The number of 3-SAT functions*

L. Ilinca and J. Kahn

Abstract

With $G_k(n)$ the number of functions of n boolean variables definable by k-SAT formulas, we prove that $G_3(n)$ is asymptotic to $2^{n+\binom{n}{3}}$. This is a strong form of the case $k = 3$ of a conjecture of Bollobás, Brightwell and Leader stating that for fixed k, $\log_2 G_k(n) \sim \binom{n}{k}$.

1 Introduction

Let $X_n = \{x_1, \ldots, x_n\}$ be a collection of Boolean variables. Each variable x is associated with a *positive* literal, x, and a *negative* literal $\bar{x}$. Recall that a *k-SAT formula* (in disjunctive normal form) is an expression $\mathscr{C}$ of the form

$$C_1 \vee \cdots \vee C_t, \tag{1}$$

with t a positive integer and each C_i a *k-clause*; that is, an expression $y_1 \wedge \cdots \wedge y_k$, with $y_1, \ldots, y_k$ literals corresponding to different variables. A formula (1) defines a Boolean function of $x_1, \ldots, x_n$ in the obvious way; any such function is a *k-SAT function.* Though we will be concerned here almost exclusively with the case $k = 3$, we leave the discussion general for the moment.

Following [4], we write $G_k(n)$ for the number of k-SAT functions of n variables. Of course $G_k(n)$ is at most $\exp_2[2^k\binom{n}{k}]$, the number of k-SAT formulas; on the other hand it's easy to see that

$$G_k(n) > 2^n(2^{\binom{n}{k}} - n2^{\binom{n-1}{k}}) \sim 2^{n+\binom{n}{k}} \tag{2}$$

(all formulas obtained by choosing $y_i \in \{x_i, \bar{x}_i\}$ for each i and a set of clauses using precisely the literals $y_1, \ldots, y_n$ give different functions).

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The problem of estimating $G_3(n)$ was suggested by Bollobás, Brightwell and Leader [4]. They showed

$$G_k(n) \leq \exp_2[(2\sqrt{\pi})\tbinom{n}{k}], \tag{3}$$

for $k < n/2$ and conjectured that

$$\log_2 G_k(n) < (1+o(1))\tbinom{n}{k}. \tag{4}$$

for any fixed k. Even $k = 2$ is not easy; here (4) was proved in [4], and the precise asymptotics—

$$G_2(n) \sim \exp_2[n + \tbinom{n}{2}] \tag{5}$$

—conjectured in [4] were proved in [1] and (later) in [11]. As is often the case, nothing from this earlier work seems to be of much help in treating larger k.

Here, for $k = 3$, we prove (4) and more, again showing (as in (5)) that (2) gives the asymptotics not just of $\log G_3(n)$, but of $G_3(n)$ itself:

Theorem 1.1. $G_3(n) \sim 2^{n+\binom{n}{3}}$.

For a formula $\mathscr{C}$ as in (1) we may identify the associated function, say $f_{\mathscr{C}}$, with the set (henceforth also referred to as a "k-SAT function") $F(\mathscr{C}) \subseteq \{0,1\}^n$ of satisfying assignments for $\mathscr{C}$ (that is, $F(\mathscr{C}) = f_{\mathscr{C}}^{-1}(1)$). For our purposes it will also usually be convenient to think of $\mathscr{C}$ as the set $\{C_1, \ldots, C_t\}$ of clauses. Then $F(\mathscr{C}') \subseteq F(\mathscr{C})$ whenever $\mathscr{C}' \subseteq \mathscr{C}$, and we say $\mathscr{C}$ is *irredundant* if it is a minimal formula giving $F(\mathscr{C})$; that is, if $F(\mathscr{C}') \subset F(\mathscr{C})$ for each $\mathscr{C}' \subset \mathscr{C}$. Of course each 3-SAT function F corresponds to at least one irredundant $\mathscr{C}$, so that, with $I(n) = I_3(n)$ denoting the number of irredundant formulas on X_n, Theorem 1.1 is contained in

Theorem 1.2. $I(n) \sim 2^{n+\binom{n}{3}}$.

This (together with (2)) says that in fact most F's admit only one irredundant formula. We regard this simple idea as one of the keys to the present work: it allows us to forget about functions and work directly with formulas, which are easier (though to date still not easy) to handle.

Notice that $\mathscr{C}$ is irredundant iff for each $C \in \mathscr{C}$ there is some (not necessarily unique) *witness* $\mathsf{w}_C \in \{0,1\}^n$ that satisfies C but no other clause in $\mathscr{C}$ (i.e. $\mathsf{w}_C \in F(\mathscr{C}) \setminus F(\mathscr{C} \setminus \{C\})$). Such witnesses will be central to our analysis. *For the rest of this paper, we use "formula" to mean "irredundant formula"* (but we will still sometimes retain the "irredundant" for emphasis).

We feel sure that the analogues of Theorems 1.1 and 1.2 hold for any fixed k in place of 3; that is (with $I_k(n)$ the number of irredundant k-SAT formulas of n variables), we should have

Conjecture 1.3. *For each fixed k, $G_k(n) \sim I_k(n) \sim 2^{n+\binom{n}{k}}$.*

While we do think it should be possible to prove this along the present lines, the best we can say for now is that our argument can probably be generalized to reduce Conjecture 1.3 to a finite problem for any given k; see the remarks following Corollary 6.3. For example, at this writing we are pretty sure we could do $k = 4$; but as this doesn't contribute anything very interesting beyond what's needed for $k = 3$, it seems not worth adding to the present, already very long argument.

On the other hand, if we retreat to $k = 2$ then much of the present proof evaporates—in particular hypergraph regularity becomes ordinary Szemerédi regularity—leaving perhaps the easiest verification of (5) to date. (Of course—if one cares—anything based on regularity must give far slower convergence than the argument of [11].)

From now on we will be concerned only with the case $k = 3$, and will say "clause" for "3-clause," "formula" for "(irredundant) 3-SAT formula," and so on. Let us try to say what we can about the proof at this point. The argument proceeds in two phases. The first of these—which, incidentally, gives the asymptotics of $\log I(n)$, though the proof doesn't need to say this—is based on the Hypergraph Regularity Lemma (HRL) of P. Frankl and V. Rödl [8], a pioneering extension to 3-uniform hypergraphs of the celebrated (graph) Regularity Lemma of E. Szemerédi [17]. (See e.g. [15], [9] for more on the spectacular recent developments on this topic.)

A mild adaptation of some of the material in [8] shows that each irredundant $\mathscr{C}$ is "compatible" with some "extended partition" $\mathcal{P}^*$ (defined in Section 3). On the other hand we show—this is Lemma 3.1, the upshot of this part of the argument—that the set of $\mathscr{C}$'s compatible with $\mathcal{P}^*$ is small unless $\mathcal{P}^*$ is "coherent." Since the number of $\mathcal{P}^*$'s is itself negligible relative to what we are aiming at, this allows us to restrict our attention to $\mathscr{C}$'s compatible with coherent $\mathcal{P}^*$'s.

Coherence of $\mathcal{P}^*$ turns out to imply that there is some $z \in \{0,1\}^n$ so that for *any* $\mathscr{C}$ compatible with $\mathcal{P}^*$ *every* witness for $\mathscr{C}$ mostly agrees (in the obvious sense) with z. Once we have this we are done with $\mathcal{P}^*$ and the HRL, and, in the second phase, just need to bound the number of $\mathscr{C}$'s admitting a z as above, so for example the number of $\mathscr{C}$'s for which every witness is at least 99% zeros (note we expect that a typical such $\mathscr{C}$ uses mostly *positive* literals.) While this can presumably be handled as a stand-alone statement, we instead give a recursive bound (see (20)) that includes minor terms involving earlier values of I.

The paper is organized as follows. Section 2 fills in what we need from hypergraph regularity. Once we have this we can, in Section 3, make the preceding mumble concrete and complete the proof of Theorem 1.2 assuming various supporting results. These are proved in the remaining sections: after some preliminaries in Section 4, Sections 5 and 6 implement the first part of the above sketch (proving Lemma 3.1); the easy Section 7 then produces the above-mentioned z associated with a coherent $\mathcal{P}^*$; and the final part of the argument (proving (20)) is carried out in Section 8.

Usage

Throughout the paper we use log and exp for $\log_2$ and $\exp_2$, and H for binary entropy. We use "$x = 1 \pm y$" for "$x \in (1-y,(1+y))$." With the exception of (22) (in Section 3) we always assume that n is large enough to support our assertions. Following a common abuse, we usually pretend that all large numbers are integers, and, pushing this a little, we will occasionally substitute, e.g., "at most a" for "at most $a+1$" in situations where the extra 1 is clearly irrelevant.

2 Regularity

In this section we recall what we need from [8] and slightly adapt what they do to our situation. Our notation follows theirs as much as possible.

For a bipartite graph $G = (A \cup B, E)$, $A' \subseteq A$ and $B' \subseteq B$, the *density* of the pair (A', B') is

$$d(A', B') = d_G(A', B') = |E(A', B')|/(|A'||B'|)$$

(where $E(A', B')$ is the set of edges joining A' and B'). In particular, the *density* of G is $d(A, B)$. The graph G (or the pair (A, B)) is *ε-regular* if $|d(A', B') - d(A, B)| < \varepsilon$ for all $A' \subseteq A$ and $B' \subseteq B$ with $|A'| > \varepsilon|A|$ and $|B'| > \varepsilon|B|$.

For a set V write $[V]^2$ for the collection of 2-element subsets of V. An *$(l, t, \varepsilon_1, \varepsilon_2)$-partition* $\mathcal{P}$ of $[V]^2$ consists of an auxiliary partition

$$V = V_0 \cup V_1 \cup \cdots \cup V_t \tag{6}$$

with $|V_0| < t$ and $|V_1| = \cdots = |V_t| =: m$, together with a system of edge-disjoint bipartite graphs

$$P^{ij}_{\alpha}, \quad 1 \le i < j \le t, \, 0 \le \alpha \le l_{ij} \le l, \tag{7}$$

satisfying

(a) $\cup_{\alpha=0}^{l_{ij}} P_\alpha^{ij} = K(V_i, V_j) \ := \{\{x,y\} : x \in V_i, y \in V_j\} \ \ \forall\, 1 \le i < j \le t$, and

(b) all but at most $\varepsilon_1\binom{t}{2}m^2$ pairs $\{v_i, v_j\}$, $v_i \in V_i$, $v_j \in V_j$, $1 \le i < j \le t$, are edges of ε_2-regular bipartite graphs P_α^{ij}.

A partition $\mathcal{P}$ as above is *equitable* if for all but at most $\varepsilon_1\binom{t}{2}$ pairs i,j, with $1 \le i < j \le t$, we have

$$|P_0^{ij}| < \varepsilon_1 m^2$$

and

$$|d_{P_\alpha^{ij}}(V_i, V_j) - l^{-1}| < \varepsilon_2 \quad \forall\, 1 \le \alpha \le l_{ij}. \tag{8}$$

Note this implies $(1+\varepsilon_2 l)^{-1} l < l_{ij} \ \ (\le l)$, so in fact

$$l_{ij} = l \tag{9}$$

if $\varepsilon_2 < l^{-2}$, as will be true below.

It will be convenient to refer to $V_1, \ldots, V_t$ (but *not* V_0) as the *blocks of* $\mathcal{P}$ and to the P_α^{ij}'s with $\alpha > 0$ as the *bundles of* $\mathcal{P}$.

From now on we take V to be X_n, our set of Boolean variables. For the following definitions we fix a partition $\mathcal{P}$ as above. To simplify notation we will often use A, B, C and so on for blocks of $\mathcal{P}$. A *triad* of $\mathcal{P}$ on a triple of (distinct) blocks (A, B, C) is $P = P_{ABC} = (P_{AB}, P_{BC}, P_{AC})$, with P_{AB} one of the bundles of $\mathcal{P}$ joining A and B, and similarly for P_{AC} and P_{BC}.[*] A *subtriad* of such a P is then $Q = (Q_{AB}, Q_{BC}, Q_{AC})$ with $Q_{AB} \subseteq P_{AB}$ and so on. Since we are fixing $\mathcal{P}$ for the present discussion, in what follows we will usually drop the stipulation "of $\mathcal{P}$."

A *triangle* of a triad P as above is a triangle in the graph with edge set $P_{AB} \cup P_{AC} \cup P_{BC}$ (usually designated by its set of vertices). We write $T(P)$ for the set of such triangles and $t(P)$ for $|T(P)|$. Triangles of a subtriad Q and $T(Q)$, $t(Q)$ are defined similarly.

For a triad P on blocks A, B, C, a *pattern* on P is $\pi : \{A, B, C\} \to \{0, 1\}$. We interpret this as associating a preferred literal, $\pi(x)$, with each (variable) $x \in A \cup B \cup C$; thus, for example, for $a \in A$, $\pi(a)$ is a if $\pi(A) = 1$ and $\bar{a}$ if $\pi(A) = 0$. We also write $\pi(a, b, c)$ for the clause $\pi(a)\pi(b)\pi(c) := \pi(a) \wedge$

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[*] This usage differs slightly from that in [8], in which triads of $\mathcal{P}$ may also use P_0^{ij}'s; the change is convenient for us and of course does not affect Theorem 2.1 (formally it makes the theorem a bit weaker).

$\pi(b) \wedge \pi(c)$ (where $a \in A, b \in B$ and $c \in C$); such a clause is said to *belong* to π.

Remark. Of course we could just define patterns directly on triples of blocks, but the current definition will turn out to be less troublesome. Note that, as above, we will often give the blocks of triad P as an *ordered* triple, which allows us to write, e.g., $\pi = (1,1,0)$ without ambiguity.

Now fix an (irredundant) formula $\mathscr{C}$, again regarded as a set of clauses. For a triad P and pattern π on P, we set

$$T_\pi = T_\pi^{\mathscr{C}} = \{\{x,y,z\} \in T(P) : \pi(x,y,z) \in \mathscr{C}\},$$

and for the analogue for a subtriad Q of P use $T_\pi(Q)$. Define the *density of* π to be

$$d_\pi = d_\pi^{\mathscr{C}} = |T_\pi|/t(P). \tag{10}$$

For a pattern π on triad P, integer r, and r-tuple $\mathcal{Q} = (Q(1), \ldots, Q(r))$ of subtriads of P, set

$$d_\pi(\mathcal{Q}) = \frac{|\cup_{s=1}^r T_\pi(Q(s))|}{|\cup_{s=1}^r T(Q(s))|}.$$

We say P is (δ, r, π)-*regular for* $\mathscr{C}$ if for every $\mathcal{Q}$ as above with $|\cup_{s=1}^r T(Q(s))| > \delta t(P)$, we have $|d_\pi(\mathcal{Q}) - d_\pi| < \delta$, and (δ, r)-*regular for* $\mathscr{C}$ if it is (δ, r, π)-regular for each of the eight patterns π on P (and (δ, r)-*irregular* otherwise).

Finally, $\mathcal{P}$ is (δ, r)-*regular for* $\mathscr{C}$ if

$$\sum\{t(P) : P \text{ is a } (\delta, r)\text{-irregular triad of } \mathcal{P}\} < \delta n^3. \tag{11}$$

Let us emphasize that in the above discussion, the quantities subscripted by π, as well as the definitions of regularity for triads and partitions, refer to the fixed $\mathscr{C}$.

Theorem 2.1. *For all δ, ε_1 with $0 < \varepsilon_1 \le 2\delta^4$ and integers t_0 and l_0, and for all integer-valued functions $r = r(t,l)$ and decreasing functions $\varepsilon_2 = \varepsilon_2(l)$ with $0 < \varepsilon_2(l) \le l^{-1}$, there are T_0, L_0 and N_0 such that any formula $\mathscr{C}$ on X_n, with $n > N_0$, admits a (δ, r)-regular, equitable $(l, t, \varepsilon_1, \varepsilon_2)$-partition $\mathcal{P}$ for some t and l satisfying $t_0 \le t < T_0$ and $l_0 \le l < L_0$.*

Proof. This is given by the proof of Theorem 3.11 in [8] (which is the same as the proof of Theorem 3.5 beginning on page 151), with some minor modifications at the outset. We just indicate what these are, omitting a couple definitions that are obvious analogues of their counterparts above. We use the initial equitable $(l_0, t_0, \varepsilon_1, \varepsilon_2(l))$-partition $\mathcal{P}_0$ (which is defined without reference to any hypergraph) to specify hypergraphs $\mathcal{H}_1, \ldots, \mathcal{H}_8$, as follows. Suppose the blocks of $\mathcal{P}_0$ are $V_1, \ldots, V_{t_0}$. For $\pi = (\pi_1, \pi_2, \pi_3) \in \{0,1\}^3$ and $x \in V_i$, $y \in V_j$ and $z \in V_k$ with $i < j < k$, set $\pi(x,y,z) = \pi_1(x)\pi_2(y)\pi_3(z)$ $(= \pi_1(x) \wedge \pi_2(y) \wedge \pi_3(z))$, where

$$\psi_(x) = \begin{cases} x & \text{if } \psi_1 = 1 \\ \bar{x} & \text{if } \psi_1 = 0, \end{cases}$$

and similarly for $\psi_2(y)$ and $\psi_3(z)$. Then let $\pi^1, \ldots, \pi^8$ be some ordering of $\{0,1\}^3$, and for $s \in [8]$ and x, y, z as above, let $\{x,y,z\} \in \mathcal{H}_s$ if (and only if) $\pi^s(x,y,z) \in \mathscr{C}$.

The (only) point here is that by starting this way we guarantee that clauses belonging to the same pattern in our eventual partition will correspond to edges of the same $\mathcal{H}_s$: Theorem 3.11 of [8] gives a partition $\mathcal{P}$ as in our Theorem 2.1 in which regularity with respect to $\mathscr{C}$ is replaced by regularity with respect to each of $\mathcal{H}_1, \ldots, \mathcal{H}_8$ (which we will not define). But for any triad P of $\mathcal{P}$ and pattern π on P, (δ, r, π)-regularity for $\mathscr{C}$ is the same as (δ, r)-regularity of P (again, we omit the definition) for the appropriate $\mathcal{H}_s$, and we are done. (To be unconscionably picky, we should slightly adjust δ, since bounds corresponding to (11) for the $\mathcal{H}_s$'s will turn into a bound $8\delta n^3$ for $\mathscr{C}$.)

■

Final remark. In applying Theorem 2.1 it will be convenient to require that in fact

$$l_{ij} = l \quad \forall\, i, j. \tag{12}$$

As noted in (9) this is automatically true for i, j satisfying (8) (again, assuming $\varepsilon_2 < l^{-2}$ which will be true below); while the assumption (equitability) that all but $\varepsilon_1\binom{t}{2}$ pairs i, j do satisfy (8) allows us to arbitrarily modify the partitions of the remaining $K(V_i, V_j)$'s—we just replace them with partitions satisfying (12)—without significantly affecting (11). (So, to be overly precise, we get this very slightly strengthened version of Theorem 2.1 by applying the original with a slightly smaller δ. Of course the message here is that pairs failing (8) are essentially irrelevant; indeed the only point of (12) is that it makes some things a little easier to say in Section 6.)

3 Skeleton

In this section we give enough in the way of additional definitions to allow us to state our main lemmas, and give the proof of Theorem 1.2 modulo the much longer proofs of these supporting results.

We will soon need to say something concrete about our many parameters, but defer this discussion to the end of the present section. Given $\delta, \varepsilon_1, t_0, l_0, r = r(t,l), \varepsilon_2 = \varepsilon_2(l)$, and associated T_0, L_0 as in Theorem 2.1, define an *extended partition* $\mathcal{P}^*$ to consist of an equitable $(l,t,\varepsilon_1,\varepsilon_2)$-partition $\mathcal{P}$, with $t \in [t_0, T_0]$, $l \in [l_0, L_0]$, together with

(a) a set $\mathcal{R}(\mathcal{P}^*)$ of triads of $\mathcal{P}$ that (i) includes no P for which some two blocks of P violate (8) or some bundle of P violates ε_2-regularity, and (ii) satisfies

$$\sum\{t(P) : P \text{ a triad of } \mathcal{P} \text{ not in } \mathcal{R}(\mathcal{P}^*)\} < 2\delta n^3 \tag{13}$$

(we will mostly ignore triads not in $\mathcal{R}(\mathcal{P}^*)$); and

(b) a value $\mathsf{d}_\pi = \mathsf{d}_\pi^{\mathcal{P}^*} \in \{0, t(P)^{-1}, \ldots, (t(P)-1)t(P)^{-1}, 1\}$ for each pattern π on some $P \in \mathcal{R}(\mathcal{P}^*)$.

We will call the triads in $\mathcal{R}(\mathcal{P}^*)$ the *triads of* $\mathcal{P}^*$. The *bundles of* $\mathcal{P}^*$ are those ε_2-regular bundles P_α^{ij} of $\mathcal{P}$ for which the pair $\{i,j\}$ satisfies (8) (so the bundles of $\mathcal{P}$ that we allow in triads of $\mathcal{P}^*$). A *triangle of* $\mathcal{P}^*$ is a triangle belonging to some triad of $\mathcal{P}^*$. Say π is a *pattern of* $\mathcal{P}^*$ if it is a pattern on some triad of $\mathcal{P}^*$ *and*

$$\mathsf{d}_\pi > 2\mathsf{d}_0, \tag{14}$$

where d_0 will be specified below. A *clause of* $\mathcal{P}^*$ is then a clause belonging to a pattern of $\mathcal{P}^*$; we use $\mathcal{K}(\mathcal{P}^*)$ for the set of such clauses.

Say a formula $\mathscr{C}$ and $\mathcal{P}^*$ are *compatible* (written $\mathscr{C} \sim \mathcal{P}^*$) if every triad P of $\mathcal{P}^*$ is (δ, r)-regular for $\mathscr{C}$, and has $d_\pi^{\mathscr{C}} = \mathsf{d}_\pi$ for each pattern π on P. It follows from Theorem 2.1 that (for large enough n) every $\mathscr{C}$ is compatible with some $\mathcal{P}^*$. (The extra "2" on the right hand side of (13) covers triangles involving pairs $\{i,j\}$ violating (8).) We say $\mathcal{P}^*$ is *feasible* if it is compatible with at least one $\mathscr{C}$ and in what follows *always assume this to be the case.* Set

$$N^*(\mathcal{P}^*) = |\{\mathscr{C} : \mathscr{C} \sim \mathcal{P}^*\}|.$$

We use N^* here because we will later work mostly with

$$\mathcal{N}(\mathcal{P}^*) = \{\mathscr{C} \cap \mathcal{K}(\mathcal{P}^*) : \mathscr{C} \sim \mathcal{P}^*\} \text{ and } N(\mathcal{P}^*) = |\mathcal{N}(\mathcal{P}^*)|.$$

Say a triad P of $\mathcal{P}^*$ is *proper* if it supports a unique pattern of $\mathcal{P}^*$—*always denoted* π_P—and $\mathsf{d}_{\pi_P} > 1/3$. Say f : {blocks of $\mathcal{P}$} $\rightarrow \{0,1\}$ and P *agree* if P is proper and $\pi_P(A) = f(A)$ for each block A of P. Finally, say $\mathcal{P}^*$ is *coherent* if there is an f as above such that (with ζ_2 discussed below)

$$\textit{all but at most } \zeta_2\tbinom{t}{3}l^3 \textit{ triads of } \mathcal{P}^* \textit{ agree with } f. \tag{15}$$

The longest part of our argument is devoted to proving, for c_2 and all of the preceding parameters as described below,

Lemma 3.1. *If*

$$\log N^*(\mathcal{P}^*) > (1-c_2)\tbinom{n}{3} \tag{16}$$

then $\mathcal{P}^*$ *is coherent.*

The argument then proceeds as follows. Fix δ, ε_1, t_0, l_0, r, ε_2 (again, see below for settings; note r and ε_2 are functions). As noted above, Theorem 2.1 implies that each (irredundant) $\mathscr{C}$ is compatible with some extended partition $\mathcal{P}^*$. The number of possiblities for $\mathcal{P}^*$ is, for large enough n, less than (say) $\exp[(\log L_0)n^2]$. (There are, very crudely, at most: T_0^n choices for the partition $\{V_i\}$; $\exp[(\log L_0)\binom{n}{2}]$ for the bundles P_α^{ij}; and $\exp[(1+8\log m^3)\binom{T_0}{3}L_0^3]$ for $\mathcal{R}(\mathcal{P}^*)$ and the d_π's.) Combining this with Lemma 3.1 we have, for any constant $c' < c_2$ and large enough n,

Corollary 3.2. *All but at most* $\exp[(1-c')\binom{n}{3}]$ *irredundant* $\mathscr{C}$*'s satisfy*

$$\mathscr{C} \sim \mathcal{P}^* \textit{ for some } \text{coherent } \mathcal{P}^*. \tag{17}$$

We next need a bound on the number of $\mathscr{C}$'s that do satisfy (17). Define the *multiplicity*, $m(y) = m_{\mathscr{C}}(y)$, of the *literal* y in $\mathscr{C}$ to be the number of clauses of $\mathscr{C}$ containing y. Say $\mathscr{C}$ is *positive* if $m(x) \geq m(\bar{x})$ for each variable x. The $\mathcal{P}^*$'s will disappear from our argument once we establish (with ζ again TBA)

Lemma 3.3. *If* $\mathscr{C}$ *is positive and* $\mathscr{C} \sim \mathcal{P}^*$ *for some coherent* $\mathcal{P}^*$*, then*

$$\textit{any witness } \mathsf{w} \textit{ for any clause in } \mathscr{C} \textit{ has fewer than } \zeta n \text{ 1's.} \tag{18}$$

The easy proof is given in Section 7.

Write $\mathcal{I}^*$ for the collection of (irredundant) positive $\mathscr{C}$'s satisfying (18). According to Corollary 3.2 and Lemma 3.3 we have

$$I(n) < \exp[(1-c')\tbinom{n}{3}] + 2^n|\mathcal{I}^*|. \tag{19}$$

In Section 8 we will show, for large enough n and an appropriate positive constant c,

$$\begin{aligned} |\mathcal{I}^*| &< 2^{\binom{n}{3}} + \exp[(1-\mathsf{c})\tbinom{n}{2}]I(n-1) \\ &\quad + \exp[(1-\mathsf{c})3\tbinom{n}{2}]I(n-3) + \exp[\tbinom{n}{3} - \mathsf{c}n]. \end{aligned} \tag{20}$$

The proof of Theorem 1.2 is then completed as follows. Combining (19) and (20) and setting $B(n) = 2^{n+\binom{n}{3}}$, we have (again, for large enough n)

$$\begin{aligned} I(n) &< (1+\exp[-\mathsf{c}'n])B(n) + \exp[(1-\mathsf{c}')\tbinom{n}{2}]I(n-1) \\ &\quad + \exp[(1-\mathsf{c}')3\tbinom{n}{2}]I(n-3) \end{aligned} \tag{21}$$

(where the change from c to c' takes care of some factors 2^n and allows us to absorb the first term on the r.h.s. of (19) in the term $\exp[-\mathsf{c}'n]B(n)$).

We show by induction that (21) implies that, for some constant Δ and *all* n,

$$I(n) \leq (1+\Delta\cdot 2^{-\mathsf{c}'n})B(n) \tag{22}$$

(which proves Theorem 1.2).

For (22), choose n_0 large enough so that (21) holds for $n \geq n_0$, and then choose $\Delta > 2$ (say) so that (22) holds for $n \leq n_0$. Assuming (22) holds up to $n-1$ ($\geq n_0$), we have (omitting the little calculation for the second inequality)

$$\begin{aligned} I(n) - B(n) &< 2^{-\mathsf{c}'n}B(n) + \exp[(1-\mathsf{c}')\tbinom{n}{2}](1+\Delta 2^{-\mathsf{c}'(n-1)})B(n-1) \\ &\quad + \exp[(1-\mathsf{c}')3\tbinom{n}{2}](1+\Delta 2^{-\mathsf{c}'(n-3)})B(n-3) \\ &< \{2^{-\mathsf{c}'n} + \exp[(-\mathsf{c}'\tbinom{n}{2}+n](1+\Delta 2^{-\mathsf{c}'(n-1)})\}B(n) \end{aligned}$$

This gives (22) for n.

∎

Parameters

Before proceeding we should say something about relations between parameters. Our task in Section 8 is to prove (20) with *some* positive c. This requires an upper bound on the ζ produced by Lemma 3.3 (see (68) and (69), which involve some additional parameters), which in turn, *via* Lemma 3.3, forces ζ_2 in (15) to be small (namely it should satisfy (67)).

Of course for Lemma 3.1 to hold, we then need c_2 to be small. Specific requirements (which, for whatever it's worth, can be satisfied e.g. with c_2 some smallish multiple of ζ_2^6) are given in Section 6 (see (50)-(52)). These again involve some auxiliaries, mainly ζ_1 and c_1, which play roles in Lemma 6.4 analogous to those of ζ_2 and c_2 in Lemma 3.1. (The subscripts are arranged in this way because we think of ζ_1 and c_1 as appearing earlier in the argument, Lemma 6.4 being the final intermediate step in the proof Lemma 3.1.)

We then take d_0 to be small compared to c_2 (the smallest of the preceding parameters), and all of $\delta, \varepsilon_1, t_0^{-1}, l_0^{-1}$ small compared to d_0 (where "small" means small enough to support our arguments; here we won't spell out the requirements, but it will be clear as we proceed that there is no difficulty in arranging this). Though unnecessary, it will be slightly convenient to set

$$\delta = t_0^{-1} = l^{-1} \tag{23}$$

(but we retain the names to preserve the flavor of Theorem 2.1). Finally, we take r $(= r(t,l)) = l^6$ and ε_2 $(= \varepsilon_2(l)) = l^{-40}$. (The value of r is needed in Section 5 and then the rather severe value of ε_2 is dictated by Lemma 4.7 (whose h will eventually turn into r).)

We will use the usual asymptotic notation $\alpha = O(\beta)$, even when α and β are themselves (usually very small) constants, the interpretation being that $\alpha < C\beta$ for some C that could be fixed in advance of any of our arguments. But we will also sometimes use inequalities with explicit constants, where this seems to make the exposition clearer.

4 Basics

Here we collect some general observations, first (Section 4.1) for regular *graphic* partitions, and then (Sections 4.2 and 4.3) for feasible $\mathcal{P}^*$'s. These will be used in establishing, in Section 5, limits on legal configurations of patterns, the technical basis for the proof of Lemma 3.1. We begin with some

Conventions.

From this point through the end of Section 6 we fix a feasible $\mathcal{P}^*$ (for which we will eventually prove Lemma 3.1) together with some $\mathscr{C} \sim \mathcal{P}^*$. Triads, clauses and patterns are then understood to be triads, clauses and patterns *of* $\mathcal{P}^*$, and we will drop the latter specification.

As noted above, Section 4.1 deals only with graphic aspects of $\mathcal{P}^*$, so does not really require feasibility. Most of the remaining sections do require feasibility, and it is to make use of this assumption that we need $\mathscr{C}$; that is, we are not really interested in $\mathscr{C}$ itself at this point, but only in the implications for $\mathcal{P}^*$ that can be derived from its compatibility with $\mathscr{C}$. For the duration of this discussion (that is, through Section 6), notation involving patterns (e.g. T_π) and choices of witnesses will always refer to $\mathscr{C}$.

We will also assume, here and in Section 5, that we have fixed a bundle P^{ij}_α *of* $\mathcal{P}^*$ for any pair of blocks $\{V_i, V_j\}$ used by some triad involved in our discussion; thus if two of these triads share a pair of blocks, then they use the same bundle from this pair. The bundles and triads under discussion may then by specified by their blocks: for simplicity we will usually rename blocks $A, B, C, \ldots$ and use P_{AB} for the (fixed) bundle joining A and B and P_{ABC} for the triad on $\{A, B, C\}$. To avoid repeated specification, *we will always take* $a, a_i \in A$ *and so on.*

We will also adopt the following abusive but convenient notation. For blocks A, B, C and $X \subseteq A$, $Y \subseteq B$, $Z \subseteq C$, we will write XY for the set of edges of P_{AB} joining X and Y, and XYZ for the set of triangles of the subtriad (XY, XZ, YZ) of P_{ABC}.

Finally, for a graph G on V, $Y \subseteq V$ and $x_1, \ldots, x_k \in V \setminus Y$, we set $Y(x_1, \ldots, x_k) = \{y \in Y : y \sim x_i \ \forall\, i \in [k]\}$ (where, as usual, $x \sim y$ means $xy \in E(G)$).

4.1 Decency

We first need a few easy consequences of *graphic* regularity, beginning with the following basic (and standard) observation (see e.g. Fact 1.3 in [13]).

Proposition 4.1. *If* (A, B) *is* ε*-regular with density* d*, then for any* $B' \subseteq B$ *of size at least* $\varepsilon|B|$*,*

$$|\{a \in A : |B'(a)| \neq (d \pm \varepsilon)|B'|\}| < 2\varepsilon|A|.$$

Now suppose that $Y_1, \ldots, Y_k$ are (distinct) blocks of $\mathcal{P}^*$ and, for $1 \leq i < j \leq k$, P_{ij} is a bundle of $\mathcal{P}^*$ joining Y_i and Y_j (so in particular P_{ij} is ε_2-regular with density $l^{-1} \pm \varepsilon_2$). For distinct $x_1, \ldots, x_s \in \cup Y_i$ and $Y_j(x_i : i \in I)$ defined by the P_{ij}'s, say $\{x_1, \ldots, x_s\}$ is *decent* (with respect to $Y_1, \ldots, Y_k$ and the P_{ij}'s, but we will drop this specification when the meaning is clear) if for all $I \subseteq [s]$,

$$|Y_j(x_i : i \in I)| = (l^{-1} \pm 2\varepsilon_2)^{|I|} m \quad (= (1 \pm 2\varepsilon_2 l)^s m l^{-s})$$

whenever the left side is defined; that is, whenever $x_i \notin Y_j \ \forall\, i \in I$.

The next easy observation is similar to, e.g., [13, Fact 1.4].

Proposition 4.2. *With notation as above, if s is fixed and $\{x_1, \ldots, x_s\}$ is decent, then for any $u \in [k]$,*

$$|\{x \in Y_u : \{x_1, \ldots, x_s, x\} \text{ is indecent}\}| < 2^{s+1}k\varepsilon_2 m.$$

(Actually we will always have $k \leq 4$, but it is no harder to give the general statement. In fact s need not be fixed: we just need $(l^{-1} - 2\varepsilon_2)^s > \varepsilon_2$. It may also be worth noting that the constant $2^{s+1}k$ can always be improved; but all we ever really need from Proposition 4.2 is a bound of the form $O(\varepsilon_2 m)$, so there's no reason to be careful here.)

Proof. If $x \in Y_u$ and $\{x_1, \ldots, x_s, x\}$ is indecent, then there are $j \in [k] \setminus \{u\}$ and $I \subseteq [s]$ such that $x_i \notin Y_j \; \forall\, i \in I$ and

$$|Y_j(x) \cap Y_j(x_i : i \in I)| \neq (l^{-1} \pm 2\varepsilon_2)|Y_j(x_i : i \in I)|.$$

But by Proposition 4.1 (using $|Y_j(x_i : i \in I)| > (l^{-1} - 2\varepsilon_2)^s m > \varepsilon_2 m$), the number of such x's for a given j and I is less than $2\varepsilon_2 m$.

∎

In line with the conventions given at the beginning of this section, we will in what follows always assume that "decency" refers to the set of blocks under discussion, and will tend to drop the specification "with respect to $Y_1, \ldots, Y_k$."

From now until the end of Section 4.2 we work with blocks A, B, C, employing the conventions discussed earlier and setting $P = P_{ABC}$. The following definitions are given with A, B, C in particular roles, but of course are meant to also apply when these roles are permuted. Set (for $a \in A$)

$$L(a) = L_P(a) = \{bc : \; \{a, b, c\} \in T(P)\}$$

(L for "link"), and, similarly, for an edge ab,

$$L(ab) = L_P(ab) = \{c : \; \{a, b, c\} \in T(P)\}$$

(where, recall, we assume $a \in A$ and so on).

The next proposition, in which decency is with respect to A, B, C, is immediate from the definitions

Proposition 4.3. (a) *If a is decent then $|L(a)| = (1 \pm 2\varepsilon_2 l)^3 m^2 l^{-3}$;*

(b) *If ab is a decent edge, then $|L(ab)| = (1 \pm 2\varepsilon_2 l)^2 m l^{-2}$;*

Finally, we need to say something about triangle counts (compare e.g. [8, Fact A, p. 139]):

Proposition 4.4. *If X, Y, Z are subsets of A,B,C (resp.) with each of $|Y|, |Z|$ at least $(1-2\varepsilon_2 l)^{-1}\varepsilon_2 lm$, then*

$$(1-\tfrac{2\varepsilon_2 m}{|X|})(1-2\varepsilon_2 l)^3|X||Y||Z|l^{-3} < |XYZ| < |X||Y||Z|l^{-3} + 5\varepsilon_2 m^3.$$

In particular,

$$(1-7\varepsilon_2 l)m^3 l^{-3} < t(P) < (1+5\varepsilon_2 l^3)m^3 l^{-3}$$

Proof. Lower bound: There are at least $|X| - 2\varepsilon_2 m = (1-\frac{2\varepsilon_2 m}{|X|})|X|$ a's in X with $|Y(a)| > (1-2\varepsilon_2 l)|Y|l^{-1}$ and $|Z(a)| > (1-2\varepsilon_2 l)|Z|l^{-1}$, and for each of these a's we have (now fully using the lower bounds on $|Y|$ and $|Z|$) $|Y(a)Z(a)| > (1-2\varepsilon_2 l)|Y(a)||Z(a)|l^{-1}$.

Upper bound: There are at most $2\varepsilon_2 m$ a's with $|Y(a)| > (1+2\varepsilon_2 l)|Y|l^{-1}$ or $|Z(a)| > (1+2\varepsilon_2 l)|Z|l^{-1}$ (or both), while for *any* a we have $|Y(a)Z(a)| < \max\{(1+2\varepsilon_2 l)|Y(a)||Z(a)|l^{-1}, \varepsilon_2 m^2\}$. This gives (crudely)

$$|XYZ| < (1+2\varepsilon_2 l)^3|X||Y||Z|l^{-3} + 4\varepsilon_2 m^3.$$

∎

4.2 Triads

We continue to work with blocks A, B, C and $P = P_{ABC}$, and now fix a pattern π on P. Note in particular that "decency" in this section is with respect to these three blocks (and P). Set (e.g.)

$$L^\pi(a) = \{bc : \{a,b,c\} \in T_\pi\}$$

(where, recall, T_π is $T_\pi^{\mathscr{C}}$ for our fixed $\mathscr{C}$) and, for an edge ab,

$$L^\pi(ab) = \{c : \{a,b,c\} \in T_\pi\}.$$

Say a is *good* for π—or for now simply *good*—if, with $\delta_1 = \sqrt{\delta}$,

(i) a is decent, and

(ii) for any $B_1, \ldots, B_r \subseteq B(a)$ and $C_1, \ldots, C_r \subseteq C(a)$, if the edge sets $G_s := B_s C_s$ satisfy $|\cup_{s=1}^r G_s| > \delta_1 m^2 l^{-3}$, then

$$|L^\pi(a) \cap (\cup_{s=1}^r G_s)| = (d_\pi \pm \delta)|\cup_{s=1}^r G_s|. \tag{24}$$

(Note that (ii) implies the formally more general statement where the number of B_i's and C_i's is *at most* r, since we can add some empty sets to the list.)

For a good a, say $b \in B(a)$ is *nice to* a (with respect to π, but again we'll drop this specification) if $\{a, b\}$ is decent and

$$|L^{\pi}(ab)| = (d_{\pi} \pm 2\delta)ml^{-2} \tag{25}$$

An edge ab is then *good* if a and b are good and nice to each other. A triangle $\{a, b, c\}$ is *good* if its edges are all good and *great* if it is good and belongs to T_{π}. Finally, we say a vertex is *great* if it belongs to at least $\mathsf{d}_0 m^2 l^{-3}$ great triangles and an edge is *great* if it belongs to at least $\mathsf{d}_0 ml^{-2}$ great triangles.

Let $\delta_2 = 4\varepsilon_2 l + 3\delta_1$, $\delta_3 = 12\varepsilon_2 + 3\delta_1$, $\delta_4 = 114\varepsilon_2 l^3 + 4\delta_2 + 4\delta_3$, and $\gamma = 2\delta_4/\mathsf{d}_0$ $(= \Theta(\sqrt{\delta}/\mathsf{d}_0))$. We will use these ugly expressions in the statement and proof of the next lemma, but will then immediately pass to the relaxed version, Corollary 4.6, at which point $\delta_2, \delta_3, \delta_4$ will disappear from the discussion.

Lemma 4.5. (a) *At least* $(1 - \delta_2)m$ *vertices of* A *are good.*

(b) *If* a *is good, then* $|\{b \in B(a) : b \text{ is not nice to } a\}| < \delta_3 ml^{-1}$*; thus at least* $(1 - 2\varepsilon_2 l - \delta_3)ml^{-1}$ *vertices of* $B(a)$ *are nice to* a*.*

(c) *At most* $\delta_4 m^3 l^{-3}$ *members of* $T(P)$ *are not good. It follows that* $T(P)$ *contains at least* $(1 - 7\varepsilon_2 - \delta_4)m^3 l^{-3}$ *good triangles and at least* $(d_{\pi} - 7\varepsilon_2 l - \delta_4)m^3 l^{-3}$ *great triangles.*

(d) *At least* $(1 - \gamma)m$ *vertices of* A *are great, and at least* $(1 - \gamma)(m^2 l^{-1})$ *edges of* P_{AB} *are great.*

Corollary 4.6. (a) *At least* $(1 - \gamma)m$ *vertices of* A *are good.*

(b) *If* a *is good, then* $|\{b \in B(a) : b \text{ is not nice to } a\}| < \gamma ml^{-1}$*, and at least* $(1 - \gamma)ml^{-1}$ *vertices of* $B(a)$ *are nice to* a*.*

(c) *At most* $\gamma m^3 l^{-3}$ *members of* $T(P)$ *are not good. At least* $(1 - \gamma)m^3 l^{-3}$ *triangles of* $T(P)$ *are good, and at least* $(d_{\pi} - \gamma)m^3 l^{-3}$ *are great.*

(d) *(Repeating:) At least* $(1 - \gamma)m$ *vertices of* A *are great and at least* $(1 - \gamma)m^2 l^{-1}$ *edges of* P_{AB} *are great.*

Proof of Lemma 4.5. We use "bad" for "not good" and for the proofs of (a) and (b) set $G = P_{BC}$.

(a) By Proposition 4.1, at most $4\varepsilon_2 m$ vertices of A are indecent; so failure of (a) implies that there is a set A_0 of at least $(3/2)\delta_1 m$ decent vertices of A satisfying either (i) for each $a \in A_0$ there are $B_1(a), \ldots, B_r(a) \subseteq B(a)$ and $C_1(a), \ldots, C_r(a) \subseteq C(a)$ such that, with $G_s(a) = B_s(a)C_s(a)$, we have $|\cup_{s=1}^r G_s(a)| > \delta_1 m^2 l^{-3}$, and

$$|L^\pi(a) \cap (\cup_{s=1}^r G_s(a))| < (d_\pi - \delta)|\cup_{s=1}^r G_s(a)|,$$

or (ii) the corresponding statement with "$<(d_\pi - \delta)$" replaced by "$>(d_\pi + \delta)$." Assuming the first (the argument for the second is identical) and setting $G_s = \cup_{a\in A_0} G_s(a)$, $H_s = \cup_{a\in A_0}\{ab : b \in B_s(a)\}$ and $K_s = \cup_{a\in A_0}\{ac : c \in C_s(a)\}$, we find that for the subtriads $Q_s = (G_s, H_s, K_s)$ of P we have

$$|\cup_{s=1}^r T(Q_s)| = \sum_{a\in A_0} |\cup_{s=1}^r G_s(a)| > |A_0|\delta_1 m^2 l^{-3} \geq \delta t(P)$$

(using the upper bound on $t(P)$ in Proposition 4.4), while

$$\begin{aligned} |\cup_{s=1}^r T_\pi(Q_s)| &= \sum_{a\in A_0} |L^\pi(a) \cap (\cup_{s=1}^r G_s(a))| \\ &< \sum_{a\in A_0} (d_\pi - \delta)|\cup_{s=1}^r G_s(a)| = (d_\pi - \delta)|\cup_{s=1}^r T(Q_s)|, \end{aligned}$$

contradicting the (δ, r, π)-regularity of P.

(b) Since a is decent, each of $|B(a)|, |C(a)|$ is at least $(1 - 2\varepsilon_2 l)ml^{-1}$; in particular the second assertion in (b) follows from the first. By Proposition 4.2, $|\{b \in B(a) : ab \text{ is indecent}\}| < 12\varepsilon_2 m$; so we will be done if we show that at most $3\delta_1 ml^{-1}$ b's violate (25). Suppose instead (e.g., the other case again being similar) that there is $B_0 \subseteq B(a)$ of size at least $(3/2)\delta_1 ml^{-1}$ with

$$|L^\pi(ab)| < (d_\pi - 2\delta)ml^{-2} \quad \forall\, b \in B_0.$$

Then with $G_1 = B_0 C(a)$ we have

$$|G_1| > |B_0||C(a)|(l^{-1} - 2\varepsilon_2) > |B_0|(1 - 2\varepsilon_2 l)^2 ml^{-2} > \delta_1 m^2 l^{-3},$$

while

$$|L^\pi(a) \cap G_1| = \sum_{b\in B_0} |L^\pi(ab)| < |B_0|(d_\pi - 2\delta)ml^{-2} < (d_\pi - \delta)|G_1|,$$

contradicting the assumption that a is good.

(c) Of the triangles $\{a,b,c\}$ of $T(P)$ at most $114\varepsilon_2 m^3$ are indecent (by Proposition 4.2; the constant is of course a bit excessive); at most $3\delta_2(1+2\varepsilon_2 l)^3 m^3 l^{-3} < 4\delta_2 m^3 l^{-3}$ are bad because at least one of a,b,c is decent but bad (by (a) and Proposition 4.3(a)); and at most $3m(\delta_3 ml^{-1})(1+2\varepsilon_2 l)^2 ml^{-2} < 4\delta_3 m^3 l^{-3}$ are decent but bad because one of a,b,c fails to be nice to another (by (b) and Proposition 4.3(b)). This gives the first assertion; the second and third then follow from Proposition 4.4, the latter since the number of great triangles of P is at least

$$|T_\pi| - \delta_4 m^3 l^{-3} = d_\pi t(P) - \delta_4 m^3 l^{-3} > [d_\pi - 7\varepsilon_2 l - \delta_4] m^3 l^{-3}.$$

(d) Set $\eta = 7\varepsilon_2 l + \delta_4$; thus (c) says that the number of great triangles is at least $(d_\pi - \eta) m^3 l^{-3}$.

We first consider great vertices a. A good a belongs to at most $(d_\pi + 2\delta) m^2 l^{-3}$ great triangles (namely, $|L^\pi(a)| < (d_\pi+\delta)|(B(a)C(a))| < (d_\pi+\delta)(l^{-1}+2\varepsilon_2)^3 m^2 < (d_\pi+2\delta)m^2 l^{-3}$). Thus, with s the number of non-great a's (note a bad vertex is in no great triangles), the number of great triangles is at most

$$(m-s)(d_\pi+2\delta)m^2 l^{-3} + s\mathsf{d}_0 m^2 l^{-3},$$

and combining these bounds gives (using (14)) $s < (2\delta+\eta)/(d_\pi+2\delta-\mathsf{d}_0)m < \gamma m$.

The argument for edges is similar. A good edge belongs to at most $(d_\pi+2\delta)ml^{-2}$ great triangles, so if s is the number of non-great ab's then the number of great triangles is at most $((1+\varepsilon_2 l)m^2 l^{-1} - s)(d_\pi+2\delta) + s\mathsf{d}_0 ml^{-2}$. Again combining with (c) bounds s by roughly $(\delta_1/\mathsf{d}_0)m^2 l^{-1}$, and the (second) statement in (c) follows since $|AB| > (1-\varepsilon_2 l)m^2 l^{-1}$.

■

4.3 More basics

We continue to work with $P = P_{ABC}$, and a fixed π on P. For the next lemma we add a fourth block, say D, which only appears incognito: "decency" in Lemma 4.7 means with respect to A,B,C,D.

Lemma 4.7. *For* $\mathcal{T} \subseteq T(P)$ *with* $|T(P)\setminus\mathcal{T}| < 5\gamma m^3 l^{-3}$ *and* h *such that* $h^6\varepsilon_2 l^2 << d_\pi$, *there are distinct* a_i, b_{ij} *and* c_{ij}, $i,j \in [h]$ *satisfying*

(i) $\{a_i, b_{ij}, c_{ij}\} \in \mathcal{T}$ *is great for all* i,j, *and* (ii) *any set of four of the vertices* a_i, b_{ij}, c_{ij} *is decent.*

In practice $\mathcal{T}$ will consist of all members of $T(P)$ avoiding some set of pathologies that are known to be rare by the results of Section 4.2.

Proof. We first observe that, with $\alpha = d_\pi$ and $\mathcal{T}^*$ the set of great triples from $\mathcal{T}$, we have (using Proposition 4.4 and Corollary 4.6(c))

$$|\mathcal{T}^*| > \alpha t(P) - 5\gamma m^3 l^{-3} - \gamma m^3 l^{-3} > (\alpha - 6\gamma - 7\varepsilon_2 l^3) m^3 l^{-3}. \tag{26}$$

Say an edge ab is *fine* if $|\{c : abc \in \mathcal{T}^*\}| > \frac{1}{2}\alpha m l^{-2}$, and a is *fine* if ab is fine for at least $\frac{1}{2} m l^{-1}$ b's. We assert that

$$\text{at most } 40(\gamma/\alpha)m \ a\text{'s are not fine.} \tag{27}$$

Proof of (27). Writing s for the number of non-fine ab's we find (with explanations to follow) that $|\mathcal{T}^*|$ is at most

$$3\gamma m^3 l^{-3} + ((1 + 2\varepsilon_2 l)m^2 l^{-1} - s)(\alpha + 2\delta)ml^{-2} + (1/2)s\alpha m l^{-2}. \tag{28}$$

Here the first term covers triangles on edges ab that are either indecent or for which $|L^\pi(ab)| > (\alpha + 2\delta)ml^{-2}$. (By Proposition 4.2 there are at most $O(\varepsilon_2 m^2)$ ab's of the first type, a minor term since ε_2 is much smaller than γl^{-3}. On the other hand, ab decent with $|L^\pi(ab)| > (\alpha + 2\delta)ml^{-2}$ implies that either a is bad, or a is good and b is not nice to a; by Corollary 4.6(a) and (b), there are essentially at most $2\gamma m^2 l^{-1}$ such ab's; decency gives $|L^\pi(ab)| < (1 + 2\varepsilon_2 l)^2 ml^{-2}$.) The expression $(1 + 2\varepsilon_2 l)m^2 l^{-1}$ is an upper bound on the number of decent edges ab, and the rest of (28) is self-explanatory.

Combining (28) and (26) gives (say) $s < 19(\gamma/\alpha)m^2 l^{-1}$. It follows (using $|P_{AB}| > (1 - 2\varepsilon_2 l)m^2 l^{-1}$) that for the number, say u, of fine ab's, we have

$$u > (1 - 19\gamma/\alpha)m^2 l^{-1}. \tag{29}$$

But we also have, with v the number of non-fine a's,

$$u < 2\varepsilon_2 m^2 + (m - v)(1 + 2\varepsilon_2 l)ml^{-1} + (1/2)vml^{-1} < (m - v/2)ml^{-1} + 4\varepsilon_2 m^2,$$

and combining this with (29) gives (27). ♢

We now turn to producing the sequences described in the lemma. First, from the set of at least $(1 - 40\gamma/\alpha)m$ fine a's, choose (distinct) $a_1, \ldots, a_h$ such that

$$\text{any 4-subset of the } a_i\text{'s is decent.} \tag{30}$$

This is possible because, by Proposition 4.2, once we have $a_1, \ldots, a_i$, (30) rules out at most $O(i^3 \varepsilon_2 m)$ choices for a_{i+1}.

Second, for $i = 1, \ldots, h$, do: for $j = 1, \ldots, h$ choose (distinct) b_{ij}, c_{ij} with $a_i b_{ij} c_{ij} \in \mathcal{T}^*$ such that (ii) holds for all a's, b's and c's chosen to this point (that is, any set of at most four vertices from $\{a_1, \ldots, a_h\} \cup \bigcup\{\{b_{kl}, c_{kl}\} : k < i \text{ or } [k = i \text{ and } l \le j]\}$ is decent). We can do this because (again using Proposition 4.2) when we come to j: from an initial set of at least $(1/2)ml^{-1}$ b's for which $a_i b$ is fine, at most $O(h^6 \varepsilon_2 m)$ are disallowed because they introduce a violation of (ii) or are equal to some earlier b_{kl}; and similarly, given b_{ij}, there are at least $(1/2)\alpha m l^{-2} - O(h^6 \varepsilon_2 m)$ choices for c_{ij}.

$\blacksquare$

In Section 5 we will use sequences as in Lemma 4.7 to prove the impossibility of certain combinations of patterns. The underlying mechanism, provided by Lemma 4.9, is again similar to uses of (δ, r)-regularity in [8]. We first need the elementary

Proposition 4.8. *If $S_1, \ldots, S_h$ are sets of size at least p with $|S_i \cap S_j| < q \;\; \forall\, i \neq j$, then for any $k \le h$ we have*

$$|\cup S_i| \ge |\cup_{i=1}^k S_i| \ge kp - \binom{k}{2} q.$$

In particular, if $h \ge p/q$ then taking $k = p/q$ gives $|\cup S_i| \ge p^2/(2q)$.

Lemma 4.9. (a) *Suppose $X_i \subseteq A$ and $Y_i \subseteq B$, $i = 1, \ldots, h$ with $h > (\lambda/\kappa)^2 l^{c+d-a-b}$ satisfy*

$$|X_i| > \lambda m l^{-a}, \quad |Y_i| > \lambda m l^{-b} \qquad \forall \tag{31}$$

and

$$|X_i \cap X_j| < \kappa m l^{-c}, \quad |Y_i \cap Y_j| < \kappa m l^{-d} \qquad \forall\, i \neq j, \tag{32}$$

where $\lambda > \varepsilon_2 \max\{l^a, l^b\}$. Then

$$|\cup X_i Y_i| > \tfrac{\lambda^4}{3\kappa^2} m^2 l^{c+d-2a-2b-1}. \tag{33}$$

(b) *If $X_i \subseteq A$, $Y_i \subseteq B$ and $Z_i \subseteq C$, $i = 1, \ldots, h > (\lambda/\kappa)^3 l^{d+e+f-a-b-c}$ satisfy*

$$|X_i| > \lambda m l^{-a}, \quad |Y_i| > \lambda m l^{-b}, \quad |Z_i| > \lambda m l^{-c} \qquad \forall \forall\, i$$

and

$$|X_i \cap X_j| < \kappa m l^{-d}, \quad |Y_i \cap Y_j| < \kappa m l^{-e}, \quad |Z_i \cap Z_j| < \kappa m l^{-f} \qquad \forall\, i \neq j,$$

where (say) $\lambda > 40\varepsilon_2 \max\{l^a, l^b, l^c\}$ *and* $\kappa > (20\varepsilon_2 l^{d+e+f})^{1/3}$*, then*

$$|\cup X_iY_iZ_i| > \tfrac{\lambda^6}{3\kappa^3} m^3 l^{d+e+f-2a-2b-2c-3}. \tag{34}$$

Remark. The assumptions on λ and κ, as well as the precise expressions involving them in (33) and (34), are best ignored. In practice both will be large compared to l^{-1} (*a fortiori* to ε_2), so that the assumptions will be automatic and their roles in the conclusions minor. In some of our applications we could improve the constants in these conclusions by using, e.g., different λ's in the two bounds of (31).

Proof of (a). We have (by (31) and ε_2-regularity)

$$|X_iY_i| > (1 - 2\varepsilon_2 l)\lambda^2 m^2 l^{-a-b-1} \quad \forall\, i \tag{35}$$

and

$$|X_iY_i \cap X_jY_j| = |(X_i \cap X_j)(Y_i \cap Y_j)| < (1+2\varepsilon_2 l)\kappa^2 m^2 l^{-c-d-1} \quad \forall\, i \neq j, \tag{36}$$

where the second inequality follows from ε_2-regularity and (32) when each of $|X_i \cap X_j|$, $|Y_i \cap Y_j|$ is at least $\varepsilon_2 m$, and from $|(X_i \cap X_j)(Y_i \cap Y_j)| \leq \varepsilon_2 m^2$ otherwise. Combining these and applying Proposition 4.8 (and sacrificing a factor like 3/2 to take care of the terms with $\varepsilon_2 l$'s) gives (33).

The proof of (b) is similar and we won't repeat the argument. Here the lower bound on $|X_iY_iZ_i|$ corresponding to (35) and the upper bound on $|X_iY_iZ_i \cap X_jY_jZ_j|$ corresponding to (36) are given by Proposition 4.4. ∎

5 Configurations

We continue to follow the conventions given at the beginning of Section 4.

We will use (for example)

	A	B	C	D
π	σ_A	σ_B	σ_C	-
π'	τ_A	τ_B	-	τ_D

to mean that π and π' are patterns on P_{ABC} and P_{ABD} respectively, with $\pi(A) = \sigma_A$ $(\in \{0,1\})$ and so on. A combination of patterns—called a *configuration* and usually involving more than two patterns—is *legal* if it can arise in a feasible $\mathcal{P}^*$.

Two configurations are *isomorphic* if they can be obtained from each other by interchanging rows, interchanging columns, and/or interchanging

0's and 1's within a column (so by renaming blocks or patterns, or by interchanging the roles of positive and negative literals within a block). Of course legality is an isomorphism invariant.

This long section is devoted to showing illegality of certain configurations in a feasible $\mathcal{P}^*$. To use the feasibility assumption we will (of course) fix some $\mathscr{C} \sim \mathcal{P}^*$ and then, as usual, our notation (e.g. L^π, T_π, witnesses) refers to $\mathscr{C}$. We will make repeated use of Lemmas 4.7 and 4.9, always with $h = r$ ($= l^6$), $\lambda = \mathsf{d}_0$, and $\kappa \approx 1$. Usefulness of the bounds (33) and (34) then requires several lower bounds on d_0, the strongest of which is

$$\mathsf{d}_0^8 > 10\delta. \tag{37}$$

Most of our configurations will involve four blocks, but we begin with a pair of patterns using just three, say A, B, C, and abbreviate $P_{ABC} = P$.

Lemma 5.1. *Any two patterns for P differ on at most one of A,B,C.*

Corollary 5.2. *There are at most two patterns on P.*

Proof of Lemma 5.1.

Suppose instead that the patterns π_1 and π_2 differ on at least two of A, B and C, say (w.l.o.g.) $\pi_1(A) = \pi_1(B) = \pi_1(C) = 1$ and $\pi_2(B) = \pi_2(C) = 0$. There are then two cases:

Case 1	A	B	C
π_1	1	1	1
π_2	1	0	0

Case 2	A	B	C
π_1	1	1	1
π_2	0	0	0

Case 1. According to Lemma 4.7 we can find a ($\in A$) and disjoint pairs (b_i, c_i) ($\in B \times C$) for $i \in [r]$ satisfying:

(i) each $\{a, b_i, c_i\}$ is great for π_1;

(ii) a is good for π_2;

(iii) each set of three of the vertices a, b_i, c_i is decent.

To see this, let $\mathcal{T}$ in Lemma 4.7 consist of those $\{a, b, c\} \in T(P)$ for which a is good for π_2. Then Proposition 4.2 (with $s = 0$), Corollary 4.6(a) and Proposition 4.3(a) give

$$|T(P) \setminus \mathcal{T}| < O(\varepsilon_2 m^3) + \gamma m(1 + 2\varepsilon_2 l)^3 m^2 l^{-3} < 5\gamma m^3 l^{-3}.$$

(Of course Lemma 4.7 gives more than what we use here.)

Let w_i be a witness for $\pi_1(a, b_i, c_i)$ $(= ab_ic_i)$ and set

$$B_i = L^{\pi_1}(ac_i) \setminus \{b_i\}, \quad C_i = L^{\pi_1}(ab_i) \setminus \{c_i\}.$$

Then for each i we have

$$|B_i|, |C_i| > \mathsf{d}_0 ml^{-2} \tag{38}$$

(since by (i) and the definition of "pattern of $\mathcal{P}^*$" we have $|B_i|, |C_i| > (d_{\pi_1} - 2\delta)ml^{-2} - 1 > \mathsf{d}_0 ml^{-2}$) and, by the definition of "witness,"

$$B_i, C_i \subseteq \mathsf{w}_i^{-1}(0)$$

which implies that

$$L^{\pi_2}(a) \cap B_iC_i = \emptyset \tag{39}$$

(since $bc \in L^{\pi_2}(a) \cap B_iC_i$ would mean that w_i satisfies the clause $a\bar{b}\bar{c} \in \mathscr{C}$, contradicting the assumption that w_i is a witness for ab_ic_i). On the other hand (iii) says $|B_i \cap B_j|, |C_i \cap C_j| < (1+2\varepsilon_2 l)^3 ml^{-3}$ $(\forall\, i \neq j)$, so that, in view of (38) and (37), Lemma 4.9(a) gives $| \cup B_iC_i| > \frac{1}{3}\mathsf{d}_0^4(1 + 2\varepsilon_2 l)^{-6} m^2 l^{-3} > \delta_1 m^2 l^{-3}$. But then (39) contradicts (ii).

Case 2. By Lemma 4.7 (with $\mathcal{T} = T(P)$) we can find triples $\{a_i, b_i, c_i\}$, $i \in [r]$, satisfying:

(i) each $\{a_i, b_i, c_i\}$ is great for π_1;

(ii) each set of four of the vertices a_i, b_i, c_i is decent.

Let w_i be a witness for $\pi_1(a_i, b_i, c_i)$ $(= a_ib_ic_i)$ and set

$$A_i = L^{\pi_1}(b_ic_i) \setminus \{a_i\}, \quad B_i = L^{\pi_1}(a_ic_i) \setminus \{b_i\}, \quad C_i = L^{\pi_1}(a_ib_i) \setminus \{c_i\}.$$

Then for each i we have

$$|A_i|, |B_i|, |C_i| > \mathsf{d}_0 ml^{-2} \tag{40}$$

and

$$A_i, B_i, C_i \subseteq \mathsf{w}_i^{-1}(0),$$

the latter implying

$$T_{\pi_2} \cap A_iB_iC_i = \emptyset. \tag{41}$$

On the other hand Lemma 4.9(b) with (40) and (ii) (which implies that each of $|A_i \cap A_j|, |B_i \cap B_j|, |C_i \cap C_j|$ is at most $(1 + 2\varepsilon_2 l)^4 ml^{-4}$) gives

$$| \cup A_iB_iC_i| > \frac{1}{3}\mathsf{d}_0^6(1 + 2\varepsilon_2 l)^{-12} m^3 l^{-3} > \delta t(P)$$

(where the second inequality uses (37) and the upper bound in Proposition 4.4), so that (41) contradicts the assumption that π_2 is a pattern.

∎

We now turn to configurations on four blocks, say A, B, C, D. At one point in the argument we will need the next result, which is contained in Lemma 4.2 of [8] (the "Counting Lemma").

Lemma 5.3. *Let* π_1, π_2, π_3 *and* π_4 *be patterns on* P_{ABC}, P_{ABD}, P_{ACD} *and* P_{BCD}, *respectively. Then for any* $\mathscr{C} \sim \mathcal{P}^*$ *there are* $a \in A$, $b \in B$, $c \in C$ *and* $d \in D$ *so that* $\pi_1(a,b,c)$, $\pi_2(a,b,d)$, $\pi_3(a,c,d)$ *and* $\pi_4(b,c,d)$ *are all clauses of* $\mathscr{C}$.

glossaryname=consistent Say a configuration is *consistent* if any two of its patterns agree on their common blocks. Our main technical result is Lemma 5.5, which in particular says that, up to isomorphism, the only *inconsistent* legal configuration comprised of patterns on three distinct triads from a given set of four blocks is

Conf 0	A	B	C	D
π_1	1	1	1	-
π_2	1	1	-	1
π_3	0	-	1	1

(To elaborate a little: any configuration of the type described is isomorphic to some

	A	B	C	D
π_1	1	1	1	-
π_2	1	$*$	-	1
π_3	$*$	-	$*$	$*$

(where the $*$'s are 0's and 1's); and then either the $*$'s are all 1's (and we have coherence), or the configuration is isomorphic to Configuration 0 above or to one of the first eight configurations of Lemma 5.5, the only slightly nonobvious case here being the isomorphism

	A	B	C	D
π_1	1	1	1	-
π_2	1	0	-	1
π_3	0	-	0	1

$\cong$

Conf 4	A	B	C	D
π_1	1	1	1	-
π_2	1	0	-	1
π_3	0	-	1	0

gotten by interchanging the first two rows, the last two columns, and the 0 and 1 in the second column.) A convenient rephrasing of the above assertion regarding Configuration 0 (which, again, will follow from Lemma 5.5) is

Corollary 5.4. *In a legal configuration consisting of patterns on three different triples from a set of four blocks, no column can contain a 0, a 1 and a blank.*

Lemma 5.5. *The following configurations are illegal.*

Conf 1	A	B	C	D
π_1	1	1	1	-
π_2	1	1	-	1
π_3	1	-	0	0

Conf 2	A	B	C	D
π_1	1	1	1	-
π_2	1	1	-	1
π_3	0	-	0	0

Conf 3	A	B	C	D
π_1	1	1	1	-
π_2	1	0	-	1
π_3	1	-	0	0

Conf 4	A	B	C	D
π_1	1	1	1	-
π_2	1	0	-	1
π_3	0	-	1	0

Conf 5	A	B	C	D
π_1	1	1	1	-
π_2	1	0	-	1
π_3	0	-	0	0

Conf 6	A	B	C	D
π_1	1	1	1	-
π_2	1	1	-	1
π_3	0	-	1	0

Conf 7	A	B	C	D
π_1	1	1	1	-
π_2	1	0	-	1
π_3	0	-	1	1

Conf 8	A	B	C	D
π_1	1	1	1	-
π_2	1	1	-	1
π_3	1	-	1	0

Conf 9	A	B	C	D
π_1	1	1	1	-
π_2	1	1	0	-
π_3	0	0	-	1

Conf 10	A	B	C	D
π_1	1	1	1	-
π_2	1	1	0	-
π_3	1	0	-	1

Remarks. The full list of forbidden configurations in Lemma 5.5 is slightly more than what we'll eventually need, but it seems worth recording precisely what's going on here. Though the arguments are fairly repetitive—and we will accordingly give less detail in the later ones—we don't see a way to consolidate. An outlier is Configuration 8, which is easily handled by Lemma 5.3 but doesn't seem susceptible to an argument like those for the other cases.

Proof of Lemma 5.5. Excepting those for Configurations 7 and 8, each of the following arguments begins with a set of variables satisfying certain properties, with existence again given by Lemma 4.7. We only discuss this for Configurations 1 and 6 (see also Case 1 of Lemma 5.1), arguments in the remaining cases being similar to (usually easier than) that for Configuration 1. Note that, without further mention, *we assume in each case that the specified variables are distinct.*

Configuration 1. Let a, $b_1, \ldots, b_r$ and c satisfy:

(i) each $\{a, b_i, c\}$ is great for π_1;

(ii) each $\{a, b_i\}$ is good for π_2;

(iii) a is good for π_3;

(iv) each set of three of the vertices a, b_i, c is decent

(Existence: Take $\mathcal{T}$ in Lemma 4.7 to consist of all $\{a, b, c\} \in T(P)$ for which ab and a are good for π_2 and π_3 respectively. Corollary 4.6(a,b) bounds the number of a's that are bad for π_2 or π_3 by $2\gamma m$; the number of b's that are bad for π_2 by γm; and the number of $\{a, b\}$'s with a, b good for π_2 but ab bad for π_2 by $\gamma m^2 l^{-1}$. Thus Propositions 4.2 and 4.3 give

$$|T(P) \backslash \mathcal{T}| < O(\varepsilon_2 m^3) + \gamma[m^3 l^{-3} + 3m(1+2\varepsilon_2 l)^3 m^2 l^{-3} + m^2 l^{-1}(1+2\varepsilon_2 l) m l^{-2}],$$

which is less than $5\gamma m^3 l^{-3}$.)

Let w_i be a witness for $\pi_1(a, b_i, c)$ $(= ab_i c)$ and set

$$C_i = L^{\pi_1}(ab_i) \setminus \{c\} \quad \text{and} \quad D_i = L^{\pi_2}(ab_i)$$

Then $C_i, D_i \subseteq \mathsf{w}_i^{-1}(0)$, implying

$$L^{\pi_3}(a) \cap C_i D_i = \emptyset. \tag{42}$$

On the other hand,

$$|C_i|, |D_i| > \mathsf{d}_0 m l^{-2}$$

(given by (i) and (ii)) and (iv) (which bounds each of $|C_i \cap C_j|, |D_i \cap D_j|$ by $(1+2\varepsilon_2 l)^3 m l^{-3}$ for $i \neq j$) imply (using Lemma 4.9) that $|\cup C_i D_i| > \delta_1 m^2 l^{-3}$. But then (42) contradicts (iii).

Configuration 2. Choose triples $\{a_i, b_i, c_i\}$, $i \in [r]$, satisfying:

(i) each $\{a_i, b_i, c_i\}$ is great for π_1;

(ii) each $\{a_i, b_i\}$ is good for π_2;

(iii) each set of three of the vertices a_i,b_i,c_i is decent.

Let w_i be a witness for $\pi_1(a_i, b_i, c_i)$ $(= a_ib_ic_i)$ and set

$$A_i = L^{\pi_1}(b_ic_i) \setminus \{a_i\}, \quad C_i = L^{\pi_1}(a_ib_i) \setminus \{c_i\}, \quad D_i = L^{\pi_2}(a_ib_i).$$

Then for each i we have

$$|A_i|, |C_i|, |D_i| > \mathsf{d}_0 ml^{-2}$$

(by (i) and (ii)) and

$$A_i, C_i, D_i \subseteq \mathsf{w}_i^{-1}(0).$$

The latter implies

$$T_{\pi_3} \cap A_iC_iD_i = \emptyset, \tag{43}$$

while the former, with (iii) and Lemma 4.9 (using (37) and Proposition 4.4 as in Case 2 of Lemma 5.1) gives $|\cup A_iC_iD_i| > \delta m^3 l^{-3}$, and these together contradict the assumption that π_3 is a pattern.

Configuration 3. Choose a and pairs $\{b_i, c_i\}$, $i \in [r]$, satisfying:

(i) each $\{a, b_i, c_i\}$ is great for π_1;

(ii) a is good for π_2 and π_3;

(iii) each set of three of the vertices a, b_i, c_i is decent.

Let w_i be a witness for $\pi_1(a, b_i, c_i)$ $(= ab_ic_i)$ and set

$$B_i = L^{\pi_1}(ac_i) \setminus \{b_i\}, \quad C_i = L^{\pi_1}(ab_i) \setminus \{c_i\}$$

and

$$D_i^\tau = \mathsf{w}_i^{-1}(\tau) \cap D(a), \quad \tau \in \{0, 1\}.$$

Then for each i we have (by (i))

$$|B_i|, |C_i| > \mathsf{d}_0 ml^{-2} \text{ and } B_i, C_i \subseteq \mathsf{w}_i^{-1}(0).$$

W.l.o.g. there are at least $h/2$ i's—say those in I—for which $|D_i^1| > \frac{1}{3}ml^{-1}$, so that Lemma 4.9 (with (iii), and just using $|D_i^1| > \mathsf{d}_0 ml^{-1}$) gives

$$|\cup B_iD_i^1| \ \geq \ |\cup_{i\in I} B_iD_i^1| \ > \ \delta_1 m^2 l^{-3}.$$

But we also have

$$L^{\pi_2}(a) \cap B_iD_i^1 = \emptyset,$$

so we contradict the assumption that a is good for π_2.

Configuration 4: Let $\{a_{ij}, c_i, d_{ij}\}$, $i, j \in [r]$, satisfy

(i) each $\{a_{ij}, c_i, d_{ij}\}$ is great for π_3;

(ii) each c_i is good for π_1;

(iii) each set of four of the a_{ij}'s, c_i's and d_{ij}'s is decent.

Let w_{ij} be a witness for $\pi_3(a_{ij}, c_i, d_{ij})$ $(= \bar{a}_{ij} c_i \bar{d}_{ij})$ and set

$$A_{ij} = L^{\pi_3}(c_i d_{ij}) \setminus \{a_{ij}\}, \quad D_{ij} = L^{\pi_3}(a_{ij} c_i) \setminus \{d_{ij}\}$$

and

$$B^{\tau}_{ij} = \mathsf{w}_{ij}^{-1}(\tau) \cap B(c_i), \quad \tau \in \{0, 1\}.$$

Then for all i, j we have

$$|A_{ij}|, |D_{ij}| > \mathsf{d}_0 m l^{-2} \quad \text{and} \quad A_{ij}, D_{ij} \subseteq \mathsf{w}_{ij}^{-1}(1),$$

the latter implying in particular that

$$L^{\pi_1}(c_i) \cap A_{ij} B^1_{ij} = \emptyset. \tag{44}$$

Suppose first that there is an i for which $|B^1_{ij}| > \frac{1}{3} m l^{-1}$ for at least $h/2$ j's, say those in J. Then combining our lower bounds on $|A_{ij}|$ and $|B^1_{ij}|$ with (iii) and applying Lemma 4.9 gives

$$| \cup_{j \in J} A_{ij} B^1_{ij} | > \delta_1 m^2 l^{-3}.$$

But then (44) contradicts the assumption that c_i is good for π_1.

We may thus suppose (at least) that for each i there is *some* $j(i)$ with $|B^0_{i,j(i)}| > \frac{1}{3} m l^{-1}$. We then drop the remaining j's and relabel $a_i = a_{i,j(i)}$, $d_i = d_{i,j(i)}$, $\mathsf{w}_i = \mathsf{w}_{i,j(i)}$, $A_i = A_{i,j(i)}$, $D_i = D_{i,j(i)}$ and $B_i = B^0_{i,j(i)}$.

Since $A_i, D_i \subseteq \mathsf{w}_i^{-1}(1)$ and $B_i \subseteq \mathsf{w}_i^{-1}(0)$ we have

$$T_{\pi_2} \cap (\cup A_i B_i D_i) = \emptyset \quad \forall\, i. \tag{45}$$

But our lower bounds on sizes (to repeat, these are $|A_i|, |D_i| > \mathsf{d}_0 m l^{-2}$ and $|B_i| > \frac{1}{3} m l^{-1}$) together with (iii) imply (*via* Lemma 4.9; note that here the $|A_i \cap A_j|$'s and $|D_i \cap D_j|$'s are all at most about $m l^{-4}$)

$$| \cup A_i B_i D_i | > \delta m^3 l^{-3},$$

so that (45) contradicts the assumption that π_2 is a pattern.

Configuration 5: Let $\{a_i, b_{ij}, c_{ij}\}$, $i, j \in [r]$, satisfy

(i) each $\{a_i, b_{ij}, c_{ij}\}$ is great for π_1;

(ii) each a_i is good for π_2;

(iii) each set of four of the a_i's, b_{ij}'s and c_{ij}'s is decent.

Let w_{ij} be a witness for $\pi_1(a_i, b_{ij}, c_{ij})$ $(= a_i b_{ij} c_{ij})$ and set

$$A_{ij} = L^{\pi_1}(b_{ij}, c_{ij}) \setminus \{a_i\}, \quad B_{ij} = L^{\pi_1}(a_i c_{ij}) \setminus \{b_{ij}\}, \quad C_{ij} = L^{\pi_1}(a_i b_{ij}) \setminus \{c_{ij}\}$$

and

$$D_{ij}^{\tau} = \mathsf{w}_{ij}^{-1}(\tau) \cap D(a_i), \quad \tau \in \{0, 1\}.$$

Then

$$|A_{ij}|, |B_{ij}|, |C_{ij}| > \mathsf{d}_0 m l^{-2}$$

and

$$B_{ij}, C_{ij} \subseteq \mathsf{w}_{ij}^{-1}(0) \quad \forall\, i, j,$$

implying in particular that

$$L^{\pi_2}(a_i) \cap (\cup_j B_{ij} D_{ij}^1) = \emptyset. \tag{46}$$

If there is an i such that $|D_{ij}^1| > \frac{1}{3} m l^{-1}$ for at least $h/2$ j's, then Lemma 4.9 (with (iii) and our lower bound on $|B_{ij}|$) gives

$$| \cup_j B_{ij} D_{ij}^1 | > \delta_1 m^2 l^{-3},$$

so that (46) contradicts (ii).

We may thus suppose that for each i there is some $j(i)$ with $|D_{i,j(i)}^0| > \frac{1}{3} m l^{-1}$, and relabel $\mathsf{w}_i = \mathsf{w}_{i,j(i)}$, $A_i = A_{i,j(i)}$, $C_i = C_{i,j(i)}$ and $D_i = D_{i,j(i)}^0$. Then $A_i, C_i, D_i \subseteq \mathsf{w}_i^{-1}(0)$ implies

$$T_{\pi_3} \cap (\cup A_i C_i D_i) = \emptyset \quad \forall\, i,$$

while Lemma 4.9 gives

$$| \cup A_i C_i D_i | > \delta m^3 l^{-3},$$

contradicting the assumption that π_3 is a pattern.

Configuration 6: Let c and the pairs $\{a_i, b_i\}$, $i \in [r]$, satisfy

(i) $\{a_i, b_i, c\}$ is great for π_1;

(ii) $|L^{\pi_2}(a_ib_i) \cap D(c)| > \mathsf{d}_0 ml^{-3}$;

(iii) c is good for π_3;

(iv) each set of three of the vertices a_i, b_i, c is decent.

(For existence we use Lemma 4.7 with $\mathcal{T}$ consisting of all $\{a, b, c\} \in T(P)$ for which $|L^{\pi_2}(ab) \cap D(c)| > \mathsf{d}_0 ml^{-3}$ and c is good for π_3. (In showing $\mathcal{T}$ is large we restrict to ab's that are good for π_2, but this is not needed once we have existence.)

The number of $\{a, b, c\}$'s with ab bad for π_2 or c bad for π_3 is bounded, as in the argument for Configuration 1, by $5\gamma m^3 l^{-3}$. On the other hand, if ab is good for π_2, then ε_2-regularity (of P_{CD}) gives $|L^{\pi_2}(ab) \cap D(c)| > (d_{\pi_2} - 2\delta)(1 - \varepsilon_2 l)ml^{-3} > \mathsf{d}_0 ml^{-3}$ for all but at most $\varepsilon_2 m$ c's.)

Let w_i be a witness for $\pi_1(a_i, b_i, c)$ $(= a_ib_ic)$ and set

$$A_i = L^{\pi_1}(b_ic) \setminus \{a_i\} \quad \text{and} \quad D_i = L^{\pi_2}(a_ib_i) \cap D(c).$$

Then $A_i, D_i \subseteq \mathsf{w}_i^{-1}(0)$ implies

$$L^{\pi_3}(c) \cap A_iD_i = \emptyset; \tag{47}$$

but

$$|A_i| > \mathsf{d}_0 ml^{-2} \;\text{ and }\; |D_i| > \mathsf{d}_0 ml^{-3}$$

(given by (i) and (ii)) and (iv) imply (using Lemma 4.9 and (iv); note here $|D_i \cap D_j|$ is at most about ml^{-5}) $|\cup A_iD_i| > \delta_1 m^2 l^{-3}$, so that (47) contradicts (iii).

Configuration 7. For a pattern π on P_{ABC}, say c is *good for* π *relative to* d if $L^{\pi}(c) \cap A'B' \neq \emptyset$ whenever $A' \subseteq A(c, d)$ and $B' \subseteq B(c, d)$ are each of size at least $\mathsf{d}_0 ml^{-2}$; of course "d good for π' relative to c" for a pattern π' on P_{ABD} is defined similarly.

To rule out Configuration 7 it will be enough to show that there is *some* $\{a, c, d\}$ that is great for π_3 and satisfies

(i) c is good for π_1 relative to d;

(ii) d is good for π_2 relative to c;

(iii) $\{a, c, d\}$ is decent.

Given such a triple, choose a witness w for $\pi_3(a, c, d)$ and set

$$A' = L^{\pi_3}(c, d) \setminus \{a\} \quad (\subseteq \mathsf{w}^{-1}(1))$$

and

$$B^\tau = \mathsf{w}^{-1}(\tau) \cap B(c,d), \quad \tau \in \{0,1\}.$$

We then have $|A'| > \mathsf{d}_0 ml^{-2}$ (since cd is good for π_3) and, w.l.o.g., $|B^1| > \frac{1}{2}(1-2\varepsilon_2 l)^2 ml^{-2}$, contradicting (i) (since $L^{\pi_1}(c) \cap A'B^1 = \emptyset$).

For existence of a, c, d as above, we may argue as follows. We know from Corollary 4.6(c) that at least $\mathsf{d}_0 m^3 l^{-3}$ triangles $\{a,c,d\}$ are great for π_3, so just need to show that the number that fail to satisfy (i)-(iii) is smaller than this. The number that violate (iii) is (by Proposition 4.2, as usual) $O(\varepsilon_2 m^3)$. We will bound the number of violations of (i), and of course the same bound applies to (ii).

By Corollary 4.6(a) at most γm c's are not good for π_1. On the other hand, we assert that if c *is* good for π_1 then the size of $D' := \{d \in D(c) :$ c is not good for π_1 relative to $d\}$ is $O(h\varepsilon_2 m)$. For suppose this is false and choose $d_1, \dots, d_h \in D'$ with all triples $\{c, d_i, d_j\}$ decent. (For existence of the d_i's just note that, as in Lemma 4.7, the number of d's that *cannot* be d_{i+1} is at most $O(i\varepsilon_2 m)$; of course this is where we use the assumption that D' is large.) For each $i \in [r]$ let $A_i \subseteq A(c, d_i)$ and $B_i \subseteq B(c, d_i)$ be sets of size at least $\mathsf{d}_0 ml^{-2}$ with $L^{\pi_1}(c) \cap A_iB_i = \emptyset$; then Lemma 4.9 (using decency to guarantee that the $|A_i \cap A_j|$'s and $|B_i \cap B_j|$'s are small) gives $|\cup A_iB_i| > \delta_1 m^2 l^{-3}$, so that $L^{\pi_1}(c) \cap \cup A_iB_i = \emptyset$ says that in fact c was *not* good for π_1 (so we have our assertion). Thus the number of triangles $\{a,c,d\}$ for which $\{c,d\}$ is decent but violates (i) is at most

$$[\gamma m(1+2\varepsilon_2 l)ml^{-1} + O(h\varepsilon_2)m^2](1+2\varepsilon_2 l)^2 ml^{-2} < 4\gamma m^3 l^{-3}.$$

$$\Diamond$$

Configuration 8. As mentioned earlier, this one doesn't seem to follow from an argument like those above, but is an easy consequence of Lemma 5.3, according to which there are a, b, c, d such that each of $\pi_1(a,b,c) = abc$, $\pi_2(a,b,d) = abd$ and $\pi_3(a,c,d) = ac\bar{d}$ belongs to $\mathscr{C}$. But this is impossible, since a witness w for abc must satisfy either abd (if $\mathsf{w}(d) = 1$) or $ac\bar{d}$ (if $\mathsf{w}(d) = 0$).

Configuration 9. Choose d and $\{a_i, b_i\}$, $i \in [r]$ satisfying

(i) each $\{a_i, b_i, d\}$ is great for π_3;

(ii) each set of four of the vertices a_i, b_i, d is decent.

Let w_i be a witness for $\pi_3(a_i, b_i, d)$ $(= \bar{a}_i\bar{b}_i d)$ and set

$$A_i = L^{\pi_3}(b_i d) \setminus \{a_i\}, \quad B_i = L^{\pi_3}(a_i d) \setminus \{b_i\}$$

and

$$C_i^\tau = \mathsf{w}_i^{-1}(\tau) \cap C(a_i, b_i), \quad \tau \in \{0,1\}.$$

W.l.o.g. $|C_i^1| > \frac{1}{3}ml^{-2}$ for at least $h/2$ i's. But then $A_i, B_i \subseteq \mathsf{w}_i^{-1}(1)$ and $|A_i|, |B_i| > \mathsf{d}_0 ml^{-2}$ imply $|\cup A_i B_i C_i^1| > \delta m^3 l^{-3}$, so that

$$T_{\pi_1} \cap \cup A_i B_i C_i^1 = \emptyset$$

contradicts the assumption that π_1 is a pattern.

Configuration 10. Choose a and $\{b_i, d_i\}$, $i \in [r]$, satisfying

(i) each $\{a, b_i, d_i\}$ is great for π_3;

(ii) a is good for π_1 and π_2;

(iii) each set of four of the vertices a, b_i, d_i is decent.

Let w_i be a witness for $\pi_3(a, b_i, d_i)$ $(= a\bar{b}_i d_i)$ and set

$$B_i = L^{\pi_3}(a_i d_i) \setminus \{b_i\}$$

and

$$C_i^\tau = \mathsf{w}_i^{-1}(\tau) \cap C(a), \quad \tau \in \{0,1\}.$$

W.l.o.g. $|C_i^1| > \frac{1}{3}ml^{-1}$ for at least $h/2$ i's. But then $B_i \subseteq \mathsf{w}_i^{-1}(1)$ and $|B_i| > \mathsf{d}_0 ml^{-2}$ give $|\cup B_i C_i^1| > \delta_1 m^2 l^{-3}$ and

$$L^{\pi_1}(a) \cap (\cup B_i C_i^1) = \emptyset,$$

contradicting (ii).

∎

6 Coherence

Here we complete the proof of Lemma 3.1. We continue to work with a fixed feasible $\mathcal{P}^*$ (so that "triad" and so on continue to mean "of $\mathcal{P}^*$" unless otherwise specified). As usual in applications of regularity, we will eventually have to say that we can more or less ignore some minor effects, here those associated with clauses not belonging to patterns of $\mathcal{P}^*$; but we delay dealing with this for as long as possible (until we come to "Proof of Lemma 3.1" below).

In addition to the "auxiliary" parameters ζ_1 and c_1 mentioned earlier (at the end of Section 3) we use $\varphi = .05$, chosen to satisfy

$$\varphi < (1 - H(1/3))/2 \tag{48}$$

and

$$\varphi < \min\{10 - \mathsf{a} - \mathsf{b}\log 3 : \mathsf{a}, \mathsf{b} \in \mathbf{N}, \mathsf{a} + \mathsf{b}\log 3 < 10\}. \tag{49}$$

We then require

$$\zeta_1 << \zeta_2^2, \tag{50}$$

meaning $\zeta_1 < \varepsilon\zeta_2^2$ for a suitable small ε which we will not specify;

$$10c_1\varphi^{-1} < (\zeta_1/6)^2; \tag{51}$$

and

$$\zeta_1 > 2c_2c_1^{-1}. \tag{52}$$

(Given ζ_2 we may successively choose ζ_1, c_1, c_2 small enough to achieve (50), (51) and (52) respectively.)

Define a *bundle configuration* (BC) of $\mathcal{P}^*$ to be any $\beta = (\beta_{ij} : \{i,j\} \in \binom{[t]}{2}) \in [l]^{\binom{[t]}{2}}$. Similarly, for $I \subseteq [t]$, an *I-bundle* is some $\beta = (\beta_{ij} : \{i,j\} \in \binom{I}{2}) \in [l]^{\binom{I}{2}}$. In this case we call the blocks indexed by I the *blocks of* β; say β is a *k-bundle* if $|I| = k$; and for $J \subseteq I$ set $\beta[J] = (\beta_{ij} : \{i,j\} \in \binom{J}{2})$—a *subbundle* or *$|J|$-subbundle* of β. In any case we call the $P^{i,j}_{\beta_{ij}}$'s (i,j in $[t]$, I or J as appropriate) the *bundles of* β (or, in the last case, $\beta[J]$). Of course those for which $\{i,j\}$ violates (8) or $P^{i,j}_{\beta_{ij}}$ is not ε_2-regular are essentially irrelevant; but they are useful for bookkeeping purposes.

The next few definitions parallel the discussion leading to Lemma 3.1. The *patterns* and *clauses* of a BC or k-bundle β are those patterns and clauses of $\mathcal{P}^*$ that are supported on (bundles of) β. We use $\mathcal{K}(\beta)$ for the set of clauses of β (so the set of members of $\mathcal{K}(\mathcal{P}^*)$ supported on β), $\mathcal{N}(\beta) = \{\mathscr{C} \cap \mathcal{K}(\beta) : \mathscr{C} \sim \mathcal{P}^*\}$ and $N(\beta) = |\mathcal{N}(\beta)|$.

In contrast we will take a *triad of* β to be any *triad of* $\mathcal{P}$ (the partition underlying $\mathcal{P}^*$) supported on β. But note that as soon as a triad supports a pattern it is necessarily a triad of $\mathcal{P}^*$; in particular a *proper* triad of β will be a proper triad of $\mathcal{P}^*$ supported on β.

It will now also be helpful to define

$$h(\beta) = [(1 + 5\varepsilon_2 l^3)m^3 l^{-3}]^{-1} \log N(\beta) \tag{53}$$

and $h(\mathcal{P}^*) = [(1+5\varepsilon_2 l^3)m^3 l^{-3}]^{-1}\log N(\mathcal{P}^*)$, the expression in square brackets being the upper bound on $t(P)$ given by Proposition 4.4 (for any triad P of $\mathcal{P}^*$). This is a convenient normalization: for a pattern π of $\mathcal{P}^*$, say on triad P, the number of possibilities for the restriction of a $\mathscr{C} \sim \mathcal{P}^*$ to π is at most

$$\binom{t(P)}{\mathsf{d}_\pi t(P)} < \exp[H(\mathsf{d}_\pi)t(P)] \tag{54}$$

(recall H is binary entropy), so that the aforementioned upper bound gives

$$h(\beta) \le \sum\{H(\mathsf{d}_\pi) : \pi \text{ a pattern of } \beta\}.$$

For β a given I-bundle, $J \subseteq I$, and $A, \ldots, Z$ the blocks indexed by J, we will also write $h(A, \ldots, Z)$ for $h(\beta[J])$.

For a fixed k, say a k-bundle β is *coherent* if there is $f_\beta : \{\text{blocks of } \beta\} \to \{0,1\}$ such that each triad P of β agrees with f_β (which, recall, includes the requirement that P be proper). The definition for coherence of a BC is defined is similar to that for an extended partition; precisely: a *BC* β is *coherent* if there is some $f = f_\beta : \{\text{blocks of } \mathcal{P}^*\} \to \{0,1\}$ such that

$$\textit{all but at most } \zeta_1\binom{t}{3} \textit{ triads of } \beta \textit{ agree with } f_\beta. \tag{55}$$

In outline the proof of Lemma 3.1 goes as follows. First, the forbidden configuration results of Section 5 are used to prove

Lemma 6.1. *For a 4-bundle β, any legal configuration consisting of one pattern on each of the four triads of β is consistent.*

(Recall consistency was defined (in the natural way) following the statement of Lemma 5.3.)

Using this and, again, the results of Section 5, we obtain what we may think of as a "local" version of Lemma 3.1, *viz.*

Lemma 6.2. *A 5-bundle β with*

$$h(\beta) > 10 - \varphi \tag{56}$$

is coherent.

Corollary 6.3. *For any 5-bundle β, $h(\beta) \le 10$.*

Remarks. Note that the analogues of Corollary 6.3 and Lemma 6.2 for *4-bundles* β (namely that $h(\beta)$ is at most 4 and that $h(\beta)$ close to 4 implies coherence) are not true; rather, $h(\beta)$ can be as large as $3\log 3$, as shown by adding the pattern $\pi_6 = (1,1,0)$ on (B,C,D) to Configuration 11 in the proof of Lemma 6.2 below. It is for this reason that we need to work with 5-bundles.

For extension of the present results from 3 to larger k, it is getting to a suitable analogue of Lemma 6.2 that so far requires k-specific treatment, though a general argument does not seem out of the question. Notice for example that for $k = 4$, the "5" in Lemma 6.2 will become "7," since (compare the preceding paragraph) there can be 6-bundles β with $h(\beta) > 15$ $(= \binom{6}{4})$. Here one should of course substitute [15] for [8], which does not seem to cause any difficulties. The rest of the argument (i.e. from Lemma 6.2 onwards) seems to go through without much modification.

Once we have Lemma 6.2 (and Corollary 6.3) we are done with all that's come before, and may derive Lemma 3.1 from these last two results. A convenient intermediate step is

Lemma 6.4. (a) *For any BC* β, $h(\beta) \le \binom{t}{3}$.

(b) *Any BC* β *with*

$$h(\beta) > (1-c_1)\binom{t}{3} \tag{57}$$

is coherent.

Before turning to proofs we need some quick preliminaries. We first recall *Shearer's Lemma* [5], which we will need here and again in Section 8. For a set W, $A \subseteq W$ and $\mathcal{F} \subseteq 2^W$, the *trace* of $\mathcal{F}$ on A is $\mathrm{Tr}(\mathcal{F}, A) = \{F \cap A : F \in \mathcal{F}\}$. For a hypergraph $\mathcal{H}$ on W—that is, a collection (possibly with repeats) of subsets of W—we use, as usual, $d_{\mathcal{H}}(x)$ for the degree of $x \in W$ in $\mathcal{H}$; that is, the number of members of $\mathcal{H}$ containing x. The original statement of Shearer's lemma (though his proof gives a more general entropy version) is

Lemma 6.5. *Let* W *be a set and* $\mathcal{F} \subseteq 2^W$, *and let* $\mathcal{H}$ *be a hypergraph on* W *with* $d_{\mathcal{H}}(v) \ge k$ *for each* $v \in W$. *Then*

$$\log|\mathcal{F}| \le \frac{1}{k}\sum_{A\in\mathcal{H}} \log|\mathrm{Tr}(\mathcal{F}, A)|.$$

Applications of Lemma 6.5 in the present section will be instances of

Corollary 6.6. (a) *Suppose* $3 \leq k < q$*; let* I *be a* q*-subset of* $[t]$ *and* β *an* I*-bundle. Then*

$$h(\beta) \leq \tbinom{q-3}{k-3}^{-1} \sum \{h(\beta[J]) : J \in \tbinom{I}{k}\}.$$

(b) $h(\mathcal{P}^*) \leq l^{-\binom{t}{2}+3} \sum h(\beta)$*, where the sum runs over BC's* β *(of* $\mathcal{P}^*$*).*

Proof. For (a) apply Lemma 6.5 with $W = \mathcal{K}(\beta)$, $\mathcal{F} = \mathcal{N}(\beta)$ and $\mathcal{H} = \{\mathcal{K}(\beta[J]) : J \in \binom{I}{k}\}$. Then $\mathrm{Tr}(\mathcal{F}, \mathcal{K}(\beta[J])) = \mathcal{N}(\beta[J])$ and $d_{\mathcal{H}}(C) = \binom{q-3}{k-3}$ for each $C \in W$, and the statement follows.

The proof of (b) is similar and is omitted. ∎

We will also make some use of the following easy (and presumably well-known) observation, whose proof we omit.

Lemma 6.7. *Any graph* G *with* s *vertices and at least* $(1-\alpha)\binom{s}{2}$ *edges (where* $0 \leq \alpha < 1/2$*) has a component of size at least* $(1-\alpha)s$*.*

Finally, we recall that (as in (54)), for any m and $\alpha \in [0, 1/2]$,

$$\binom{m}{\alpha m} < \exp[H(\alpha)m].$$

Proof of Lemma 6.1 A counterexample would be a configuration of the form

	A	B	C	D
π_1	$*$	$*$	$*$	-
π_2	$*$	$*$	-	$*$
π_3	$*$	-	$*$	$*$
π_4	-	$*$	$*$	$*$

(where the $*$'s are 0's or 1's), in which we may assume (invoking isomorphism) that each column contains at most one 0. Since the configuration is incoherent there is at least one 0, say (w.l.o.g.) $\pi_1(A) = 0$. But then Corollary 5.4 says that the configuration consisting of π_1, π_2 and π_4 is illegal (as is the full configuration). ∎

Proof of Lemma 6.2.

Suppose A, B, C are blocks of β, with P the corresponding triad of β. Of course $h(A, B, C)$ is zero if there is no pattern (of β) on (A, B, C), and at most 1 if there is exactly one such pattern. We assert that

$$h(A, B, C) \leq \log 3 \tag{58}$$

in any case (really meaning when there are exactly two patterns on (A, B, C); see Corollary 5.2). To see this, suppose (w.l.o.g.) $\pi = (1,1,1)$ and $\pi' = (1,1,0)$ are patterns on (A, B, C) and, for a fixed pair a, b ($a \in A, b \in B$), consider the possibilities for the links $L^{\pi}(ab) = L^{\pi}_{\mathscr{C}}(ab)$ and $L^{\pi'}(ab) = L^{\pi'}_{\mathscr{C}}(ab)$ (with $\mathscr{C} \sim \mathcal{P}^*$). We cannot have $c \in L^{\pi}(ab) \cap L^{\pi'}(ab)$ unless each of these links consists *only* of c (since e.g. a witness for abc' ($c' \neq c$) would agree with one of abc, $ab\bar{c}$). Thus $(L^{\pi}(ab), L^{\pi'}(ab))$ is either a pair of disjoint subsets of $C(a, b)$ ($= L_P(ab)$) or two copies of the same singleton, whence the number of possibilities for this pair is less than $\exp_3[|C(a,b)|] + |C(a,b)|$. This nearly gives (58) since $\sum |C(a,b)| = t(P)$; to keep the clean expression in (58) (which of course is not really necessary), one may use the fact that $\mathscr{C} \sim \mathcal{P}^*$ requires that $\sum_{ab} |L^{\pi}(ab)| = \mathsf{d}_{\pi} t(P)$, but we leave this detail to the reader. (We could also get around this by slightly shrinking the coefficient of $\log N(\beta)$ in (53).) $\diamondsuit$

It follows, using Lemma 6.1 and Corollary 5.2, that if A, B, C, D are blocks of β, indexed by J say, with $h(\beta[J]) > 3 + H(1/3)$ ($> 2 + \log 3$), then either $\beta[J]$ is coherent or exactly three of its triads support patterns, and at least two of them support two patterns. It's also easy to see, using Corollary 5.4, that if we do have the latter possibility, say with two patterns on each of (A, B, D) and (A, C, D) and at least one on (B, C, D), then up to isomorphism (the set of patterns of) $\beta[J]$ contains the configuration

Conf 11	A	B	C	D
π_1	1	1	-	1
π_2	1	1	-	0
π_3	1	-	1	1
π_4	1	-	1	0
π_5	-	1	1	1

We next assert that if β is incoherent (and satisfies (56)), then

$$\text{some 4-subbundle } \beta' \text{ of } \beta \text{ is incoherent with } h(\beta') > 3 + H(1/3), \tag{59}$$

so, according to the preceding discussion, contains Configuration 11. For the assertion, notice that incoherence of β implies incoherence of at least one of its 4-subbundles; so if (59) fails, then Corollary 6.6 (and the fact that $h(\beta') \leq 4$ for a coherent 4-bundle β') gives

$$h(\beta) \leq \tfrac{1}{2}[4 \cdot 4 + 3 + H(1/3)] < 10 - \varphi,$$

contradicting (56).

Assume then that β contains Configuration 11; let E be the fifth block of β; and let a be the number of triads of β that support exactly one pattern, and b the number that support exactly two. Then

$$h(\beta) \leq \mathsf{a} + \mathsf{b}\log 3,$$

implying in particular (using (56) and (49)) that

$$\mathsf{a} + \mathsf{b}\log 3 \geq 10. \tag{60}$$

Corollary 5.4 now says: (i) there is no pattern on $\{A, B, C\}$ (since such a pattern together with (e.g.) π_1 and π_4 would violate the corollary); (ii) there is either no pattern on $\{A, B, E\}$ or no pattern on either of $\{A, D, E\}$, $\{B, D, E\}$ (since if π is a pattern on $\{A, B, E\}$ and π' a pattern on either of $\{A, D, E\}$, $\{B, D, E\}$, then π and π' together with one of π_1, π_2 violate the corollary); and similarly (iii) there is either no pattern on $\{A, C, E\}$ or no pattern on either of $\{A, D, E\}$, $\{C, D, E\}$.

It follows that $\mathsf{a} + \mathsf{b} \leq 7$, which with (60) implies $\mathsf{b} \geq 6$, so that there is a set of four blocks from $\{A, B, C, D, E\}$ three of whose triads support two patterns apiece (since if $S_1, \ldots, S_6$ are 3-subsets of a 5-set S, then some 4-subset of S contains at least three S_i's). But we have already seen, in the derivation of Configuration 11, that any configuration consisting of five of these patterns must be isomorphic to Configuration 11, whence it follows easily that (up to isomorphism) β contains Configuration 11 together with

	A	B	C	D
π_6	-	1	1	0

The discussion in the preceding paragraph then shows that there is either no pattern on $\{B, C, E\}$ or no pattern on either of $\{B, D, E\}$, $\{C, D, E\}$; and combining this with (i)-(iii) above gives $\mathsf{a} + \mathsf{b} \leq 6$, contradicting (60).

∎

Proof of Lemma 6.4.

(a) This is immediate from Corollaries 6.6(a) (with $q = t$, $I = [t]$) and 6.3.

(b) We first assert that (for β as in (57))

$$\text{all but at most } 10c_1\varphi^{-1}\binom{t}{5} \text{ 5-bundles of } \beta \text{ are coherent.} \tag{61}$$

Proof. By Lemma 6.2, the number of incoherent 5-bundles of β is at most

$$s := |\{I \in \binom{[t]}{5} : h(\beta[I]) < 10 - \varphi\}|.$$

Thus, again using Corollaries 6.6(a) and 6.3, we have

$$h(\beta) \le \binom{t-3}{2}^{-1}\left[\left(\binom{t}{5} - s\right)10 + s(10-\varphi)\right] = \binom{t-3}{2}^{-1}\left[10\binom{t}{5} - \varphi s\right],$$

which, combined with (57), gives $s < 10c_1\varphi^{-1}\binom{t}{5}$. $\diamondsuit$

We may then finish *via* the following simple lemma. Let k, l be integers with $k < l$ and W a set of size t. Suppose that for each $R \in \binom{W}{k}$ we are given some $\sigma_R : R \to \{0,1\}$, and for $R, S \in \binom{W}{k}$ write $R \sim S$ if σ_R and σ_S agree on $R \cap S$. Say $L \in \binom{W}{l}$ is *consistent* if $R \sim S \ \forall \ R, S \in \binom{L}{k}$.

Lemma 6.8. *For all k, l as above and* $\varepsilon > 0$ *there is a* $\xi > 0$ *such that (with notation as above) if at least* $(1-\xi)\binom{t}{l}$ *l-subsets of W are consistent, then there is some* $f : W \to \{0,1\}$ *such that* $\sigma_R \equiv f|_R$ *for all but at most* $\varepsilon\binom{t}{k}$ *k-subsets R of W.*

We will prove this only for $k = 3$ and $l = 5$, in which case we may take $\xi = (\varepsilon/6)^2$. The proof of the general case, an induction on k, is in a similar vein, though not exactly a generalization of the argument given here.

Of course to get Lemma 6.4(b) from (the case $k = 3, l = 5$ of) Lemma 6.8 we take W to be the set of blocks of $\mathcal{P}^*$, set $\sigma_R = \pi_P$ whenever P is a proper triad and R its set of blocks, and define σ_R arbitrarily for the remaining R's. (Here we use (51).) ∎

Proof of Lemma 6.8 (for $k = 3, l = 5$). Let ξ be as above, set $\alpha = \frac{1}{8}\sqrt{\xi}$, and say $x \in W$ is *bad* if there are at least $\alpha(t)_4$ pairs $\{R, S\}$ with: $R, S \in \binom{W}{3}$; $R \cap S = \{x\}$; and $R \not\sim S$. If the number of bad x's is b then the number of inconsistent 5-sets is at least $\frac{1}{15}b\alpha(t)_4$, so we have $b < \frac{15}{\alpha(t)_4}\xi\binom{t}{5} < \frac{\xi}{8\alpha}t$.

If, on the other hand, x is not bad then (by Lemma 6.7) there is $f(x) \in \{0,1\}$ such that $\sigma_R(x) = f(x)$ for at least (say) $(1-8\alpha)\binom{t-1}{2}$ 3-sets $R \ni x$. So extending this f arbitrarily to the bad x's we find that the number of 3-sets R that fail to satisfy $\sigma_R \equiv f|_R$ is at most $t \cdot 8\alpha\binom{t-1}{2} + b\binom{t-1}{2} < (8\alpha + \frac{\xi}{8\alpha})t\binom{t-1}{2} = \varepsilon\binom{t}{3}$.

∎

Proof of Lemma 3.1. We first show that clauses not belonging to $\mathcal{K}(\mathcal{P}^*)$ are more or less irrelevant. We are interested in the number of possibilities for $\mathscr{C} \setminus \mathcal{K}(\mathcal{P}^*)$ with $\mathscr{C} \sim \mathcal{P}^*$. Members of $\mathscr{C} \setminus \mathcal{K}(\mathcal{P}^*)$ are either

(i) clauses not supported on triads of $\mathcal{P}^*$ or

(ii) clauses belonging to patterns π that are supported on triads of $\mathcal{P}^*$, but that are not patterns of $\mathcal{P}^*$ (i.e. for which $\mathsf{d}_\pi \leq 2d_0$).

The total number of *possible* clauses of the first type is $O(\delta + \varepsilon_1 + t^{-1})n^3 = O(\delta n^3)$ (see (23)), where the first term, given by (13), is for clauses supported on triads of the underlying partition $\mathcal{P}$ that are not triads of $\mathcal{P}^*$. (The other two terms bound the number of clauses that use either V_0 or some P_0^{ij}, or that meet some block more than once.) On the other hand, no $\mathscr{C} \sim \mathcal{P}^*$ contains more than $16d_0\binom{n}{3}$ clauses of type (ii). Thus we have (using $\sum_{i \leq k}\binom{m}{i} \leq \exp[H(k/m)m]$)

$$N^*(\mathcal{P}^*) < \exp[8H(2\mathsf{d}_0)\tbinom{n}{3} + O(\delta)n^3]N(\mathcal{P}^*). \tag{62}$$

Thus (16) implies

$$\begin{aligned} h(\mathcal{P}^*) &> [(1+5\varepsilon_2 l)m^3 l^{-3}]^{-1}[(1-c_2)\tbinom{n}{3} - 8H(2\mathsf{d}_0)\tbinom{n}{3} - O(\delta)n^3] \\ &> (1-2c_2)\tbinom{t}{3}l^3 \end{aligned} \tag{63}$$

(where we used $c_2 >> \max\{H(2\mathsf{d}_0), \delta, \varepsilon_2 l\}$ $(= H(2\mathsf{d}_0))$ and $\binom{n}{3} > \binom{t}{3}m^3$).

We next observe that (63) (and so (16)) implies

$$\text{all but at most } 2c_2c_1^{-1}l^{\binom{t}{2}} \text{ BC's of } \mathcal{P}^* \text{ are coherent.} \tag{64}$$

Proof. This is similar to the proof of (61). By Lemma 6.4(b), the number of incoherent BC's of $\mathcal{P}^*$ is at most

$$s := |\{\beta : \beta \text{ a BC of } \mathcal{P}^*;\ h(\beta) < (1-c_1)\tbinom{t}{3}\}|.$$

Thus Corollary 6.6(b) and Lemma 6.4(a) give

$$\begin{aligned} h(\mathcal{P}^*) &\leq l^{-\binom{t}{2}+3}\sum\{h(\beta) : \beta \text{ a BC of } \mathcal{P}^*\} \\ &< l^{-\binom{t}{2}+3}((l^{\binom{t}{2}} - s)\tbinom{t}{3} + s(1-c_1)\tbinom{t}{3}), \end{aligned}$$

which with (63) implies $s < 2c_2c_1^{-1}l^{\binom{t}{2}}$. $\diamondsuit$

For the rest of this argument β ranges over BC's (of $\mathcal{P}^*$), P and Q over *triads of* $\mathcal{P}$, and A, B, C over blocks. For each coherent β we fix some f_β as in (55) and assign an arbitrary (convenient but irrelevant) $f_\beta : \{\text{blocks}\} \to \{0,1\}$ to each incoherent β.

Say P and Q *disagree* at a common block A if at least one of P, Q is not proper or (both are proper and) $\pi_P(A) \neq \pi_Q(A)$. (Here one should think of

P and Q as having just the one block in common; effects due to pairs with larger overlap will be insignificant.) We now proceed roughly as follows. An averaging argument shows that for most blocks A there are few pairs P, Q that disagree at A. When this happens there must be a value for $f(A)$ that agrees with most of the triads using A. The remaining few f-values are then of no concern and may be assigned arbitrarily.

To say this properly, write $P \not\sim_A Q$ if P and Q disagree at A and have no other block in common. Write $P \not\sim_A \beta$ if P is a triad of β and either P is improper or π_P disagrees with f_β at the block A of P, and $P \not\sim \beta$ if $P \not\sim_A \beta$ for some block A of P. Setting

$$M = |\{(\beta, P, Q, A) : P, Q \text{ triads of } \beta;\ P \not\sim_A \beta \text{ or } Q \not\sim_A \beta\}|,$$

we have

$$\begin{aligned} M &\leq 2\tbinom{t-1}{2}|\{(\beta, P, A) : P \not\sim_A \beta\}| \\ &\leq 6\tbinom{t-1}{2}|\{(\beta, P) : P \not\sim \beta\}| \\ &\leq 6\tbinom{t-1}{2}(2\zeta_1)l^{\binom{t}{2}}\tbinom{t}{3} < O(\zeta_1 t^5 l^{\binom{t}{2}}), \end{aligned}$$

where we use ζ_1 to bound both the fraction of incoherent β's (see (64) and (52)) and the fraction of triads that disagree with f_β when β is coherent. But we also have

$$M \geq |\{(A, P, Q) : P \not\sim_A Q\}| l^{\binom{t}{2}-6};$$

thus

$$\sum_A |\{(P, Q) : P \not\sim_A Q\}| = |\{(A, P, Q) : P \not\sim_A Q\}| < O(\zeta_1 t^5 l^6),$$

implying

$$|\{(P, Q) : P \not\sim_A Q\}| < \sqrt{\zeta_1} t^4 l^6 \tag{65}$$

for all but at most $O(\sqrt{\zeta_1} t)$ A's.

For A satisfying (65) we again appeal to Lemma 6.7, applied to the graph $G = G_A$ having vertices the triads (of $\mathcal{P}$) that use A, and PQ an edge if P, Q are proper and $\pi_P(A) = \pi_Q(A)$ (so improper triads become isolated vertices). We have $|V(G)| = \binom{t}{2} l^3$ and $|E(\overline{G})| < \sqrt{\zeta_1} t^4 l^6 + t^3 l^6$ (the negligible second term being a bound on the number of pairs P, Q that share at least one additional block); so the lemma says there is some $f(A) \in \{0, 1\}$ such that $\pi_P(A) = f(A)$ for all but at most $O(\sqrt{\zeta_1} t^2 l^3)$ triads P using A.

Finally, extending this f arbitrarily to A's failing (65), we find that the number of triads (of $\mathcal{P}$) that are improper or disagree with f—so in particular the number (needed for (15)) that *are* proper and disagree with f—is less than $O(\sqrt{\zeta_1}t^3l^3)$; so, in view of (50), $\mathcal{P}^*$ is coherent.

∎

7 Proof of Lemma 3.3

It will now be convenient to work with triangles rather than triads, which we can arrange, e.g., by observing that (15) implies

$$\textit{all but at most } 2\zeta_2\tbinom{t}{3}m^3 \textit{ triangles belong to triads that agree with } f \quad (66)$$

(by (13), since δ is much smaller than ζ_2).

We first need to show that f as in (66) is mostly 1. Say (just for the present argument) that a block V_i is "bad" if at least $.05\binom{t-1}{2}m^3$ triangles belong to triads that disagree with f at V_i. Let M be the number of bad V_i's and N the number of pairs (V_i, K) with V_i a block of $\mathcal{P}^*$ and K a triangle belonging to a triad that disagrees with f at V_i. Then

$$6\zeta_2\tbinom{t}{3}m^3 \geq N \geq .05M\tbinom{t-1}{2}m^3$$

gives $M \leq 40\zeta_2 t$.

Suppose, on the other hand, that V_i is good (i.e. not bad). Then the number of clauses (of $\mathscr{C}$) that agree with f at V_i is at least $\frac{1}{3}(.95)\binom{t-1}{2}m^3$ (since each triad P that agrees with f at V_i is proper and thus contributes at least $\frac{1}{3}t(P)$ such clauses), while the number that disagree is at most $4(.05+\mathsf{d}_0)\binom{t-1}{2}m^3$. There is thus (since $\frac{1}{3}(.95) > 4(.05+\mathsf{d}_0)$) some $x \in V_i$ that belongs to more clauses that agree with f at x than that disagree, so that $m(x) \geq m(\bar{x})$ implies that $f(V_i) = 1$. So we have shown that

$$|f^{-1}(0)| \leq 40\zeta_2 t.$$

Now suppose for a contradiction that w is a witness for some $C \in \mathscr{C}$ and $|\mathsf{w}^{-1}(1)| > \zeta n$. Then for the set, say $\mathcal{W}$, of blocks V_i satisfying

$$f(V_i) = 1 \text{ and } |\mathsf{w}^{-1}(1) \cap V_i| > \zeta m/2,$$

we have

$$\zeta n \;<\; |\mathsf{w}^{-1}(1)| \;<\; 40\zeta_2 n + |\mathcal{W}|m + \zeta n/2,$$

whence

$$|\mathcal{W}| \ \geq \ (\zeta/2-40\zeta_2)n/m \ \geq \ (\zeta/2-40\zeta_2)t.$$

It then follows from (66), using (say)

$$(\zeta/2-40\zeta_2)^3 > 3\zeta_2, \tag{67}$$

that there is some triad P that agrees with f, all three of whose blocks are in $\mathcal{W}$ (which, note, implies $\pi_P \equiv 1$). But then

$$(1-8\varepsilon_2 l)(\zeta/2)^3 > \delta(1+5\varepsilon_2 l^3)$$

(implied by (67)) and (δ, r)-regularity of P imply that there is some $C \neq xyz \in \mathscr{C}$ supported by P, so that w cannot have been a witness. (In more detail: Suppose the blocks of P are V_i, V_j, V_k, and let $V_u' = \mathsf{w}^{-1}(1) \cap V_u$. Then using Proposition 4.4 (both the upper and lower bounds), we find that for the subtriad Q of P spanned (in the obvious sense) by V_i', V_j', V_k', we have

$$|T(Q)| > (1-8\varepsilon_2 l)(\zeta/2)^3 m^3 l^{-3} > \delta(l^{-3}+5\varepsilon_2)m^3 > \delta t(P);$$

thus (δ, r)-regularity (here $r=1$ would suffice) gives $\mathsf{d}_{\pi_P}(Q) > \mathsf{d}_{\pi_P} - \delta$, implying the existence of xyz as above.

∎

8 Recursion

Here we prove (20). From this point we write simply X for X_n (the set of variables), and use $a, b, c, u, v, w, x, y, z$ for members of X. We call a clause *positive (negative)* if it contains only positive (negative) literals, and *non-positive* if it contains at least one negative literal. We assume throughout that all $\mathscr{C}$'s under discussion belong to $\mathcal{I}^*$ (and, as usual, that n is large enough to support our assertions).

As the form of (20) suggests, the proof will proceed by removing from $\mathcal{I}^*$ $\mathscr{C}$'s exhibiting various "pathologies," eventually leaving only (a subset of all) $\mathscr{C}$'s containing only positive clauses; these account for the main term, $2^{\binom{n}{3}}$, on the right hand side of (20).

The arguments again involve interplay of a number of small constants, and we begin by naming these and specifying what we will assume in the way of relations between them. In addition to c (from (20)) and ζ (from

(18)), we will use constants α, ϑ and ξ, assumed to satisfy the (satisfiable) relations

$$0 < \mathsf{c} < \min\{\xi, \vartheta^3 - 7H(2\zeta), \tfrac{2-\log 3}{12} - 3H(\varrho/3)\} = \vartheta^3 - 7H(2\zeta), \qquad (68)$$

where $\varrho = \sqrt{2\alpha} + \zeta$, and

$$\begin{aligned} \xi \;\; < \;\; & \min\{\alpha - 2\vartheta, \sqrt{.04 - 2\vartheta} - \vartheta, 0.1 - 7H(2\zeta), \\ & \qquad 1 - \tfrac{1}{3}H(\tfrac{1}{10}) - 0.3\log 7 - 7H(2\zeta + \alpha)\} = \alpha - 2\vartheta. \qquad (69) \end{aligned}$$

(These hold if all parameters are small and, for example, $\alpha > 2\xi > 5H(\vartheta)$ and $\vartheta > 7H(2\zeta)$.)

Step 0. Let

$$\mathcal{I}_1^* = \{\mathscr{C} \in \mathcal{I}^* : \text{each variable is used at least } \tfrac{1}{10}\tbinom{n-1}{2} \text{ times in } \mathscr{C}\}.$$

Then

$$|\mathcal{I}^* \setminus \mathcal{I}_1^*| < \exp[.8\tbinom{n}{2}]I(n-1). \qquad (70)$$

Proof. There are at most

$$n\sum\{\tbinom{8\binom{n-1}{2}}{t} : t \leq \tfrac{1}{10}\tbinom{n-1}{2}\} \;\; < \;\; \exp[H(\tfrac{1}{80})8\tbinom{n-1}{2}] \;\; < \;\; \exp[.8\tbinom{n}{2}]$$

ways to choose a variable x to be used fewer than $\frac{1}{10}\binom{n-1}{2}$ times, together with the clauses that use x, and the collection of clauses of $\mathscr{C}$ not using x is an (irredundant) formula on the $n-1$ remaining variables. ♢

Step 1. If $\mathscr{C} \in \mathcal{I}^*$ then for any two variables u, v there are at most ζn variables w for which $uv\bar{w} \in \mathscr{C}$. The same bound applies to w's with $u\bar{v}\bar{w} \in \mathscr{C}$ and those with $\bar{u}\bar{v}\bar{w} \in \mathscr{C}$.

Proof. If w is a witness for $uv\bar{w} \in \mathscr{C}$ then any $x \neq w$ with $uv\bar{x} \in \mathscr{C}$ must lie in $\mathsf{w}^{-1}(1)$. The other cases are similar. ♢

In particular:

(a) for any u, $\mathscr{C}$ contains at most ζn^2 clauses of each of the forms $uv\bar{w}$, $u\bar{v}\bar{w}$, $\bar{u}v\bar{w}$, $\bar{u}\bar{v}\bar{w}$;

(b) $\mathscr{C}$ contains at most (say) $2\zeta n^3$ non-positive clauses;

(c) if $\mathscr{C} \in \mathcal{I}_1^*$ then, for any u, $\mathscr{C}$ contains at least (say) $0.02n^2$ positive clauses using u (by (a), since $\mathscr{C} \in \mathcal{I}_1^*$ implies $m(u) \geq \frac{1}{20}\binom{n-1}{2}$).

Step 2. Let $\mathcal{I}_2^*$ consist of those $\mathscr{C} \in \mathcal{I}_1^*$ that satisfy

$$\text{for each } u, \mathscr{C} \text{ contains at most } \alpha n^2 \text{ clauses } \bar{u}vw. \tag{71}$$

Then

$$|\mathcal{I}_1^* \setminus \mathcal{I}_2^*| < \exp[(1-c)\tbinom{n}{3}] + \exp[(1-c)\tbinom{n}{2}]I(n-1). \tag{72}$$

Proof. We should show that the number of $\mathscr{C}$'s in $\mathcal{I}_1^*$ violating (71) is at most the right hand side of (72). Given such a $\mathscr{C}$ we fix u violating (71) and set $Y = X \setminus \{u\}$,

$$R = \{\{a,b\} \subseteq Y : uab \in \mathscr{C}\}, \;\; B = \{\{a,b\} \subseteq Y : \bar{u}ab \in \mathscr{C}\},$$

$$S = \{a \in Y : d_R(a) \le \vartheta n\}, \;\; T = \{a \in Y : d_B(a) \le \vartheta n\}$$

(where we regard R and B as graphs on Y and use d for degree) and $Z = Y \setminus (S \cup T)$.

The main point here is that, because $\mathscr{C}$ is irredundant,

$$\text{if } ab \in R \text{ and } ac \in B \text{ (and } b \ne c) \text{ then } abc \notin \mathscr{C}. \tag{73}$$

Since the number of clauses $\bar{u}vw$, which we are assuming to be at least αn^2, is at most $(n-|T|)n + |T|\vartheta n \le (|S|+|Z|)n + \vartheta n^2$, we must have *either* $|Z| > \vartheta n$ *or* $|Z| \le \vartheta n$ and $|S| > \xi n$ (see (69)).

Suppose first that $|Z| > \vartheta n$. In this case, once we have specified Z and the R- and B-edges meeting Z, (73) gives at least $\vartheta n \cdot \vartheta n \cdot (\vartheta n - 1)/6$ positive clauses abc that are known to *not* belong to $\mathscr{C}$. We may thus (crudely) bound the number of possibilities for $\mathscr{C}$ of this type by the product of the factors: n (corresponding to the choice of u); 2^n (choose Z); $\exp[n^2]$ (for the R- and B-edges meeting Z); $\exp[H(2\zeta) \cdot 7\binom{n}{3})]$ (for the remaining non-positive members of $\mathscr{C}$ (i.e. those not of the form $\bar{u}vw$); here we use (b) of Step 1); and $\exp[(1-\vartheta^3)\binom{n}{3}]$ (for the remaining positive members of $\mathscr{C}$). This product is less than the first term on the right hand side of (72).

Next suppose $|Z| \le \vartheta n$ and $|S| > \xi n$. We first observe that $n - |S|$ can't be *too* small: the number of positive clauses of $\mathscr{C}$ using u is at least $0.02n^2$ (by (c) of Step 1), but also at most $|S|\vartheta n + \binom{n-|S|}{2}$, which, after a little calculation, gives $n - |S| > \sqrt{.04 - 2\vartheta}\; n$. Thus in the present case we must have $|T| > (\sqrt{.04 - 2\vartheta} - \vartheta)n > \xi n$.

We may specify a $\mathscr{C}$ of the present type (i.e. with $|Z| \le \vartheta n$ and $|S| > \xi n$, so also $|T| > \xi n$) by choosing: (i) u; (ii) S and T (so also Z); (iii) the R-edges meeting $S \cup Z$ and the B-edges meeting $T \cup Z$; (iv) the R-edges contained in $T' := T \setminus S$ and the B-edges contained in $S' := S \setminus T$; and (v) the clauses not involving the variable u. The numbers of choices in (i), (ii) and (v) are at most n, 4^n and $I(n-1)$ (respectively), while those for for (iii) and (iv) are bounded by

$$\exp[2\vartheta n^2 + (|S|+|T|)H(\vartheta)n + \tbinom{|S'|}{2} + \tbinom{|T'|}{2}].$$

Combining these bounds with the easy

$$\tbinom{|S'|}{2} + \tbinom{|T'|}{2} < \tbinom{n-1}{2} - \xi(1-\xi)n^2,$$

we find that the number of $\mathscr{C}$'s in question is less than

$$n4^n \exp[\tbinom{n}{2} - (\xi(1-\xi) - 2\vartheta - 2H(\vartheta))n^2]I(n-1),$$

which is less than the second term on the right hand side of (72). $\diamondsuit$

Note that $\mathscr{C} \in \mathcal{I}_2^*$ implies (by (a) of Step 1) that for any u,

$$\mathscr{C} \text{ contains at most } (4\zeta+\alpha)n^2 \text{ non-positive clauses using } u \text{ or } \bar{u}. \tag{74}$$

Step 3. For a variable u, set $\mathbf{X}_u = \{\{v,w\} : uvw \in \mathscr{C}\}$ and $\bar{\mathbf{X}}_u = \binom{X\setminus\{u\}}{2} \setminus \mathbf{X}_u$. Let $\mathcal{I}_3^*$ consist of those $\mathscr{C} \in \mathcal{I}_2^*$ with the property that for any three variables u, v, w,

$$\text{each of } |\mathbf{X}_u \cap \mathbf{X}_v \cap \mathbf{X}_w|,\ |\mathbf{X}_u \cap \mathbf{X}_v \cap \bar{\mathbf{X}}_w|,\ |\mathbf{X}_u \cap \bar{\mathbf{X}}_v \cap \bar{\mathbf{X}}_w| \\ \text{and } |\bar{\mathbf{X}}_u \cap \bar{\mathbf{X}}_v \cap \bar{\mathbf{X}}_w| \text{ is at least } 0.1\tbinom{n}{2}. \tag{75}$$

(The "0.1" is just a convenient constant smaller than 1/8.) We assert that

$$|\mathcal{I}_2^* \setminus \mathcal{I}_3^*| < \exp[(1-\mathsf{c})3\tbinom{n}{2}]I(n-3). \tag{76}$$

Proof. We may choose $\mathscr{C} \in \mathcal{I}_2^* \setminus \mathcal{I}_3^*$ by choosing:

(i) u, v, w violating (75);

(ii) the non-positive clauses involving at least one of u, v, w;

(iii) the positive clauses involving u, v, w;

(iv) the clauses not involving u, v, w.

The numbers of possibilities for the choices in (i), (ii) and (iv) may be bounded by $\binom{n}{3}$, $\exp[3H((4\zeta+\alpha)n^2)/(7\binom{n}{2}) \cdot 7\binom{n}{2}] < \exp[21H(2\zeta+\alpha)\binom{n}{2}]$ (see (74)) and $I(n-3)$ respectively. The main point is the bound for the number of choices in (iii), which, apart from the $2^{O(n)}$ possibilities for clauses involving at least two of u, v, w, is bounded by the number of choices for an ordered partition of $\binom{X\setminus\{u,v,w\}}{2}$ into eight parts, at least one of which has size less than $0.1\binom{n}{2}$. We assert (a presumably standard observation) that this number is less than $8\exp[(H(.1) + .9\log 7)\binom{n}{2}]$. which finishes Step 3 since the product of the preceding bounds is less than the right hand side of (76).

For the assertion, notice that the log of the number of (ordered) partitions $[m] = Z_1 \cup \cdots \cup Z_8$ with $|Z_1| < 0.1m$ is $H(Y_1, \dots, Y_m) \leq \sum H(Y_i)$, where we choose $(\mathbf{Z}_1, \dots, \mathbf{Z}_8)$ uniformly from the set of such partitions and set $Y_i = j$ if $i \in \mathbf{Z}_j$. (The inequality, an instance of Lemma 6.5, is a basic (easy) property of entropy; see e.g. [6, Theorem 2.6.6].) Setting $p_i(j) = \Pr(Y_i = j)$ $(= \Pr(i \in \mathbf{Z}_j))$ and $\bar{p}_j = m^{-1}\sum_i p_i(j)$, we have

$$\begin{array}{rcl} \sum H(Y_i) & = & \sum_j \sum_i p_i(j) \log \frac{1}{p_i(j)} \\ & \leq & m \sum_j \bar{p}_j \log \frac{1}{\bar{p}_j} = mH(\bar{p}_1, \dots, \bar{p}_8) \end{array}$$

(by Jensen's Inequality) and

$$H(\bar{p}_1, \dots, \bar{p}_8) \leq H(\bar{p}_1) + (1 - \bar{p}_1)\log 7 < H(0.1) + 0.9\log 7$$

(using $H(X) \leq \log|\text{range}(X)|$ for the first inequality). ♢

Step 4. Let

$$\mathcal{I}_4^* = \{\mathscr{C} \in \mathcal{I}_3^* : \text{no clause of } \mathscr{C} \text{ uses more than one negative literal.}\}$$

Then

$$|\mathcal{I}_3^* \setminus \mathcal{I}_4^*| < \exp[(1-c)\tbinom{n}{3}]. \tag{77}$$

Proof. We first observe that $\mathscr{C} \in \mathcal{I}_3^*$ *cannot* contain a clause with exactly two negative literals. For suppose $\bar{u}\bar{v}w \in \mathscr{C}$. Since $\mathscr{C} \in \mathcal{I}_3^*$, there is some pair $\{a, b\}$ with $abu, abv, abw \in \mathscr{C}$; but this is impossible, since a witness for abw must agree with at least one of abu, abv, $\bar{u}\bar{v}w$.

While the preceding argument doesn't quite work to exclude negative clauses, the assumption that $\bar{u}\bar{v}\bar{w} \in \mathscr{C}$ is extremely restrictive, since it says that whenever $\{a, b\} \in \mathbf{X}_u \cap \mathbf{X}_v \cap \mathbf{X}_w$, there cannot be *any* $c \notin \{u, v, w\}$ with $abc \in \mathscr{C}$ (since a witness for abc would have to agree with one of abu, abv, abw, $\bar{u}\bar{v}\bar{w}$). So we may bound the number of $\mathscr{C}$'s that do contain negative clauses by the product of: n^3 (choose u, v, w); $\exp[n^2]$ (choose $\mathbf{X}_u \cap \mathbf{X}_v \cap \mathbf{X}_w$); $\exp[7H(2\zeta)\binom{n}{3} + O(n^2)]$ (for clauses that either are non-positive or involve u, v or w; here we again use (b)); and $\exp[\binom{n-3}{3} - 0.1\binom{n}{2}(n -$

3)/3] $<\exp[.9\binom{n}{3}]$ (for the remaining positive clauses; here the subtracted term corresponds to triples known to contain members of $\mathbf{X}_u \cap \mathbf{X}_v \cap \mathbf{X}_w$). And again, the product of these bounds is less than $\exp[(1-c)\binom{n}{3}]$. $\diamondsuit$

Step 5. Finally, we set

$$\mathcal{I}_5^* = \{\mathscr{C} \in \mathcal{I}_4^* : \mathscr{C} \text{ contains no clause with exactly one negative literal}\}$$

(so $\mathcal{I}_5^* \subseteq \{\mathscr{C} \in \mathcal{I}^* : \mathscr{C}$ contains only positive clauses$\}$) and show

$$|\mathcal{I}_4^* \setminus \mathcal{I}_5^*| < \exp[\tbinom{n}{3} - cn]. \tag{78}$$

Proof. We show that for any $\mathtt{t} > 0$ (by (b) of Step 1 $\mathtt{t}$ will be at most ζn^3, but we don't need this),

$$|\{\mathscr{C} \in \mathcal{I}_4^* : \mathscr{C} \text{ has exactly } \mathtt{t} \text{ non-positive clauses}\}| < \exp[\tbinom{n}{3} - c'n] \tag{79}$$

for a suitable c'; this gives (78) for any $c < c'$.

Fix $\mathtt{t}$ and suppose $\mathscr{C}$ is as in (79). The main point driving the argument (which, however, will take us a while to get to) is:

$$\text{if } \bar{u}vw \in \mathscr{C} \text{ and } a \notin \{u,v,w\}, \text{ then } |\mathscr{C} \cap \{auv, avw\}| \le 1 \tag{80}$$

(since a witness for avw must agree with either auv or $\bar{u}vw$).

Let $\mathscr{C}'$ be the set of non-positive clauses in $\mathscr{C}$. It will be helpful to introduce an auxiliary collection: for each $C \in \mathscr{C}'$, we will fix an ordering of the three literals in C with the negative literal first, and write $\mathscr{C}''$ for the resulting collection of ordered triples. We assert that we can do this so that

$$|\{w : (\bar{u}, v, w) \in \mathscr{C}''\}| \le \sqrt{\alpha/2}\, n \qquad \forall\, u, v. \tag{81}$$

This will follow from

Proposition 8.1. *Any (simple) graph admits an orientation with all out-degrees at most* $\sqrt{|E(G)|/2}$.

Proof (sketch). A precise statement (due to Hakimi [10]; see also [16, Theorem 61.1, Corollary 61.1b]) is: for any graph $G = (V, E)$ and $c : V \to \mathbf{N}$, there is an orientation with $d_v^+ \le c_v \ \forall\, v$ (where, of course, d_v^- is the out-degree of v) iff for every $W \subseteq V$, $|E(G[W])| \le \sum\{c_v : v \in W\}$; in particular, there is an orientation with $d_v^+ \le c \ \forall\, v$ iff $c \ge \max\{|E(G[W])|/|W| : W \subseteq V\}$, which is easily seen to hold with $c = \lceil \sqrt{|E(G)|/2}\, \rceil$.

(Alternatively it's easy to see that orienting each edge toward the end of larger degree (breaking ties arbitrarily) gives maximum out-degree less than $\sqrt{2|E(G)|}$, which would also be fine for present purposes.) $\diamondsuit$

To get (81) from Proposition 8.1, regard, for a given u, $\{vw : \bar{u}vw \in \mathscr{C}'\}$ as the edge set of a graph G_u on $X \setminus \{u\}$, and choose an orientation of $E(G_u)$ as in the proposition. We have $|E(G_u)| \leq \alpha n^2$ (by (71)); so interpreting orientation of vw toward w as specifying $(\bar{u}, v, w) \in \mathscr{C}'$ gives (81). $\diamondsuit$

Of course there will typically be many choices of $\mathscr{C}''$ as above, and we fix one such for each $\mathscr{C}'$. Given $\mathscr{C}''$, set $\mathcal{G} = \mathcal{G}(\mathscr{C}') = \{\{\{u,v\},\{v,w\}\} : (\bar{u}, v, w) \in \mathscr{C}''\}$. Regard $\mathcal{G}$ as a multigraph on the vertex set $\binom{X}{2}$, and let ν and τ denote its matching and (vertex) cover numbers. Then

$$2\nu \geq \tau \geq \lceil \frac{\mathtt{t}}{\varrho n} \rceil \tag{82}$$

(where, recall, $\varrho = \sqrt{2\alpha} + \zeta$). Here the first inequality is standard (and trivial) and the second follows from the fact that $\mathcal{G}$ has $\mathtt{t}$ edges and maximum degree at most ϱn, the latter by (81) and Step 1.

We now consider the number of possibilities for $\mathscr{C}$ with given a $\mathtt{t}$, τ and ν. We first specify $\mathscr{C}'$ by choosing a vertex cover $\mathcal{T}$ for the associated $\mathcal{G}$ and then a collection of $\mathtt{t}$ clauses, each using (the variables from) at least one member of $\mathcal{T}$. The number of possibilities for these choices is at most $\binom{n^2}{\tau}\binom{3\tau n}{\mathtt{t}}$.

We now suppose $\mathscr{C}'$ has been determined and consider possibilities for the set, say $\mathscr{C}_0$ $(= \mathscr{C} \setminus \mathscr{C}')$, of positive clauses of $\mathscr{C}$. Let $\mathcal{M}$ be some maximum matching of $\mathcal{G}$, say $\mathcal{M} = \{\{\{u_i, v_i\}, \{v_i, w_i\}\} : i \in [\nu]\}$. (We could specify $\bar{u}_i v_i w_i \in \mathscr{C}'$, but this is now unnecessary.)

Let $\mathcal{J}$ be the set of all pairs of 3-sets $\{\{a, u_i, v_i\}, \{a, v_i, w_i\}\}$ such that $\{\{u_i, v_i\}, \{v_i, w_i\}\} \in \mathcal{M}$ and $a \notin \{u_i, v_i, w_i\}$, and let $\mathcal{K}$ be the set of 3-sets belonging to pairs in $\mathcal{J}$. Then $\mathcal{J}$ is a set of at least $\nu(n-3)/2$ pairs of 3-sets (a given pair $\{\{x,y,z\},\{x,y,w\}\}$ can arise with x in the role of v_i and y in the role of a or vice versa) with the property that no 3-set belongs to more than three members of $\mathcal{J}$ (since $\mathcal{M}$ is a matching); so in particular $|\mathcal{K}| \geq \nu(n-3)/3$.

We assert that the number of possibilities for $\mathscr{C}_0 \cap \mathcal{K}$ is at most

$$\exp[\tbinom{n}{3} - \tfrac{1}{6}\nu(n-3)(2 - \log 3)].$$

Proof. This is another (somewhat more interesting) application of Lemma 6.5. Let $W = \binom{X}{3}$ (thought of as the collection of possible positive clauses);

let $\mathcal{F}$ be the collection of possible $\mathscr{C}_0$'s (compatible with the given $\mathscr{C}'$); and let $\mathcal{H}$ consist of all pairs from $\mathcal{J}$ (note these are now pairs of elements of W) together with, for each $T \in W$, $3 - \eta(T)$ copies of the singleton $\{T\}$, where $\eta(T) \leq 3$ is the number of times T appears as a member of some pair in $\mathcal{J}$. As noted earlier the key point is (80), which in the present language says that no member of $\mathcal{F}$ contains any $\{S, T\} \in \mathcal{J}$. This implies in particular that for each such $\{S, T\}$, we have $|\mathrm{Tr}(\mathcal{F}, \{S, T\})| \leq 3$, so that Lemma 6.5 gives

$$\begin{aligned} \log|\mathcal{F}| &\leq \tfrac{1}{3}\left[\textstyle\sum_{T \in W}(3 - \eta(T)) + |\mathcal{J}|\log 3\right] \\ &\leq \tbinom{n}{3} - \tfrac{1}{6}\nu(n-3)(2 - \log 3) \end{aligned}$$

(since $\sum \eta(T) = 2|\mathcal{J}|$ and $|\mathcal{J}| \geq \nu(n-3)/2$). ◇

Finishing the proof of (78) is now easy. We have shown that the number of possibilities for $\mathscr{C}$ with given t, τ and ν is at most

$$\begin{aligned} &\tbinom{n^2}{\tau}\tbinom{3\tau n}{\mathsf{t}}\exp[\tbinom{n}{3} - \tfrac{1}{6}\nu(n-3)(2 - \log 3)] \\ &\qquad < \exp\left[\tbinom{n}{3} + \left\{\log\tfrac{en^2}{\tau} + 3nH(\varrho/3) - \tfrac{(n-3)(2-\log 3)}{12}\right\}\tau\right] \end{aligned}$$

(where we used (82) (second and first inequalities respectively) for the last two terms in the exponent), and summing over τ and ν shows that the left side of (79) is less than $\exp[\binom{n}{3} - c'n]$ for any $c' < (2 - \log 3)/12 - 3H(\varrho/3)$. ◇

Finally, combining (70), (72), (76), (77) and (78) (and, of course, the fact that $|\mathcal{I}_5^*| \leq \exp[\binom{n}{3}]$) gives (20) (where we again absorb terms $\exp[(1-c)\binom{n}{3}]$ from (72) and (77) in the term $\exp[\binom{n}{3} - \mathsf{c}n]$).

■

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Department of Mathematics
Rutgers University
Piscataway NJ 08854 USA
ilinca@math.rutgers.edu
jkahn@math.rutgers.edu