presenting two more Corollaries:

Corollary 4.16. *Let $(M,0)$ be a real analytic elliptic Bishop surface with the Bishop invariant vanishing and the Moser invariant $s < \infty$ at 0. Then any element in $aut_0(M)$ is a holomorphic automorphism of $(M,0)$.*

Corollary 4.17. *Let M be defined by a real analytic function of the following form:*

$$w = z\overline{z} + z^s + \overline{z^s} + \sum_{k,l\ge 0; k+l>s; k-l=0 \ mod\ s} a_{kl} z^k \overline{z^l}.$$

Then M is biholomorphically equivalent to its normal form

$$w = z\overline{z} + z^s + \overline{z^s}.$$

Corollary 4.16 is an immediate consequence of Theorem 4.11. Corollary 4.17 is a consequence of Corollary 1.4 (d) and (e); for $(z,w) \to (e^{i\theta}z, w)$ is an automorphism of $(M,0)$ whenever $e^{is\theta} = 1$. Notice that Corollary 1.4(e) is an application of Theorem 1.1 and the convergence result in Theorem 4.11.

Example 4.18. Let M be defined by $w = z\overline{z} + z^3 + \overline{z^3} + z^6 + \overline{z^6}$, which is in the Moser pseudo-normal form. Then by Corollary 4.17 and Theorem 1.1, M can be transformed to the model surface defined by $w = z\overline{z} + z^3 + \overline{z^3}$ through a unique transformation of the form $F = (z,w) + O(|(z,w)|^2)$. By Theorem 4.11, F is convergent. However, if just working on the formal power series without using the hyperbolic geometry from the attached holomorphic disks, we do not see how to achieve a convergence proof for F. Also, without using the characterization of the model by its automorphism group, it does not seem to be easy to see that the normal form of M is $w = z\overline{z} + z^3 + \overline{z^3}$.

References

[AG] P. Ahern and X. Gong, Real analytic submanifolds in $\mathbb{C}^n$ with parabolic complex tangents along a submanifold of codimension one, preprint, 2006.

[BER] S. Baouendi, P. Ebenfelt and L. Rothschild, Local geometric properties of real submanifolds in complex space, *Bull. Amer. Math. Soc.* (N.S.) 37 (2000), no. 3, 309–33.

[BMR] S. Baouendi, N. Mir and L. Rothschild, Reflection ideals and mappings between generic submanifolds in complex space, *J. Geom. Anal. 12* (2002), 543-580.

[BG] E. Bedford and B. Gaveau, Envelopes of holomorphy of certain 2-spheres in $\mathbb{C}^2$, *Amer. J. Math.* (105), 975-1009, 1983.

[Bis] E. Bishop, Differentiable manifolds in complex Euclidean space, *Duke Math. J.* (32), 1-21, 1965.

[Car] É. Cartan, Sur les variétés pseudo-conformal des hypersurfaces de l'espace de deux variables complexes, *Ann. Mat. Pura Appl. (4) 11*, 17-90(1932).

[CM] S. S. Chern and J. K. Moser, Real hypersurfaces in complex manifolds, *Acta Math. 133*, 219-271(1974).

[Eli] Y. Eliashberg, Filling by holomorphic discs and its applications, Geometry of Low-Dimensional Manifolds, *London Math. Soc. Lecture Notes Vol. 151*, 1997.

[For] F. Forstneric, Most real analytic Cauchy-Riemann manifolds are nonalgebraizable, *Manuscripta Math. 115* (2004), 489–494.

[Gon1] X. Gong, On the convergence of normalizations of real analytic surfaces near hyperbolic complex tangents, *Comment. Math. Helv.* 69 (1994), no. 4, 549–574.

[Gon2] X. Gong, Normal forms of real surfaces under unimodular transformations near elliptic complex tangents, *Duke Math. J. 74* (1994), no. 1, 145–157.

[Gon3] X. Gong, Existence of real analytic surfaces with hyperbolic complex tangent that are formally but not holomorphically equivalent to quadrics, *Indiana Univ. Math. J. 53* (2004), no. 1, 83–95.

[Grov] M. Gromov, Pseudo holomorphic curves in symplectic geometry, *Invent Math. Vol. 82*, 1985, 307-347.

[Hu1] X. Huang, Local Equivalence Problems for Real Submanifolds in Complex Spaces, *Lecture Notes in Mathematics 1848 (C.I.M.E. series)*, Springer-Verlag, pp 109-161, Berlin-Heidelberg-New York, 2004.

[Hu2] X. Huang, On some problems in several complex variables and Cauchy-Riemann Geometry, Proceedings of ICCM (edited by L. Yang and S. T. Yau), *AMS/IP Stud. Adv. Math. 20*, 383-396, 2001.

[Hu3] X. Huang, On an n-manifold in $\mathbb{C}^n$ near an elliptic complex tangent, *J. Amer. Math. Soc.* (11), 669–692, 1998.

[HK] X. Huang and S. Krantz, On a problem of Moser, *Duke Math. J.* (78), 213-228, 1995.

[Ji] S. Ji, Algebraicity of real analytic hypersurfaces with maxium rank, *Amer. Jour. Math. Vol 124*, 255-264, 2002.

[KW1] C. Kenig and S. Webster, The local hull of holomorphy of a surface in the space of two complex variables, *Invent. Math. 67*, 1-21, 1982.

[KW2] C. Kenig and S. Webster, On the hull of holomorphy of an n-manifold in $\mathbb{C}^n$, *Annali Scoula Norm. Sup. de Pisa IV Vol. 11 (No. 2)*, 261-280, 1984

[MMZ] F. Meylan, N. Mir and D. Zaitsev, Approximation and convergence of formal CR-mappings, *International Mathematics Research Notices* 2003, no. 4, 211-242.

[Mir] N. Mir, Convergence of formal embeddings between real-analytic hypersurfaces in codimension one, *J. Differential Geom.* 62 (2002), 163–173.

[Mos] J. Moser, Analytic surfaces in $\mathbb{C}^2$ and their local hull of holomorphy, *Annales AcademiæFennicae Series A.I. Mathematica* (10), 397-410, 1985.

[MW] J. Moser and S. Webster, Normal forms for real surfaces in $\mathbb{C}^2$ near complex tangents and hyperbolic surface transformations, *Acta Math.* (150), 255-296, 1983.

[Po] H. Poincaré, Les fonctions analytiques de deux variables et la représentation conforme, *Ren. Cire. Mat. Palermo, II. Ser. 23*, 185-220, 1907.

[Sto] L. Stolovitch, Family of intersecting totally real manifolds of $(\mathbb{C}^n, 0)$ and CR-singularities, preprint, 2006.

[We] S. Webster, Pairs of Intersecting Real Manifolds in Complex Space, *Asian Jour. Math* Vol. 7 (No. 4), 449-462, 2003.

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Appendix

Lemma 2.4 is completely parallel to Lemma 2.3. Its proof is also an identical copy of the proof for Lemma 2.3 presented in §2 except a few obvious and totally trivial changes to fit into the situation in Lemma 2.4 that $\mathrm{Ord}(f) = 2t+1$ not $2t$. Thus we did not repeat the detail for its proof in the paper. Nevertheless, we use this opportunity to include a detailed proof in this separate appendix by copying exactly the same argument that we gave in the proof of Lemma 2.3 in Section 2 of the paper, which may help the reader to speed up a little the reading.

Proof of Lemma 2.4: When $m = ns + s$, $N'_s \leq m$. Thus, (2.62) holds trivially due to the presence of the term $\Theta^m_{N'_s}$. Hence, in the proof of the lemma, we always assume that $m \leq ns + s - 1$.

Notice that the lemma holds with $m = 2t+3$ in terms of (2.61) . We complete the proof of the lemma in three steps.

Step I of the proof of Lemma 2.4: This step is not needed when $s = 3$. Denote $m_0 = 2t + j(s-2) + 3$, where j is an integer with $0 \leq j \leq t$. Suppose that $m_0 \leq N_0$. We also assume that there is an integer m such that $m \geq m_0$, $m + 1 \leq 2t + (j+1)(s-2) + 2$ (such an m certainly does not exist if $s = 3$), $m + 1 \leq N_0$ and moreover the formula (2.62) holds for this m. By copying the rest argument as that in Step I of the proof of Lemma 2.3, we can show that (2.62) also holds for any m with $m_0 \leq m \leq 2t + (j+1)(s-2) + 2$ and $m \leq N_0$.

Step II of the proof of Lemma 2.4: In this step, suppose that we know that the lemma holds for $m \in [2t + j(s-2) + 3, \leq 2t + (j+1)(s-2) + 2]$ with $m \leq N_0$, where j is a certain non-negative integer bounded by $t-1$. We then proceed to prove that the lemma holds also for $m \in [2t + (j+1)(s-2) + 3, 2t + (j+2)(s-2) + 2]$, whenever $m \leq N_0$.

Suppose that $2t + (j+1)(s-2) + 2 < N_0$. By the assumption, we have

$$\begin{aligned} g_{2t+(j+1)(s-2)+2}(w) &= A(1-s)^j z^{(j+1)s}(z\bar{z} + z^s)^{t-j} + (\bar{z} + sz^{s-1} + \Theta^2_s) f_{2t+(j+1)(s-2)+1}(z,w) \\ &+ (z + s\bar{z}^{s-1} + \Theta^2_s)\overline{f_{2t+(j+1)(s-2)+1}(z,w)} + 2Re\left((b_{N_0} - a_{N_0})z^{N_0}\right) + \Theta^{2t+(j+1)(s-2)+2}_{N'_s}. \end{aligned} \tag{0.1}$$

Collecting terms of degree $2t + (j+1)(s-2) + 2$ in (0.1), we get

$$\begin{aligned} g^{(2t+(j+1)(s-2)+2)}_{nor}(z\bar{z}) &= A(1-s)^j z^{(j+1)s}(z\bar{z})^{t-j} + \hat{\mathbb{P}}^{2t+(j+1)(s-2)+2}_{N'_s} \\ &+ \bar{z} f^{(2t+(j+1)(s-2)+1)}_{nor}(z, z\bar{z}) + z\overline{f^{(2t+(j+1)(s-2)+1)}_{nor}(z, z\bar{z})}. \end{aligned} \tag{0.2}$$

Here we denote by $\hat{\mathbb{P}}^{2t+(j+1)(s-2)+2}_{N'_s}$ a certain homogeneous polynomial of degree $2t + (j+1)(s-2) + 2$ with weight at least N'_s.

Now, we solve (0.2) as follows. Denote by $\Lambda = 2t + (j+1)(s-2) + 1$. Notice that

$$I := -\hat{\mathbb{P}}^{\Lambda+1}_{N'_s} + \overline{A}(1-s)^j \bar{z}^{(j+1)s}(z\bar{z})^{t-j} + g^{(\Lambda+1)}_{nor}(z\bar{z})$$

is real valued and $I=\mathbb{P}^{\Lambda+1}_{N'_s}$. Then (0.2) can be rewritten as

$$\begin{aligned} I =&A(1-s)^j z^{(j+1)s}(z\bar{z})^{t-j}+\overline{A}(1-s)^j \bar{z}^{(j+1)s}(z\bar{z})^{t-j}\\ &+\bar{z}f^{(2t+(j+1)(s-2)+1)}_{nor}(z,z\bar{z})+z\overline{f^{(2t+(j+1)(s-2)+1)}_{nor}}(z,z\bar{z}). \end{aligned} \tag{0.3}$$

Write

$$I=\sum_{\substack{\alpha+\beta=\Lambda+1\\ \alpha+(s-1)\beta\geq N'_s}} a_{\alpha\bar{\beta}}z^{\alpha}\bar{z}^{\beta}.$$

Since $a_{\alpha\bar{\beta}}=\overline{a_{\alpha\bar{\beta}}}$, we also require that $\alpha+(s-1)\beta\geq N'_s$. We next have the following general solution of (0.1):

$$\begin{aligned} &f^{(2t+(j+1)(s-2)+1)}_{nor}(z,w)=f^{(\Lambda)}_1(z,w)+f^{(\Lambda)}_2(z,w) \quad \text{with}\\ &f^{(\Lambda)}_1(z,w)=-A(1-s)^j z^{(j+1)s+1}w^{t-j-1},\\ &f^{(\Lambda)}_2(z,w)=\textstyle\sum_{\widetilde{\alpha}+2\widetilde{\beta}=\Lambda} h_{\widetilde{\alpha}\widetilde{\beta}}z^{\widetilde{\alpha}}w^{\widetilde{\beta}}, \end{aligned} \tag{0.4}$$

where $h_{\widetilde{\alpha}\widetilde{\beta}}$ are determined by the following:

$$\sum_{\widetilde{\alpha}+2\widetilde{\beta}=\Lambda} h_{\widetilde{\alpha}\widetilde{\beta}}z^{\widetilde{\alpha}+\widetilde{\beta}}\bar{z}^{\widetilde{\beta}+1}+\sum_{\widetilde{\alpha}+2\widetilde{\beta}=\Lambda}\overline{h_{\widetilde{\alpha}\widetilde{\beta}}}z^{\widetilde{\beta}+1}\bar{z}^{\widetilde{\alpha}+\widetilde{\beta}}=\sum_{\substack{\alpha+\beta=\Lambda+1\\ \alpha+\beta(s-1)\geq N'_s}} a_{\alpha\bar{\beta}}z^{\alpha}\bar{z}^{\beta}. \tag{0.5}$$

Now, (0.5) can be handled exactly in the same way as for (2.26). (The only difference is that the role of $m-1$ is now played by Λ.) For convenience of the reader, we repeat some details as follows:

First, we have similar relations as those in (2.27)(2.28). Next, we can conclude the following:

$$wt(f^{(\Lambda)}_2(z,z\bar{z}))\geq\min_{\widetilde{\alpha}\geq 0}\{\widetilde{\alpha}+\widetilde{\beta}+(s-1)\widetilde{\beta}\}=\min_{\alpha\geq 0}\{\alpha+(s-1)\beta-s+1\}\geq N'_s-s+1, \tag{0.6}$$

$$wt\{\frac{\partial f^{(\Lambda)}_2}{\partial w}(z,z\bar{z})\}\geq\min_{\widetilde{\alpha}\geq 0}\{\widetilde{\alpha}+\widetilde{\beta}+(s-1)\widetilde{\beta}-s\}=\min_{\alpha\geq 0}\{\alpha+(s-1)\beta-s+1-s\}\geq N'_s-2s+1, \tag{0.7}$$

$$wt\{f^{(\Lambda)}_2(z,w)-f^{(\Lambda)}_2(z,z\bar{z})\}\geq N'_s-s+1, \tag{0.8}$$

$$wt\{\overline{f^{(\Lambda)}_2(z,w)-f^{(\Lambda)}_2(z,z\bar{z})}\}\geq N'_s-1, \tag{0.9}$$

$$(sz^{s-1}+\Theta^2_s)f^{(\Lambda)}_2(z,z\bar{z})+(s\bar{z}^{s-1}+\Theta^2_s)\overline{f^{(\Lambda)}_2(z,z\bar{z})}=\Theta^{\Lambda+2}_{N'_s}, \tag{0.10}$$

$$(sz^{s-1}+\Theta^2_s)f^{(\Lambda)}_2(z,w)+(s\bar{z}^{s-1}+\Theta^2_s)\overline{f^{(\Lambda)}_2(z,w)}=\Theta^{\Lambda+2}_{N'_s}, \tag{0.11}$$

$$wt\{f^{(\Lambda)}_1(z,z\bar{z})\}\geq st+1,\quad wt\{\overline{f^{(\Lambda)}_1(z,w)-f^{(\Lambda)}_1(z,z\bar{z})}\}\geq N'_s. \tag{0.12}$$

For instance, to see (0.11), it suffices to notice that by (0.6), we have

$$wt\{(sz^{s-1}+\Theta_s^2)f_2^{(\Lambda)}(z,w)+(s\bar{z}^{s-1}+\Theta_s^2)\overline{f_2^{(\Lambda)}(z,w)}\} \geq s-1+N_s'-s+1=N_s'. \tag{0.13}$$

Hence, from (0.1)-(0.12), we get

$$\begin{aligned} g_{\Lambda+2}(w)+g_{nor}^{(\Lambda+1)}(w) =&(\bar{z}+sz^{s-1}+\Theta_s^2)f_{\Lambda+1}(z,w)+(z+s\bar{z}^{s-1}+\Theta_s^2)\overline{f_{\Lambda+1}(z,w)} \\ &+\Theta_{N_s'}^{\Lambda+2}+\hat{\mathbb{P}}_{N_s'}^{\Lambda+1}+A(1-s)^j z^{(j+1)s}(z\bar{z}+z^s)^{t-j} \\ &+(\bar{z}+sz^{s-1}+\Theta_s^2)f_{nor}^{(\Lambda)}(z,w)+(z+s\bar{z}^{s-1}+\Theta_s^2)\overline{f_{nor}^{(\Lambda)}(z,w)} \\ &+2Re\left((b_{N_0}-a_{N_0})z^{N_0}\right). \end{aligned} \tag{0.14}$$

Notice that

$$g_{nor}^{(\Lambda+1)}(z\bar{z})=A(1-s)^j z^{(j+1)s}(z\bar{z})^{t-j}+\bar{z}f_{nor}^{(\Lambda)}(z,z\bar{z})+z\overline{f_{nor}^{(\Lambda)}(z,z\bar{z})}+\hat{\mathbb{P}}_{N_s'}^{\Lambda+1}. \tag{0.15}$$

Also,

$$wt\{g_{nor}^{(\Lambda+1)}(w)\}, wt\{g_{nor}^{(\Lambda+1)}(z\bar{z})\} \geq \frac{s(\Lambda+1)}{2}=ts+\frac{1}{2}s(j+1)(s-2)+s \geq ts+s+1.$$

Hence

$$g_{nor}^{(\Lambda+1)}(w)-g_{nor}^{(\Lambda+1)}(z\bar{z}) \in \Theta_{N_s'}^{\Lambda+2}. \tag{0.16}$$

Subtracting (0.15) from (0.14) and then making use of (0.16), we obtain

$$\begin{aligned} g_{\Lambda+2}(w) =&(\bar{z}+sz^{s-1}+\Theta_s^2)f_{\Lambda+1}(z,w)+2Re\left((b_{N_0}-a_{N_0})z^{N_0}\right) \\ &+(z+s\bar{z}^{s-1}+\Theta_s^2)\overline{f_{\Lambda+1}(z,w)}+\Theta_{N_s'}^{\Lambda+2}+J, \end{aligned} \tag{0.17}$$

where

$$\begin{aligned} J =&(\bar{z}+sz^{s-1}+\Theta_s^2)f_{nor}^{(\Lambda)}(z,w)+(z+s\bar{z}^{s-1}+\Theta_s^2)\overline{f_{nor}^{(\Lambda)}(z,w)} \\ &+A(1-s)^j z^{(j+1)s}(z\bar{z}+z^s)^{t-j}-A(1-s)^j z^{(j+1)s}(z\bar{z})^{t-j} \\ &-(\bar{z}f_{nor}^{(\Lambda)}(z,z\bar{z})+z\overline{f_{nor}^{(\Lambda)}(z,z\bar{z})}). \end{aligned} \tag{0.18}$$

Here we notice, by (0.10)-(0.12), that

$$\begin{aligned} &\bar{z}f_{nor}^{(\Lambda)}(z,w)+z\overline{f_{nor}^{(\Lambda)}(z,w)}-(\bar{z}f_{nor}^{(\Lambda)}(z,z\bar{z})+z\overline{f_{nor}^{(\Lambda)}(z,z\bar{z})}) \\ &\quad +A(1-s)^j z^{(j+1)s}(z\bar{z}+z^s)^{t-j}-A(1-s)^j z^{(j+1)s}(z\bar{z})^{t-j} \\ &=-A(1-s)^j\cdot z\bar{z}\cdot z^{(j+1)s}(z\bar{z}+z^s)^{t-j-1}+A(1-s)^j z^{(j+1)s}(z\bar{z})^{t-j} \\ &\quad +A(1-s)^j z^{(j+1)s}(z\bar{z}+z^s)^{t-j}-A(1-s)^j z^{(j+1)s}(z\bar{z})^{t-j}+\Theta_{N_s'}^{\Lambda+2} \\ &=-A(1-s)^j z^{(j+1)s}(z\bar{z}+z^s)^{t-j-1}\left(z\bar{z}-(z\bar{z}+z^s)\right)+\Theta_{N_s'}^{\Lambda+2} \\ &=A(1-s)^j z^{(j+2)s}(z\bar{z}+z^s)^{t-j-1}+\Theta_{N_s'}^{\Lambda+2}. \end{aligned}$$

Hence by the formular in (0.4) for $f_1^{(\Lambda)}$, we have

$$\begin{aligned} J =&(sz^{s-1} + \Theta_s^2) f_1^{(\Lambda)}(z,w) + (s\bar{z}^{s-1} + \Theta_s^2)\overline{f_1^{(\Lambda)}(z,w)} \\ &+ A(1-s)^j z^{(j+2)s}(z\bar{z} + z^s)^{t-j-1} + \Theta_{N_s'}^{\Lambda+2} \\ =&A(1-s)^{j+1} z^{(j+2)s}(z\bar{z} + z^s)^{t-j-1} + \Theta_{N_s'}^{\Lambda+2}. \end{aligned} \tag{0.19}$$

This proves the lemma when $m = 2t + (j+1)(s-2) + 3$. Now, the result obtained in the previous step completes the proof of the claim in this step.

Step III of the proof of Lemma 2.4: We now can complete the proof of the lemma by inductively using results obtained in Steps I-II. Indeed, since we know that the Lemma holds for $m = 2t + 3$, we see, by Step I, that the lemma holds for any $m \le N_0$ with $m \in [2t+3, 2t+(s-2)+2]$. Then, applying first Step II and then applying Step I again, we see the lemma holds for any $m \le N_0$ with $m \in [2t + j(s-2) + 3, 2t + (j+1)(s-2) + 2]$ and $j = 1$. Now, by an induction argument on j, we see the proof of the lemma. ∎