Math 561: Math Logic

Spring 2008

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Lecture 1.

We begin by discussing some people who tried to generalize our number system in some sense.

Sir William Hamilton (1805-1865): Invented $\mathbb{H}$, the skew field of quaternions, which generalized the numbers with regards to their arithmetic properties. We don't care.

Georg Cantor (1845-1918): Found the generalization of numbers with regards to their ordering. This gives us the ordinal numbers, which we will be discussing shortly.

Note: In what follows, *Halmos* refers to Naive Set Theory, by Paul R. Halmos, and *Levy* refers to Basic Set Theory, by Azriel Levy.

1 ZF axioms

We get started by going over the Zermelo-Fraenkel axioms of set theory.

Extensionality If two sets have the same elements, they are equal.

This one is pretty self explanatory. Halmos calls it the Axiom of Extension, and writes *Two sets are equal if and only if they have the same elements.* (pg. 2) Levy notes that it is due to Frege in 1893, and writes it as $\forall x(x \in y \leftrightarrow x \in z) \rightarrow y = z$. (pg. 5)

Union/Sum axiom If I is a set and for each $i \in I$, A_i is a set, then there is a set whose elements are exactly the members of the A_is.

Again, this one is pretty easy. Halmos writes *For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.* (pg. 12) Levy attributes it to Cantor in 1899 and Zermelo in 1908, and writes $\forall z \exists y \forall x(x \in y \leftrightarrow \exists u(x \in u \wedge u \in z))$. (pg. 18)

Powerset axiom If A is a set, then there is a set whose elements are the subsets of A.

Halmos calls this the Axiom of Powers, writing *For each set there exists a collection of sets that contains among its elements all the subsets of the given set.* (pg. 19) Levy writes it is due to Zermelo in 1908, and states it as $\forall z \exists y \forall x(x \in y \leftrightarrow x \subseteq z)$. (pg. 19)

Replacement scheme For each expression E, add the axiom: if E is a functional relation and A is a set, then there is a set whose elements are the objects of the form $E(x)$ for $x \in A$.

There is more to say about this one. First, a definition. A relation E is functional if for all x, y, y', $xEy \wedge xEy' \to y' = y$. A functional relation is a function when it is also a set. There is nothing to say that E is not actually a class, which is why the distinction is necessary. For example, the identity on the universe (i.e. the set of all sets, a class) is a functional relation, but not a function. It suffices to check that $\mathrm{dom}(E)$ is a set, as $E(A)$ is certainly no bigger than E.

Levy has this as $\forall u \forall v \forall w(\psi(u,v) \wedge \psi(u,w) \to v = w) \to \forall z \exists y \forall v(v \in y \leftrightarrow (\exists u \in z)\psi(u,v))$, where $\psi(u,v)$ is a formula, although I think he may mean to have different v's in the antecedent and consequent. It's on pg. 19. The idea is what's written above, it's just less clear this way. Halmos introduces his version of this axiom much later in his book (pg. 75), calling it the Axiom of Substitution: *If $S(a,b)$ is a sentence such that for each a in a set A the set $\{b : S(a,b)\}$ can be formed, then there exists a function F with domain A such that $F(a) = \{b : S(a,b)\}$ for each a in A.* It sounds a little weird in comparison to the other two, but the idea is the same. As he puts it, "[It] says...that anything intelligent that one can do to the elements of a set yields a set."

Finally, it was remarked that this is stronger than the Comprehension Axiom scheme, with which we may be more familiar. Deloro stated it as "a subcollection of a set is a set", and went on to say if E is a class and A is a set, then the collection of $x \in A$ that are in E is a set. (There was a little more on this in the next class.) Halmos and Levy both introduce that axiom fairly early; they do so on pg. 6 and pg. 6, respectively. Later, we'll see a model ($V_{\omega+\omega}$) in which Comprehension holds but Replacement fails.

Infinity axiom There exists an infinite set.

Later, Deloro would amend this to say "there is a limit ordinal" (ω). This simplifies things when dealing with ordinals, and allows us to avoid the Axiom of Choice which is apparently otherwise necessary. This lines up with Halmos' Axiom of Infinity: *There exists a set containing 0 and containing the successor of each of its elements.* (pg. 44, at which point terms like '0' and 'successor' had been defined, and this leads to ordinals in such a way that he's talking about ω.) Levy first writes down the axiom on pg. 23 (attributing it to Zermelo in 1908), although at this point he says it simply postulates the existence of a particular set which will be infinite once he gets around to discussing what "infinite" means. It states: $\exists z(0 \in z \wedge (\forall x \in z)(\forall y \in z)(x \cup \{y\} \in z))$. Note that according to Levy's notation, $0 = \emptyset$, not that this is weird or anything. It's the same statement as Halmos.

1.1 Other axioms

The following axioms are not in ZF normally.

Axiom of Choice If I is a set and for each $i \in I$, A_i is a non-empty set, then there is a function $h : I \to \cup_{i\in I} A_i$ such that for all i, $h(i) \in A_i$.

We will talk much more about this one later.

Axiom of Foundation If $A \neq \emptyset$, there is $B \in A$ such that $A \cap B = \emptyset$.

I find this one a little tricky, but it means that the set theory is "well-founded", so there can't be things like sets containing themselves, or infinitely descending chains of sets properly containing each other. In the case of the natural numbers, for instance, every natural number contains the empty set, so you just let that be your B. If you remove the empty set from the number, then it contains 1, which is the set containing the empty set, and you let that be your B. Etc. You see how at least in this case we have a well-founded system, since everything comes back to (is founded on) the empty set. This axiom guarantees such a thing is always true.

2 Well-orderings and ordinals

Definition 2.1. $(A, <)$ *is well-ordered if every non-empty subset has a least element, denoted* $\min(A)$.

Definition 2.2. *If* $(A, <)$ *is an ordering and* $a \in A$*, the proper initial segment before* a *is* $\{x \in A : x < a\}$*, which we denote* $S_A(a)$*.* A *is considered to be a (not proper) initial segment of* A.

Observations:

- If $(A, <)$ is well-ordered, then A is totally ordered. This is because given any two elements x and y in A, $\{x, y\} \subseteq A$, so it is well-ordered, which means in particular that the two elements are comparable using $<$, since one must be less than the other.

- If $A' \subseteq A$ with the following being true: $\forall s \in A'$, $\forall t \in A$, $t < s \Rightarrow t \in A'$, then A' is an initial segment of A.

  Proof. Assume that $X = A \setminus A' \neq \emptyset$. Let $a = \min(X)$. Then I claim $A' = S_A(a)$. If $s \in A'$, then $s < a$, since if $s > a$, then $a \in A'$ by the definition of A', which contradicts the definition of a. Further, if there were an element x in A such that $x < a$ and x is not in A', then $x \in X$, again contradicting the definition of a. So the claim holds, as A' is exactly those elements less than a. □

Definition 2.3. *An order isomorphism between $(A, <)$ and $(B, <)$ is a bijection $f : A \to B$ such that for all $a, a' \in A$, $f(a) < f(a')$ iff $a < a'$.*

Theorem 2.4. *Let A, B be well-orderings (well-ordered sets). Then either A is uniquely isomorphic to a unique proper initial segment of B or vice versa or A and B are uniquely isomorphic.*

Lemma 2.5. *If A is a well-ordering, $A' \subseteq A$ is an initial segment, and if $f : A \to A'$ is an (order) isomorphism, then $A' = A$ and $f = id_A$. (In other words, if A is (order) isomorphic to an initial segment of A, then it* is *that initial segment of A.)*

Proof. First, we prove that $\forall x \in A$, $f(x) \geq x$. Suppose not. Then $X = \{x \in A : f(x) < x\} \neq \emptyset$. Let $x_0 = \min(X)$. Since $x_0 \in X$, $f(x_0) < x_0$. Hence $f(x_0) \in X$, contradicting the definition of x_0. So X must be empty. (Note: this is a slightly different proof than the one given in class. I think it is more direct, and hopefully is still correct.) Now assume $A' \subset A$, $A' \neq A$ i.e. $A' = S_A(a)$ for some $a \in A$. Then $f(a) \in A' = S_A(a)$, meaning $f(a) < a$, a contradiction. Hence $A' = A$.

For the next part, let $Y = \{x \in A : f(x) \neq x\} = \{x \in A : f(x) > x\}$. Assume $Y \neq \emptyset$. Let $y_0 = \min(Y)$. Since $A' = A$ and f is surjective, there is a $b \in A$ with $f(b) = y_0$. $f(y_0) > y_0$ since $y_0 \in Y$, hence $f(f(b)) > f(b) \Rightarrow f(b) > b$. Hence $b \in Y$, contradicting the definition of y_0. Thus $Y = \emptyset$, meaning that $f(x) = x \ \forall x \in A \Rightarrow f = \mathrm{id}_A$. (Again, this is slightly different from the proof given in class, in much the same way as the last one was.) □

Proof of Theorem 2.4. Uniqueness of such an isomorphism is easy. Assume that $f_1 : A \to S_B(b_1)$ and $f_2 : A \to S_B(b_2)$ are both order isomorphisms of A into a proper initial segment of B. Then $S_B(b_1) \cong S_B(b_2)$, and the above lemma applies, telling us that $S_B(b_1) = S_B(b_2)$ and $f_2^{-1} \circ f_1 = \mathrm{id} \Rightarrow f_1 = f_2$. The same argument works for the other two uniqueness assertions.

Existence of such an isomorphism takes more time to prove. Let $F = \{(a, b) \in A \times B : S_A(a) \cong S_B(b)\}$. This set is nonempty, since the least elements of A and B create an ordered pair that fits the bill.

Claim: F is an injective function.

Assume $(a, b) \in F$ and $(a, b') \in F$. Then $S_B(b) \cong S_A(a) \cong S_B(b') \Rightarrow S_B(b) \cong S_B(b')$, and as before we cite the lemma to determine that $b = b'$. Hence F is a function. A symmetric argument shows that it is injective.

Claim: F is an order isomorphism between $\mathrm{dom}(F)$ and $\mathrm{im}(F)$.

Let $a, a' \in \mathrm{dom}(F)$. Then $\exists b, b' \in B$ with $S_A(a) \cong S_B(b)$ and $S_A(a') \cong S_B(b')$. Clearly then $a < a' \Leftrightarrow b < b'$.

Claim: dom(F) (im(F)) is an initial segment of A (B).
If $a \in \text{dom}(F)$, and there is $a' \in A$ with $a' < a$, then there is $f : S_A(a) \to S_B(b)$ an isomorphism for some $b \in B$, and $S_A(a') \cong S_B(f(a'))$, simply by restricting f. Hence $a' \in \text{dom}(F)$, and so dom(F) is an initial segment of A. A similar argument holds for im(F).

If dom$(F) = A$ or im$(F) = B$, we are done, since then F is an isomorphism between A and an initial segment of B or vice versa or $A \cong B$. So assume neither of those things are true.

Then dom$(F) = S_A(a)$ for some $a \in A$, and im$(F) = S_B(b)$ for some $b \in B$. But F is an isomorphism between dom(F) and dom(F), i.e. $S_A(a) \cong S_B(b)$ by the above claim. Hence $(a, b) \in F$. In particular, $a \in \text{dom}(F) = S_A(a)$, a contradiction. □

Definition 2.6. *An ordinal is a set α such that if (i) $x \in y \in \alpha$, then $x \in \alpha$, and (ii) $(\alpha, \in)$ is well-ordered. (The first property is referred to as transitivity, i.e. α is a transitive set.)*

In the remainder of the notes, α, β, $\gamma, \ldots$ will denote ordinals. Let On be their collection.

Observations:

- α is not an element of α. This is because if it were, the well-ordering would give $\alpha < \alpha$, which is impossible.

- $x \in \alpha$ iff x is a proper initial segment of x, in which case $x = S_\alpha(x)$. This again comes from the well-ordering. If $x \in \alpha$, then $y < x \Leftrightarrow y \in x$, so the set of all elements of α less than x is precisely the set of all elements of x, i.e. $x = S_\alpha(x)$. Now, assume x is a proper initial segment of α. Let γ be the least element of α not in x. (Such an element exists since α is well-ordered.) Then γ and x are both subsets of α. For x this is clear, and for γ one must note that if $\beta \in \alpha$, $\beta < \gamma$, then $\beta \in \gamma$, whence $\gamma \subset \alpha$. But from this work we see that $\gamma = S_\alpha(\gamma)$ and x have the same elements, namely those elements less than γ, and since a set is determined by its elements (Extension), we have that $x = \gamma$, and therefore $x \in \alpha$.

  It is worth noting that this condition is often used as a definition for ordinal. In Halmos (pg. 75), an ordinal is defined as *"a well-ordered set α such that $s(\xi) = \xi$ for all ξ in α; here $s(\xi)$ is...the initial segment $\{\eta \in \alpha : \eta < \xi\}$."* This definition is originally due to von Neumann, and allows one to show transitivity with relative ease, giving the same properties as our definition.

- $x \in \alpha \Rightarrow x$ is an ordinal. There is a proof of this in Levy (Prop. 3.11, pg. 52) which basically relies on the fact that $x \subset \alpha$, whence it inherits transitivity and a well-ordering.

Lemma 2.7 (Comparison)**.** *Let α, β be ordinals. Then either $\alpha \in \beta$, $\alpha = \beta$, or $\beta \in \alpha$.*

Proof. Let $\gamma = \alpha \cap \beta$. We will first show that γ is an initial segment of α (and β).

Let $s, t \in \alpha$ with $s \in \gamma$ and $t < s$. (Note: this assumes that $\gamma \neq \emptyset$. Otherwise we're done since $\emptyset$ is the inital segment of the minimum element in α (and β)). We would like to prove that $t \in \gamma$ as well.

If $\gamma = \alpha$, then $t \in \gamma$, since α is transitive. If $\gamma \neq \alpha$, $t \in s \in \gamma \subset \beta \Rightarrow s \in \beta \Rightarrow t \in \beta$, by transitivity of β, and hence $t \in \alpha \cap \beta = \gamma$. By the observations, γ is an ordinal, since it is transitive and well-ordered by $\in$. Further, γ is an initial segment of α and β.

If $\gamma = \alpha = \beta$, then we have proven the lemma. If $\gamma = \alpha$ and $\gamma \subsetneq \beta$, then $\alpha \in \beta$, since we have from above that a proper initial segment of an ordinal is an element of that ordinal. Similarly, if $\gamma = \beta$ and $\gamma \subsetneq \alpha$, we have $\beta \in \alpha$. Finally, if $\gamma \subsetneq \alpha$ and $\gamma \subsetneq \beta$, then $\gamma \in \alpha$ and $\gamma \in \beta$, whence $\gamma \in \gamma$. But this can not be since γ is an ordinal. Thus we have covered all of the cases. $\square$

Corollary. *Let α, β be ordinals isomorphic as well-orderings, say $f(\alpha) \cong \beta$. Then $\alpha = \beta$ and $f = id_\alpha$.*

Proof. By the previous lemma, one of them is an initial segment of the other. Apply the earlier result on well-orderings for just this situation. $\square$

End of Lecture 1.

Lecture 2.

Theorem 2.8. *Let A be a well-ordered set. Then A is uniquely isomorphic to a unique ordinal.*

Proof. Uniqueness follows from the corollary proved at the end of last class, specifically that any two ordinals isomorphic as well-orderings are in fact the same ordinal. With this in mind, if A were isomorphic to two ordinals, then they would be isomorphic to each other, and thus be the same.

Existence: One can argue as in the first theorem from last class regarding well-orderings being isomorphic to proper initial segments of the other. Assume that A is not isomorphic to any ordinal α. Then for any ordinal α, it can't be the case that A is isomorphic to an initial segment of α, since any initial segment of α is an element of α and hence an ordinal. However, since α is a well-ordered set, it must be a proper initial segment of A by the aforementioned theorem.

So, consider the set $\mathcal{S}$ of initial segments of A. (Note that this requires the

Powerset Axiom, as we are talking about a collection of subsets of A, which we only know is a set by that axiom.) To each initial segment B, assign the unique ordinal α if there is such a thing, $\emptyset$ otherwise. Note that this is well-defined even without the Axiom of Choice, since we know that well-ordered sets are uniquely isomorphic to proper initial segments of others, so there is no choice involved in picking α.

> **Aside**: A word on functional relations. These were mentioned in the last class when discussing the Replacement Scheme. Things may not have been clear. A functional is just a mapping (e.g. the identity map on the universe, the class of all sets). A function is a functional that is interpretable in set theory, i.e. a mapping that can be written as a set of pairs, i.e. a functional whose domain is a set. So the Replacement Scheme states if F is a functional and A is a set, $F(A)$ is a set.

Let H be the name of the functional defined in the paragraph just before the aside. Our assumption is that $H(\mathcal{S}) = \mathrm{On}$. The left side is a set by Replacement ($\mathcal{S}$ is a set), but we will prove that On is a proper class, thus arriving at a contradiction. Thus it must be the case that A is isomorphic to the initial segment of an ordinal. □

Proposition 2.9 (Burali-Forti Paradox)**.** On *is a proper class.*

Proof. Assume that On is instead a set. Then if $\beta \in \alpha \in \mathrm{On}$, β is an element of an ordinal, hence an ordinal. Thus $\beta \in \mathrm{On}$, whence On is a transitive set. Let X be a nonempty collection of ordinals. Then there exists an α in X. If α is the least element in X (ordered by $\in$, remember), then we're okay. If it is not, then the collection of ordinals in X less than α is precisely $\{\beta \in \alpha : \beta \text{ is in } X\} \neq \emptyset$. (Note that we did not write $\beta \in X$, since it would not make sense if X is a class, and we'd like this argument to work even after we show that On is a class.) This is a nonempty subset of α, hence it has a least element, since α is well-ordered. Thus we see that On is well-ordered. But then On is an ordinal (it's transitive and well-ordered), so we have $\mathrm{On} \in \mathrm{On}$, a contradiction for ordinals. □

Note: Halmos proves this somewhat differently (pg. 80); he shows that the supremum of a set of ordinals is again an ordinal (which we prove next), and uses this fact to arrive at the "paradox". Levy has this as Prop. 3.14 (pg. 53), although I don't see how it's different than Theorem 3.6 (pg. 50).

Proposition 2.10. On *is a well-ordered collection (meaning every nonempty collection of ordinals has a least element). Every subset of* On *has a supremum.*

Proof. The first point was proved in the proof of the last proposition. (Remember how we were careful not to use $\in$ when defining the "initial segment" of α?)

The next part is also easy. Let X be a set of ordinals. Then let $\alpha = \cup_{x \in X} x$.

This is a set by the Union Axiom. I begin by claiming that α is an ordinal. First, it is well-ordered. Any subset of α is a nonempty set of elements of ordinals. We have seen that elements of ordinals are again ordinals. So any subset of α is a nonempty collection of ordinals, and the first assertion tells us that it has a least element. Second, it is transitive. If $\gamma \in \beta \in \alpha$, then $\beta \in x$ for some $x \in X$. x is an ordinal, hence transitive, hence $\gamma \in x$, hence $\gamma \in \alpha$ by the definition of a union. So α is transitive and well-ordered, thus an ordinal.

Next, I claim that α is the supremum we are looking for. First, α is an upper bound (by $\in$) for the set X. If $\xi \in X$, then $\xi \in \alpha$ (def. of union), so we're done. Next, assume τ is another (ordinal) upper bound of X. Then $\xi \in \tau$ whenever $\xi \in X$, and thus either $\tau = \alpha$ because they have the exact same elements, or α is proper initial segment of τ and hence $\alpha \in \tau$. So α is the least upper bound of X, i.e. the supremum of X. □

(Note: I did this proof myself. There was some guidance from Halmos, but he sets up ordinals differently, so I couldn't just copy it or anything. Also, Levy leaves this as an exercise. What I'm saying is caveat emptor on this proof.)

Ordinals generalize the natural numbers with respect to their ordering ($<$). So it makes sense that we would want to capture some of the other properties of natural numbers using ordinals.

Definition 2.11. *Let α be an ordinal. Define $\alpha + 1 := \alpha \cup \{\alpha\}$, also known as the successor of α. α is called a successor if there is a β in* On *with $\alpha = \beta + 1$.*

Exercise: $\alpha + 1$ is an ordinal.
It is transitive: If $\beta \in \gamma \in \alpha + 1$, then either $\gamma \in \alpha \Rightarrow \beta \in \alpha \Rightarrow \beta \in \alpha + 1$, or $\gamma \in \{\alpha\} \Rightarrow \gamma = \alpha \Rightarrow \beta \in \alpha \Rightarrow \beta \in \alpha + 1$.
It is well-ordered: Simply take the well-ordering provided by α, and set $\beta < \{\alpha\}$ for all $\beta \in \alpha$. (In other words, $S_{\alpha+1}(\alpha) = \alpha$.) This is the obvious ordering.

Exercise: $\alpha + 1$ is the least ordinal greater than α.
If $\alpha < \beta$, then $\alpha \in \beta$, and clearly either $\alpha + 1 = \beta$ or $\alpha + 1 \in \beta$.

Definition 2.12. *An ordinal α is a limit ordinal if $\alpha \neq \emptyset$ and α is not a successor.*

Axiom of Infinity: There is a limit ordinal. Call the least limit ordinal ω.
Axiom of Infinity$'$: There is a non-finite ordinal. (see below)
An ordinal α is finite if $\alpha = 0$ or α is a successor and every $\beta < \alpha$ is 0 or a successor. (See Levy, pg. 56.)

3 Induction and Recursion

Theorem 3.1 (Transfinite induction)**.** *Let X be a collection of ordinals. Assume*

1. *$\emptyset$ is in X,*
2. *If α is in X, so is $\alpha + 1$,*
3. *If α is a limit ordinal and ($\forall \beta \in \alpha$, β is in X), then α is in X.*

Then X is On.

Proof. Assume not. Then the collection of ordinals not in X is a nonempty collection of ordinals. Thus we may let α denote its least element. By the assumptions, α is not $\emptyset$, it is not a successor, and it is not a limit ordinal. But this exhausts the possibilities. So the collection of ordinals not in X must be empty, i.e. X is On. □

Note that the third condition may be rewritten to talk about all $\beta < \alpha$, this looks more natural, I think, and closer to the form of induction with which we're more familiar. I'll also note that the three conditions may be put more succinctly; see Levy theorem. 3. 21, pg. 54.

Theorem 3.2 (Transfinite recursion)**.** *Let H be a functional relation defined on all functions of domain an ordinal. Then there is a unique functional relation F defined on* On *such that for all α, if α in* On, *then $F(\alpha) = H(F|_\alpha)$. ($F|_\alpha$ is F restricted to α, i.e. what was done up to and not including step α. H is defined on functions with domain an ordinal; $F|_\alpha$ fits that bill.)*

Proof. Uniqueness is, as usual, the easier thing to show. Let F,G meet the requirements. Assume there is α in On with $F(\alpha) \neq G(\alpha)$. We may assume that α is the least such ordinal. By definition, $F|_\alpha = G|_\alpha$, hence $F(\alpha) = H(F|_\alpha) = G(\alpha)$, a contradiction.

Existence: For each ordinal α, build a function f^α from α such that for all $\beta < \alpha$, $f^\alpha(\beta) = H(f^\alpha|_\beta)$. Let $f^\emptyset = \emptyset$. If f^α has been built, let

$$f^{\alpha+1}(\beta) = \begin{cases} f^\alpha(\beta) & \text{if } \beta < \alpha \\ H(f^\alpha) & \text{if } \beta = \alpha \end{cases}$$

If α is a limit ordinal and for all $\beta < \alpha$, f^β is known, let $f^\alpha(\beta) = f^{\beta+1}(\beta)$. Then for each α in On, f^α is a function having $\forall \beta < \alpha$, $f^\alpha(\beta) = H(f^\alpha|_\beta)$. This defines f^α for limit ordinals. The f^α's all extend each other, obviously.

With the above definitions in mind, have $F(\alpha) = f^{\alpha+1}(\alpha)$. F meets the requirements, since $F(\alpha) = f^{\alpha+1}(\alpha) = H(f^\alpha)$ (from above), and has domain On. □

I'd like to note that in Halmos, transfinite induction and recursion are defined before ordinals using only well-ordered sets; note that neither of the above proofs relied on the transitivity of the ordinals. Also, Levy's statement of transfinite recursion is completely inscrutable without knowing the symbols he's defined previously, which I don't.

Applications: Ordinal arithmetic, which we will see next class.

A final note: To find a model of ZF without the Axiom of Infinity (assuming there is a model of ZF), look at V_ω (mentioned last class but still undefined; roughly "what you get with finite ordinals") in which Axiom of Infinity fails.

End of Lecture 2.

Lecture 3.

Erratum from a previous class: Last class, we were given the following exercise:
Let $\mathcal{C}$ be an ordered collection in which every nonempty subcollection has a least element (i.e. $\mathcal{C}$ is a well-ordered collection). Then $\mathcal{C}$ is isomorphic to On. This is wrong.

1. Find a counterexample.
2. Find a correct statement.
3. Prove the correct statement.

Answers:

1. Take the collection On with one additional element, which we'll denote as ∞. Let this set have the same ordering as On, with the additional condition that $\alpha < \infty$ for all α in On. This is a well-ordered collection that is clearly not isomorphic to On.
2. The correct statement includes the condition that every proper initial segment in the collection is a set. Note that our counterexample fails this, since the proper initial segment of ∞ is On, which is a proper class.
3. This was left to us; I may come back to it later.

3.1 Ordinal arithmetic

Ordinal addition Our first definition is very technical. Let α be an ordinal, and let $H : (f : \gamma \in \mathrm{On} \to \mathrm{On}) \to \mathrm{On}$ be the functional given by

$$H(f) = \begin{cases} \alpha & \text{if } \gamma = \emptyset \\ f(\beta) + 1 & \text{if } \gamma = \beta + 1 \\ \sup\{f(\beta) : \beta < \gamma\} & \text{if } \gamma \text{ is a limit ordinal} \end{cases}$$

Call H α^+.
Then, let F^{α^+} be the unique functional over On with $(\forall\beta, F^{\alpha^+}(\beta) = H(F^{\alpha^+}|_\beta))$.
Define $\alpha + \beta := F^{\alpha^+}(\beta)$.

If you're like me, that definition is somewhat difficult to get your head around. Instead, we may define it inductively:

- $\alpha + \emptyset = \alpha$
- $\alpha + (\beta + 1) = (\alpha + \beta) + 1$
- if β is a limit ordinal, $\alpha + \beta = \sup\{\alpha + \gamma : \gamma < \beta\}$

Another way to "define" $\alpha+\beta$ is that $\alpha+\beta$ is α "followed by" β. This can be made rigorous, and in fact, this is how Halmos defines ordinal addition (pg. 81).

Exercise: Check + is associative.
Proof: I will be using the second definition for this. (It's trivial for the third definition.) We'd like to show that $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$, so we do induction on γ. First, assume that $\gamma = \emptyset$. Then $(\alpha+\beta)+\gamma = \alpha+\beta = \alpha+(\beta+\gamma)$. Next, assume that $1 < \gamma$ (since 1 is associative by definition), and for all $\delta < \gamma$, we have associativity. In the case that γ is a successor (of say δ),

$$\begin{aligned}(\alpha + \beta) + \gamma &= (\alpha + \beta) + (\delta + 1)\\ &= ((\alpha + \beta) + \delta) + 1\\ &= (\alpha + (\beta + \delta)) + 1\\ &= \alpha + ((\beta + \delta) + 1)\\ &= \alpha + (\beta + (\delta + 1))\\ &= \alpha + (\beta + \gamma)\end{aligned}$$

If γ is a limit ordinal, then we have

$$\begin{aligned}(\alpha + \beta) + \gamma &= (\alpha + \beta) + (\sup\{\delta : \delta < \gamma\})\\ &= \alpha + \sup\{\beta + \delta : \delta < \gamma\}\\ &= \alpha + (\beta + \gamma)\end{aligned}$$

In that second line, we were able to move β inside the supremum since we assumed associativity with everything less than γ. Thus we have shown that + is associative.

Remark: Ordinal addition generalizes addition on the natural numbers. However, ordinal addition is not commutative. $1 + \omega = \omega$, since 1 "followed by" ω looks the same as ω (or appeal to the other definitions, whatever), but $\omega + 1 \neq \omega$, since by putting a 1 after ω (a neat trick, putting something after an infinite set, but mathematically it makes sense) looks very different than just ω, which sort of trails on infinitely.

Ordinal multiplication Let α, β, and γ be ordinals. Define $H : (f : \gamma \in \mathrm{On} \to \mathrm{On}) \to \mathrm{On}$ to be the functional given by

$$H(f) = \begin{cases} \emptyset & \text{if } \gamma = \emptyset \\ f(\beta) + \alpha & \text{if } \gamma = \beta + 1 \\ \sup\{f(\beta) : \beta < \gamma\} & \text{if } \gamma \text{ is a limit ordinal} \end{cases}$$

Call H $\alpha^\times$.

Or inductively:

- $\alpha \times \emptyset = \emptyset$
- $\alpha \times (\beta + 1) = (\alpha \times \beta) + \alpha$
- if β is a limit ordinal, $\alpha \times \beta = \sup\{\alpha \times \gamma : \gamma < \beta\}$

Or define $\alpha \times \beta$ to be the ordinal of the set of pairs of α and β (the cartesian product) ordered by the right lexicographical order, i.e. $(a, b) \leq (c, d)$ if $b < d$ or ($b = d$ and $a \leq c$).

Ordinal multiplication is associative but not commutative, like addition. $2\omega = \omega$ (think of ω copies of 2 all in a row. It just looks like ω), but $\omega 2 = \omega + \omega$ (think of 2 copies of ω in a row). Halmos notes (pg. 84) that ordinal multiplication and ordinal addition do obey the left distributive law.

Ordinal exponentiation For this one, we just give the inductive definition.

- $\alpha^\emptyset = 1$
- $\alpha^{\beta+1} = (\alpha^\beta) \times \alpha$
- if β is a limit ordinal, $\alpha^\beta = \sup\{\alpha^\gamma : \gamma < \beta\}$

Exercise: Consider the set $\mathcal{F}$ of functions f from β to α such that for at most finitely many times, $f(n) \neq \emptyset$. Order this set like so: $f < g$ if for the greatest x with $f(x) \neq g(x)$, then $f(x) < g(x)$. Show that the ordinal of $\mathcal{F}$ is α^β.
I talked to Deloro about this, and have the idea (build a bijection between the set of functions and the ordinal α^β), but the devil's in the details, and I don't really feel like writing it out right now. So later, possibly. One can see pg. 126 of Levy for a presentation of this same exercise, albeit with his different notation etc.

Notice that $2^\omega = \omega$ for ordinal exponentiation. If you've seen cardinal exponentiation, this looks funny.

4 The Axiom of Choice

There are several equivalent statements to the Axiom of Choice, and we are going to show their equivalence in the section that follows. (Also see Levy Section 5.1, pg. 158.)

Choice Every product of nonempty sets is not empty.

Zorn's Lemma Every nonempty inductive poset (i.e. every chain in the poset is bounded above) has a maximal element.

Zermelo Every set can be well-ordered.

Hausdorff In a poset, every chain is included in a maximal chain.

Comparison If A,B are sets, then there is an injection $A \hookrightarrow B$ or there is an injection $B \hookrightarrow A$.

Choice′ If I is a set and the A_i's are nonempty sets, then there is $h : I \to \cup A_i$ with $h(i) \in A_i$. (This is easily shown to be equivalent to Choice.)

Choice′ ⇒ Zorn's Lemma Let A be a nonempty inductive poset. By Choice, there is a choice function $h : \mathcal{P}(A) \setminus \emptyset \to A$. Let $\mathcal{C} = \{C \subseteq A : C$ is totally ordered$\}$, which is nonempty since each chain in A is totally ordered. Let $\mathcal{C}_0 = \{c \in \mathcal{C} : c$ is strictly bounded above$\}$. If $\mathcal{C}_0 \subsetneq \mathcal{C}$, then we are done, since any of the elements in $\mathcal{C} \setminus \mathcal{C}_0$ must contain maximal elements, being chains that are bounded above but not strictly.
So assume $\mathcal{C}_0 = \mathcal{C}$. Let $g : \mathcal{C} \to A$ be given by $g(c) = h(\{$strict upper bounds of c in $A\})$. Define H to be the function with domain (functions with domain an ordinal) given by

$$H(f) = \begin{cases} \emptyset & \text{if } \operatorname{im} f \notin \mathcal{C} \\ g(\operatorname{im} f) & \text{if } \operatorname{im} f \in \mathcal{C} \end{cases}$$

Let F be the H recursive functional, i.e. $F(\alpha) = H(F|_\alpha)$.

Claim: For all ordinals α, $\operatorname{im} F|_\alpha \in \mathcal{C}$.

Proof. $\operatorname{im} F|_\emptyset = \emptyset \in \mathcal{C}$. If $\operatorname{im} F|_\alpha \in \mathcal{C}$, then $F(\alpha) = H(F|_\alpha) = g(\operatorname{im} F|_\alpha) = h(\{$strict upper bounds of $\operatorname{im} F|_\alpha\})$. Recall that h is a choice function, so that means that $F(\alpha)$ is a strict upper bound of $\operatorname{im} F|_\alpha$. So $\operatorname{im} F|_{\alpha+1}$ is the same as $\operatorname{im} F|_\alpha$ with the additional element $F(\alpha)$, which is a strict upper bound and so in particular comparable with the rest of $\operatorname{im} F|_{\alpha+1}$, meaning that $\operatorname{im} F|_{\alpha+1} \in \mathcal{C}$. If α is a limit ordinal and $\forall \beta < \alpha, \operatorname{im} F|_\beta \in \mathcal{C}$, then $\operatorname{im} F|_\alpha = \cup_{\beta<\alpha} \operatorname{im} F|_\beta$ is a chain, since a union of chains which extend the ones before it is a chain. (Any element not comparable to all the rest would have to occur in some set in the union, impossible since each is a chain.) □

Claim: F is injective.

Proof. Let $\alpha \neq \beta$ be ordinals. Without loss of generality we may assume that $\alpha \geq \beta+1$. Then $F|_\alpha = g(\operatorname{im} F|_\alpha)$, a strict upper bound for $\operatorname{im} F|_\alpha \supseteq \operatorname{im} F|_{\beta+1} \ni F(\beta)$. Hence $F(\alpha) > F(\beta)$; in particular, $F(\alpha) \neq F(\beta)$. □

Armed with these two facts, we arrive at a contradiction. Let $G\colon A \to \mathrm{On}$ be such that

$$G(a) = \begin{cases} \emptyset & \text{if a is not of the form } F(\alpha) \\ \alpha & \text{if } a = F(\alpha) \text{ (well-defined by injectivity)} \end{cases}$$

By our first claim it is easy to see that $G(A) = \mathrm{On}$. By Replacement $G(A)$ is a set, but On is a proper class, so we have a contradiction. □

Zorn's Lemma ⇒ Comparison Let A, B be sets. Let $\mathcal{F}$ ={injective functions from a subset of A to a subset of B}, which is nonempty since you can always map a single element to a single element and get an injective function. Order $\mathcal{F}$ by inclusion (if you like, think of it as extension of the functions).
I claim $\mathcal{F}$ is inductive. If $\{f_\lambda\}_{\lambda \in A}$ is a chain, $f = \cup f_\lambda$ is injective and an upper bound of the chain. So every chain has an upper bound. By Zorn's Lemma, there is a maximal element $f \in \mathcal{F}$. If $\operatorname{dom} f = A$, then we have an injection $A \hookrightarrow B$. Otherwise it must be the case that $\operatorname{im} f = B$ (else we could further extend f, contradicting maximality), and then f^{-1} is an injection $B \hookrightarrow A$. □

Comparison ⇒ Zermelo Let A be a set. Assume there is an injection $f\colon A \hookrightarrow \alpha$ for some ordinal α. Then A is well-orderable (simply "pull back" the ordering from the ordinal). Now, assume there is no such injection. Then by Comparison, for every ordinal α there is an injection $\alpha \hookrightarrow A$. Consider $F\colon \mathcal{P}(A) \to \mathrm{On}$ defined as $F(P)$=the least ordinal α with $\alpha \cong P$ if such a thing exists, $F(P) = \emptyset$ otherwise. By our assumption, $F(\mathcal{P}(A)) = \mathrm{On}$ since every ordinal is congruent to some subset of A since they all inject into A. But $F(\mathcal{P}(A))$ is a set (by the Powerset and Replacement axioms), so we have reached a contradiction. □

Zermelo ⇒ Choice′ Let $\{A_i\}$ be a family of nonempty sets. Then $\cup_{i \in I} A_i$ is a set (Union axiom), and by Zermelo we may well-order it. Then let f be the function that maps i to the least element of A_i according to this ordering. f is a choice function. □

Thus we have gone round in a circle and shown that these are all equivalent. It was left as an exercise to prove any other direction, prove equivalence with Hausdorff, and prove equivalence with
Comparison′ If A, B are sets, then there is $A \twoheadrightarrow B$ or $B \twoheadrightarrow A$.
This last one was warned to be tricky.

End of Lecture 3.

Lecture 4.

We began the class with a little talk of model theory. Don't worry too much about this stuff yet.
If you want the Axiom of Choice (AC) to hold, go to a universe (HOrd, hereditarily ordinal) in which every set admits a definition by a formula using a tuple of ordinals as parameters. The actual condition is stronger. (? notes are sketchy)
Then you have a choice functional $F\colon \mathrm{HOrd} \to \mathrm{On}$. This is stronger than AC and so AC will certainly hold.
If you want AC to fail, add ω "atoms" (i.e. sets of the form $A_i = \{A_i\}$ (violates Foundation)) and do the same construction as above, allowing the atoms as parameters as well as ordinals. Call this HOrdAt (not a canonical name) (a Fraenkel-Mostowski model). In this universe, the set of atoms can't be well-ordered, and so AC fails (being equivalent to the Well-Ordering Principle and all).
The nice thing about this was no forcing was required to create these models.

Nice Consequences of AC :

- Krull's Theorem: Every proper ideal is inside a maximal ideal.
- Hahn-Banach Theorem
- Every vector space has a basis.
- The existence of ultrafilters.

Not-so-nice Consequences of AC

Proposition 4.1 (Vitali)**.** *There is a subset of* $\mathbb{R}$ *not Lebesgue-measurable.*

Proof. Consider $[0,1]$ with the equivalence relation $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$. Use AC to pick a set E of representatives of each equivalence class (there are uncountably many). Then

$$[0,1] \subseteq \Big(\coprod_{q \in \mathbb{Q} \cap [-1,1]} q + E \Big) = F \subseteq [-1,2]$$

Assume E is Lebesgue measurable. Then so is F, and $1 \leq \mu(F) \leq 3$ by monotonicity. But F is a disjoint union of ω measurable sets, each of measure $\mu(E)$. So E can not have positive measure, nor can it have zero measure. This is a contradiction. □

Axiom: (Solovay) Every subset of $\mathbb{R}$ is Lebesgue measurable.
This is consistent with ZF, although that requires forcing to prove. Clearly by the above proposition it is not consistent with ZFC.

Axiom of Countable Choice: If I is countable and the $A_i \neq \emptyset$, then $\prod A_i \neq \emptyset$. This axiom is good enough to do most analysis, but it's not so good for most algebra (recall that Zorn's Lemma is equivalent to AC, and think about how often you use that in algebra).

Theorem 4.2 (Banach-Tarski (requires AC))**.** *There is a finite partition of S^2 which properly reassembled yields two copies of S^2.*

Notation: We write $G \circlearrowright X$ for a group G acting on a set X.

Definition 4.3. *Two subsets $A, B \subseteq X$ are G-puzzle equivalent if there is an $n \in \mathbb{N}$ with partitions $A = \coprod_{i=1}^{n} A_i$, $B = \coprod_{i=1}^{n} B_i$ and elements $g_1, \ldots, g_n \in G$ with $g_i(A_i) = B_i$. In this case, write $g = \cup g_i$ when looking at these* as a function on X. *So $g \colon A \to B$ is an equivalence relation.*

Definition 4.4. *X is G-duplicable if there are $A, B \subseteq X$ with $X = A \coprod B$ and $A \sim_G B \sim_G X$.*

Claim 1 : X is G-duplicable iff there are $A, B \subseteq X$ with $A \cap B = \emptyset$ and $A \sim B \sim X$.
($\Rightarrow$) This direction is trivial.
($\Leftarrow$) In class, this direction got messed up. I will take a crack at it, keeping in mind that we were told this was basically the same thing as the Schroeder-Bernstein Theorem (see next class).
First, we denote the similarity from A to B as f, and the similarity from B to A as g. Let $D_0 = A \setminus g(B)$. For all $n \in \mathbb{N}$, $D_{n+1} = g(f(D_n))$. Let $D = \cup_{n \in \mathbb{N}} D_n$. Consider $g(B \setminus f(D)) = g(B) \setminus g(f(D))$ since g is injective. $g(B) = A \setminus D_0$ and $g(f(C)) = C$ by the definition of C, so we have $g(B \setminus f(D)) = A \setminus C$.
...

End of Lecture 4.

Lecture 5.

5 Cardinals

Theorem 5.1 (Schröder-Bernstein (No AC required))**.** *If $f \colon A \hookrightarrow B$ and $g \colon B \hookrightarrow A$ are injections of sets, then there is $h \colon A \cong B$.*

Proof. Let $C_0 = A \setminus g(B)$, and define inductively $C_{n+1} = g(f(C_n))$. Define $C := \cup_n C_n$.
Consider

$$\begin{aligned} g(B \setminus f(C)) &= g(B) \setminus g(f(C)) \text{ by injectivity of } g \\ &= (A \setminus C_0) \setminus (\cup_{n \geq 0} C_n) \\ &= A \setminus C \end{aligned}$$

$f\colon C \to f(C)$ is clearly a bijection (technically this is a restriction of f, since $C \subset A$), and $g\colon B \setminus f(C) \to A \setminus C$ is a bijection. But clearly we may combine these two functions to create a bijection between A and B. □

Remark. On groups, we may, in fact, have two groups G, H with $G \hookrightarrow H$ and $H \hookrightarrow G$ with $G \not\cong H$. For example, $\mathcal{F}_2 \hookrightarrow \mathcal{F}_3 \hookrightarrow \mathcal{F}_2$, but the two are not isomorphic.
For algebraically closed fields, Schröder-Bernstein holds.
What other categories have SB? Find out for yourself, for funsies.

Theorem 5.2 (Cantor (No AC required))**.** *If A is a set, then there is no surjection from A to $\mathcal{P}(A)$.*

Proof. This is the classic diagonalization argument, or as Deloro put it, the Iago argument. (Iago: "I am not what I am.", Scene 1, Othello, Shakespeare) You've seen it before, but here's the rundown.

Assume there is such a surjection $f\colon A \to \mathcal{P}(A)$. Define a subset C of A as follows. For all $a \in A$, $a \in C$ iff $a \notin f(a)$. Then there can be no $a' \in A$ such that $f(a') = C$, since in this case if $a' \in f(a')$, we would have that $a' \notin f(a')$, and if $a' \notin f(a')$, then $a' \in f(a')$. This shows that f is not onto, a contradiction. □

From now on, the Axiom of Choice holds. (Zermelo form, i.e. sets are well-ordered)

Definition 5.3. *Let A be a set.* Card A *is the least ordinal α such that A can be well-ordered by α, i.e. the least α such that there is a bijection $A \cong \alpha$.*

Remark. First, note that we need the Axiom of Choice to ensure that the set of αs we have to choose from is non-empty, since if A is not well-orderable, then clearly there is no α that well-orders it. Cardinals can be defined slightly differently to avoid the Axiom of Choice (see Levy, pg. 83), but this is really delaying the inevitable, since most cardinal arithmetic requires the Axiom of Choice. Also, note that we simply require a bijection here, *not* an order-preserving isomorphism. If we were working with isomorphisms, the cardinals would be the same as the ordinals, and that would not do.

In general, κ, λ, μ will denote cardinals. Cn is the collection of cardinals.

Remark. The supremum of a set of cardinals is a cardinal.

Proof. Let $X \neq \emptyset$ be a set of cardinals and $\alpha = \sup X = \cup_{\kappa \in X} \kappa$. α is an ordinal for sure, since cardinals are ordinals and we know the supremum of a set of ordinals is an ordinal. $\kappa \subseteq \alpha$ for all $\kappa \in X$, thus Card $\kappa = \kappa \leq$ Card α, whence the union of κs is also $\leq$ Card α. Clearly Card $\alpha \leq \alpha$. But since we also know that $\alpha \subseteq \cup \kappa \leq$ Card α, we must have that $\alpha =$ Card α. Thus the supremum is also a cardinal. □

Proposition 5.4. Cn *is not a set.*

Proof. Assume Cn is a set. Then let $\kappa = \sup \mathrm{Cn} \in \mathrm{Cn}$. $\mathrm{Card}\,\mathcal{P}(\kappa)$ is a cardinal, so $\mathrm{Card}\,\mathcal{P}(\kappa) \in \kappa$. This implies that $\mathrm{Card}\,\mathcal{P}(\kappa) < \kappa$ as ordinals, which implies that there is an injection from $\mathcal{P}(\kappa)$ into κ, a contradiction to Cantor, above. □

Proposition 5.5.

- $\mathrm{Card}\,A \leq \mathrm{Card}\,B \Leftrightarrow A \hookrightarrow B$
- $\mathrm{Card}\,A = \mathrm{Card}\,B \Leftrightarrow A \cong B$
- $\mathrm{Card}\,A \leq \mathrm{Card}\,B$ *or* $\mathrm{Card}\,B \leq \mathrm{Card}\,A$

Proof.

- $\mathrm{Card}\,A \leq \mathrm{Card}\,B \Leftrightarrow \mathrm{Card}\,A \hookrightarrow \mathrm{Card}\,B$ because they are ordinals. Since $A \cong \mathrm{Card}\,A$ and $B \cong \mathrm{Card}\,B$, we are done.
- Similar to above.
- This is just Comparison for ordinals.

□

Definition 5.6. *Every finite ordinal is a cardinal.*

Subdefinition: an ordinal α is finite if $\alpha = 0$ or α is a successor ordinal and $\forall \beta \leq \alpha$, β is a successor ordinal.

The same is not true of infinite ordinals. ω is a cardinal since it cannot be put into a bijection with any smaller ordinal and clearly is in bijection with itself, but $\omega + 1$ is not, since we may put $\omega + 1$ into a bijection with ω by mapping the $+1$ element to 0 and shifting the ω part over by 1. (Roughly speaking.)

Definition 5.7. *A set is finite if its cardinal is. A set is countable if its cardinal is ω. Sometimes countable also includes finite sets.*

Exercise: A is infinite iff there is $B \subsetneq A$ with $A \cong B$.
Exercise: Check the definitions of "finite" given agree.

Definition 5.8. *If κ is a cardinal, let κ^+ be defined to be the least ordinal greater than κ. There will always be some such ordinal, since as we have seen,* $\mathrm{Card}\,\mathcal{P}(\kappa) > \kappa$.
A cardinal is a successor cardinal if it is of the form κ^+, and is a limit cardinal if it is neither 0 nor ω nor a successor cardinal.

5.1 Cardinal Arithmetic

Let κ, λ be cardinals throughout.

Definition 5.9. $\kappa + \lambda = \text{Card}(\kappa \coprod \lambda)$

Proposition 5.10.

1. *$+$ is well-defined, associative, and commutative.*

2. *for all ordinals α, β $\text{Card}(\alpha + \beta) = \text{Card}\,\alpha + \text{Card}\,\beta$*

3. *for all infinite cardinals κ, λ $\kappa + \lambda = \kappa \cup \lambda = \max\{\kappa, \lambda\}$*

Proof.

1. Well-definedness follows because there is a unique least ordinal such that $\kappa \coprod \lambda$ can be well-ordered by it. Commutativity is clear, since $\coprod$ is a commutative operation. Associativity: $\kappa + (\lambda + \mu) = \kappa + (\text{Card}(\lambda \coprod \mu)) = \text{Card}(\kappa \coprod \text{Card}(\lambda \coprod \mu)) = \text{Card}(\text{Card}(\kappa \coprod \lambda) \coprod \mu)$ by following the appropriate bijections through.

2. (Sketch of proof) If you know the "put β after α" form of ordinal addition, this is easy. Say $\text{Card}\,\alpha = \gamma$ and $\text{Card}\,\beta = \delta$. Then $\gamma + \delta = \gamma \coprod \delta$ with an appropriate ordering (that's the putting δ after γ thing) and we may use our bijections to get $\alpha \coprod \beta \cong \gamma \coprod \delta$. This ordinal must be the least such, else we would be able to find an ordinal $< \gamma$ in bijection with α (or similar for δ and β). So it must be the cardinal of $\alpha + \beta$.

3. It suffices to check that $\lambda + \lambda = \lambda$. But for $\lambda \coprod \lambda$, we may put the ordering of 2λ (not $\lambda 2$) on it, and we can see through an induction that this is always λ. (In fact, we know this is the case for 2ω already.) I may come back to this to flesh this out. In the meantime, see Levy pg. 95.

□

Principle: If α is such that $\forall x \in \alpha$, $S_\alpha(x) < \beta$, then $\alpha \leq \beta$.

Proof. Assume not, i.e. that $\alpha > \beta$. Then $\beta \in \alpha$ and thus $S_\alpha(\beta) = \beta$, a contradiction. It follows that $\alpha \leq \beta$. (This is used during the induction part of the above proof.) □

Definition 5.11. $\kappa \times \lambda = \text{Card}(\kappa \times \lambda)$*, where $\times$ on the left is cardinal multiplication and on the right is Cartesian product.*

Proposition 5.12.

1. *$\times$ is well-defined, associative, and commutative.*

2. *for all ordinals α, β $\text{Card}(\alpha \times \beta) = \text{Card}\,\alpha \times \text{Card}\,\beta$*

3. *for all infinite cardinals κ, λ $\kappa \times \lambda = \kappa + \lambda$*

Proof. I will only cover the third part this time. It suffices to show that $\lambda^2 = \lambda$. The argument is very similar to the one in the third argument above, although the ordering used is the sort of expanding diagonals that one uses to show that $\mathbb{N} \times \mathbb{N}$ is the same size as $\mathbb{N}$. Again there is an induction which pays attention to the initial segments. Again I may return to this later. □

Definition 5.13. $\kappa^\lambda = \mathrm{Card}\{$*functions from* λ *to* $\kappa\}$

Proposition 5.14.

1. *This is well-defined, not associative, and not commutative.*
2. *In general* $\mathrm{Card}(\alpha^\beta) \neq \mathrm{Card}\,\alpha^{\mathrm{Card}\,\beta}$ *(*$2^\omega = \omega$*, but* $2^{\mathrm{Card}(\omega)}$ *is much bigger.)*
3. $(\kappa^\lambda)^\mu = \kappa^{\lambda \times \mu}$
4. $\kappa^\lambda \times \kappa^\mu = \kappa^{\lambda + \mu}$

Proof. Not given in class. Consult Levy, at least for now. □

End of Lecture 5.

Lecture 6.

Proposition 5.15. *If* $\kappa \leq \lambda$ *and* λ *is infinite, then* $\kappa^\lambda = 2^\lambda$.

Proof. Clearly, $\kappa^\lambda \leq \lambda^\lambda$, which is equal to $\mathrm{Card}(\{\text{functions } \lambda \to \lambda\})$. This is a set of subsets of $\lambda \times \lambda$ (Cartesian product!), and so is $\leq \mathrm{Card}(\mathcal{P}(\lambda^2)) = \mathrm{Card}(\mathcal{P}(\lambda)) = 2^\lambda$. □

5.2 The Alephs

Definition 5.16.

- $\aleph_0 = \omega$ *(viewed as a cardinal)*
- $\aleph_{\alpha+1} = \aleph_\alpha^+$, *i.e. the least cardinal greater than* $\aleph_\alpha$
- *If* α *is a limit ordinal,* $\aleph_\alpha = \sup\{\aleph_\beta | \beta < \alpha\}$

Remark. Some write ω_α for $\aleph_\alpha$. But this is terribly confusing. What is 2^{ω_0} in this case? It has two possible different meanings, since ordinal exponentiation is different than cardinal exponentiation. So we will not use such silly notation.

Proposition 5.17. *The collection of alephs is precisely the collection of infinite cardinals.*

Proof. Clearly each $\aleph$ is an infinite cardinal. So we would like to show the other direction. Assume that there are infinite cardinals that are not alephs. Let κ denote the least of these. Let $\mu = \bigcup_{\lambda<\kappa} \lambda$, λ a cardinal. I claim that μ is an aleph.

We know that each λ is actually an aleph by our choice of κ (and the fact that κ must have some infinite cardinal less than it, else it is $\aleph_0$, a contradiction), so we have $\mu = \bigcup_{\beta\in\alpha} \aleph_\beta$ for some ordinal α (α is the ordinal that corresponds to κ).

If $\alpha = \gamma + 1$, then clearly $\mu = \aleph_\gamma$, since that alephs are contained in each other and $\aleph_\gamma$ is the largest aleph in the union. If α is a limit ordinal, then $\mu = \aleph_\alpha$ by the definition of the alephs.

> **Principle**: Let κ be a limit cardinal. Then $\kappa = \bigcup_{\lambda<\kappa} \lambda$.
>
> *Proof.* Let $\mu = \bigcup_{\lambda<\kappa} \lambda$. Then μ is a cardinal, and $\mu \leq \kappa$. If $\mu < \kappa$ and there is $\mu < \mu' < \kappa$, then μ' would be included in the union that defines μ, a contradiction. This means that if $\mu < \kappa$, $\kappa = \mu^+$, a contradiction since κ is a limit cardinal. Thus $\mu = \kappa$, as desired. $\square$

If κ is a successor cardinal, then by the definition of the alephs and the fact that every infinite cardinal less than it is an aleph, it is also an aleph, a contradiction. So κ is a limit cardinal, then the above principle says $\kappa = \mu$, which we have shown to be an aleph. Thus we have arrived at a contradiction, and it must be the case that no κ exists which is an infinite cardinal but not an aleph. $\square$

Proposition 5.18. *$\aleph_\alpha$ is a limit cardinal iff α is a limit ordinal.*

Remark. This is why we do not refer to $\aleph_0$ as a limit cardinal.

Proof. If α is a successor ordinal, then $\aleph_\alpha$ is a succesor cardinal by the definition of the alephs. So let α be a limit ordinal. Assume that $\aleph_\alpha$ is a successor cardinal, that is, $\aleph_\alpha = \aleph_\gamma^+ = \aleph_{\gamma+1}$. It is easy to show that $\aleph_\beta > \aleph_\delta \Rightarrow \beta > \delta$ (assume the opposite), so we have that $\alpha > \delta$. Since α is a limit ordinal, it follows that $\delta+1 < \delta+2 < \alpha$. This leads to the chain of inequalities $\aleph_\alpha = \aleph_{\delta+1} < \aleph_{\delta+2} \leq \aleph_\alpha$, a contradiction. Thus $\aleph_\alpha$ is a limit cardinal. $\square$

Definition 5.19.

- $\beth_0 = \aleph_0$
- $\beth_{\alpha+1} = 2^{\beth_\alpha}$
- *If α is a limit ordinal,* $\beth_\alpha = \sup\{\beth_\beta | \beta < \alpha\}$

Question: Do $\aleph$ and $\beth$ agree?

The Generalized Continuum Hypothesis (GCH) says yes. The original Continuum Hypothesis is just that $\beth_1 = \aleph_1$, i.e. $2^{\aleph_0} = \aleph_1$. Gödel proved the consistency of GCH and ZF, and Cohen proved the consistency of $\neg$GCH and

ZF using forcing.

Remark. $\beth_1 = 2^{\aleph_0}$ can be many things, but it cannot be just any cardinal, as we will soon see with the help of the next concept, cofinality.

5.3 Cofinality

Definition 5.20. *Let A be a chain. $B \subseteq A$ is cofinal in A if it is not bounded above.*

Example 5.21. $\mathbb{N} \subsetneq \mathbb{Q}$ is cofinal.

Exercise: every chain A has a cofinal subchain that is well-ordered (use AC).

First, we may assume that A does not have a greatest element, since in this case that element would be a cofinal subchain that is well-ordered.

Look at the set of all well-ordered proper subchains of A. (I'm assuming proper, because otherwise this is a silly question, since A is always cofinal in A.) This set is not empty, since it includes all singletons, for instance. Define an order on this set by saying $B \leq C$ if $B \subseteq C$ and for all $b \in B$, if $c \in C \setminus B$, then $c \geq b$ (using the order of A). Roughly speaking, C extends B.

Let $D = \bigcup_{C \text{ in chain}} C$, where I'm talking about chains of well-ordered subchains. D is clearly an upper bound to the chain, since if there were C in the chain greater than D, then C would have to contain an element not in D, which is impossible by the definition of a union. So every chain of proper well-ordered subchains has a maximal element by Zorn's Lemma (aka AC).

I claim that this maximal element is a cofinal subchain. (It's clearly well-ordered.) Assume that it were not. Then there would be an element $a \in A$ such that a is greater than every element in the maximal element. But then we could add a to the maximal subchain and still have a proper well-ordered subchain (proper since a cannot be a greatest element, so there must be elements greater than it still in A), contradicting maximality. Thus this maximal subchain is cofinal. □

Definition 5.22. *Let A be a chain. Define $\operatorname{cof} A$ to be the least ordinal α such that there is a cofinal subchain $B \cong \alpha$, i.e. the least ordinal α such that there is $f\colon \alpha \to A$ with $f(\alpha)$ not bounded.*

Example 5.23.

- $\operatorname{cof} 0 = 0$
- $\operatorname{cof} 1 = 1$
- $\operatorname{cof} 2 = 1$, since you can map 1 to the "top" element of 2 and that map is unbounded.
- In fact, the above reasoning shows that $\operatorname{cof} \alpha + 1 = 1$.

- $\operatorname{cof} \alpha = 1$ iff α is a successor ordinal.
- $\operatorname{cof} \omega = \omega$
- $\operatorname{cof} \omega + \omega = \omega$, since we can map ω to the "second" ω in $\omega + \omega$ and that is unbounded.
- $\operatorname{cof} \omega_1 + \omega = \omega$ for basically the same reason. ($\omega_1 = \aleph_1$ viewed as an ordinal.)

A few inconvenient facts about cofinality:

- $\beta < \alpha$ does not imply $\operatorname{cof} \beta < \operatorname{cof} \alpha$. For example, $\operatorname{cof} \omega > \operatorname{cof}(\omega + 1)$.
- $\operatorname{cof} \beta \leq \alpha$ does not imply $\operatorname{cof} \beta \leq \operatorname{cof} \alpha$. Again, consider ω and $\omega + 1$.
- Luckily, $\beta \leq \operatorname{cof} \alpha$ does imply $\operatorname{cof} \beta \leq \operatorname{cof} \alpha$, since $\operatorname{cof} \beta \leq \beta$ (look at the identity map on β).

Lemma 5.24. $\operatorname{cof} \alpha$ *is a cardinal, and* $\operatorname{cof} \operatorname{cof} \alpha = \operatorname{cof} \alpha$. *Also,* $\operatorname{cof} \alpha \leq \operatorname{Card} \alpha$.

Proof. Let $\kappa = \operatorname{Card}(\operatorname{cof} \alpha)$. We know that $\kappa \leq \operatorname{cof} \alpha$ by the definition of cardinality. We know there is a bijection from κ to $\operatorname{cof} \alpha$ (definition of cardinal) and an unbounded map from $\operatorname{cof} \alpha$ into α, so composing the two gives an unbounded map from κ into α. The definition of cofinality then tells us that $\kappa \geq \operatorname{cof} \alpha$, so $\operatorname{cof} \alpha = \kappa$, a cardinal. Also, since $\alpha \geq \operatorname{cof} \alpha = \kappa$, we have that $\operatorname{Card} \alpha \geq \operatorname{Card} \kappa = \kappa$, thus $\operatorname{Card} \alpha \geq \operatorname{cof} \alpha$.

It remains to prove that $\operatorname{cof} \operatorname{cof} \alpha = \operatorname{cof} \alpha$. We already know that $\operatorname{cof} \operatorname{cof} \alpha \leq \operatorname{cof} \alpha$. $\operatorname{cof} \operatorname{cof} \alpha$ has an unbounded map into $\operatorname{cof} \alpha$, which has an unbounded map into α, so composing the two gives an unbounded map from $\operatorname{cof} \operatorname{cof} \alpha$ into α, and thus by the definition of cofinality, $\operatorname{cof} \operatorname{cof} \alpha \geq \operatorname{cof} \alpha$. The two inequalities combined give the desired result. □

5.3.1 Is cof onto?

Definition 5.25. *An ordinal* α *is regular if* $\operatorname{cof} \alpha = \alpha$.

Remark. Regular ordinals are cardinals. $\operatorname{Card}(\alpha) \leq \alpha = \operatorname{cof} \alpha$, and we saw above that $\operatorname{Card} \alpha \geq \operatorname{cof} \alpha$, so $\operatorname{Card} \alpha = \operatorname{cof} \alpha = \alpha$.

Definition 5.26. *A cardinal is called singular if it is not regular.*

Lemma 5.27. *A successor cardinal is always regular. In particular,* $\operatorname{cof} \aleph_1 = \aleph_1$.

Proof. Let $\aleph_{\alpha+1}$ be a successor cardinal. Let $f\colon \kappa \to \aleph_{\alpha+1}$ be unbounded. We prove $\kappa \geq \aleph_{\alpha+1}$.

Assume not. Then $\kappa \leq \aleph_\alpha$. By assumption, $\aleph_{\alpha+1} = \bigcup_{\beta\in\kappa} f(\beta) = \bigcup_{\beta\in\kappa} S_{\aleph_{\alpha+1}}(f(\beta))$ since $f(\beta) \in \aleph_{\alpha+1}$. But $\operatorname{Card}(f(\beta)) \leq \aleph_\alpha$. Thus since we have $\kappa \leq \aleph_\alpha$ copies of $f(\beta)$s which have cardinality $\leq \aleph_\alpha$ making up $\aleph_{\alpha+1}$, this implies $\aleph_{\alpha+1} \leq \aleph_\alpha \times \aleph_\alpha = \aleph_\alpha$, a contradiction. □

Exercise: If α is a limit ordinal, then $\operatorname{cof} \aleph_\alpha = \operatorname{cof} \alpha$.

Let $f : \alpha \to \aleph_\alpha$ be the function $f(\delta) = \aleph_\delta$ for all $\delta < \alpha$. This is unbounded. Thus we may compose the unbounded map from $\operatorname{cof} \alpha$ to α with f and get an unbounded map from $\operatorname{cof} \alpha$ to $\aleph_\alpha$. Then by definition, $\operatorname{cof} \aleph_\alpha \le \operatorname{cof} \alpha$. Similarly there is an unbounded map from $\operatorname{cof} \aleph_\alpha$ to α given by mapping elements of $\operatorname{cof} \aleph_\alpha$ to $x \in \aleph_\alpha$ and then to the element of α that corresponds to the subscript of the aleph with cardinality the same as x. Thus the reverse inequality holds, and we get that $\operatorname{cof} \aleph_\alpha = \operatorname{cof} \alpha$.

Proposition 5.28. *Let κ be a cardinal $\ge \aleph_0$. Then κ is singular iff there are less than κ sets of cardinality $< \kappa$ whose union has cardinality κ.*

Remark. We will show that $\operatorname{cof} \kappa =$ the least λ such that there exists $\{A_\alpha\}_{\alpha\in\lambda}$ with $\operatorname{Card} A_\alpha < \kappa$, $\bigcup_{\alpha\in\lambda} A_\alpha = \kappa$, from which the proposition follows easily.

Proof. ($\Rightarrow$) Note that if κ is singular, it must be that $\operatorname{cof} \kappa < \kappa$. Let $f\colon \operatorname{cof} \kappa \to \kappa$ be unbounded. For all $\alpha \in \operatorname{cof} \kappa$, let $A_\alpha = f(\alpha)(= S_\kappa(f(\alpha)))$. Then $\operatorname{Card} A_\alpha < \kappa$ since A_α is a proper initial segment of κ, and $\bigcup_{\alpha\in\operatorname{cof}\kappa} A_\alpha = \kappa$.

($\Leftarrow$) Now assume that there are $\{A_\alpha\}_{\alpha\in\lambda}$ with $\operatorname{Card} A_\alpha < \kappa$, $\bigcup_{\alpha\in\lambda} A_\alpha = \kappa$ and λ is the least such ordinal. We may assume that the A_αs are disjoint. Let $A = \bigcup_{\alpha\in\lambda} A_\alpha$ be well ordered by $x < y$ if $x \in A_\alpha$, $y \in A_\beta$ and $\alpha < \beta$ or x and y are in the same A_α and $x < y$ in there.

Every initial segment of A has cardinality $< \kappa$. If it had cardinality $= \kappa$, then the largest ordinal used to index the sets in the initial segment would satisfy the condition stated above, contradicting the minimality of λ.
This implies that each initial segment is $< \kappa$ as ordinals, since if an initial segment equalled κ it would have cardinality κ, so by our principle the ordinal corresponding to $A \le \kappa$. However $\operatorname{Card} A = \kappa$, so the ordinal corresponding to A must have cardinality κ, so as a well-ordering $A = \kappa$.

Now consider $g\colon \lambda \to \kappa$ given by $g(\beta) = \min A_\beta$. (Note that this uses the fact that $\kappa = A$.) g is cofinal in κ, hence $\lambda \ge \operatorname{cof} \kappa$. □

End of Lecture 6.

Lecture 7.

We start the day by noting that cof is really only interesting for cardinals, and then move right on to a theorem. Today we will show what was promised last class, that $2^{\aleph_0}$ can't be just any cardinal.

Theorem 5.29 (König Sr.). *Let I be a set, and for each $i \in I$, A_i and B_i are sets with $\operatorname{Card} A_i < \operatorname{Card} B_i$. Then $\operatorname{Card} \cup_{i\in I} A_i < \operatorname{Card} \prod_{i\in I} B_i$.*

Proof. We may assume that the A_is are disjoint (in which case $\operatorname{Card} \cup_{i\in I} A_i = \operatorname{Card} \sum_{i\in I} A_i$) and $A_i \subset B_i$. First we prove $\operatorname{Card} \cup_{i\in I} A_i \leq \operatorname{Card} \prod_{i\in I} B_i$ by constructing an injection from one to the other.

Since $\operatorname{Card} A_i < \operatorname{Card} B_i$, $\exists b_i \in B_i \setminus A_i$. Now, consider the mapping from $A_{i_0} \to \prod B_i$ given by $a_{i_0} \mapsto (b_1, \ldots, a_{i_0}, \ldots, b_j)$ where a_{i_0} is in the i_0-th coordinate. Clearly we may use this to obtain an injective function $g\colon \coprod A_i \to \prod B_i$.

Secondly, we prove $\operatorname{Card} \sum A_i < \operatorname{Card} \prod B_i$ (strict inequality). Let $f\colon \coprod A_i \to \prod B_i$. We will show that f can not be onto.

Consider $\pi_i\colon \prod_j B_j \to B_i$, the ith canonical projection. Now look at $\pi_i \circ f|_{A_i}(A_i) \subseteq B_i$. Note that this composition must have cardinality $\leq \operatorname{Card} A_i$, since $\pi_i \circ f|_{A_i}$ can not "blow up" the size of its range past that of its domain. Hence (similar to above) $\exists b_i \in B_i \setminus \pi_i \circ f|_{A_i}(A_i)$. Let $\underline{b} := (b_i)_{i\in I}$. We claim that $\underline{b} \notin \operatorname{im} f$.

Assume the opposite, i.e. $\exists (a_j) \in \coprod A_j$ with $\underline{b} = f((a_j))$. Then $b_j = \pi_j(\underline{b}) = \pi_j \circ f((a_j)) = \pi_j \circ f|_{A_j}(a_j) \in \pi_j \circ f|_{A_j}(A_j)$, a contradiction. Note that in moving from the third term to the fourth, we are justified in the restriction since $f|_{A_j}$ only affects the jth coordinate of $\prod B_i$, and no other restriction does so. Thus we are done with the proof. □

A few notes on the proof: The business with the restriction was not explicitly treated in class, but needs to be done for his proof to work. Hopefully I did it correctly; the idea is correct in any case. As far as the structure of the proof goes, I'm not entirely sure why we couldn't just skip ahead to showing that there is no surjection $f\colon \coprod A_i \to \prod B_i$. Levy contains a similar proof on pg. 107 (which also first shows an injection), and notes that we are essentially using the diagonal argument of Cantor in the second part, so in some sense this theorem is a generalization of Cantor's.

Remark. This theorem implies the Axiom of Choice. If $B_i \neq \emptyset$ for all $i \in I$, then $\operatorname{Card} \prod_{i\in I} B_i > \operatorname{Card} \cup_{i\in I} \emptyset = 0$, so the product of nonempty sets is nonempty.

Corollary. *Let κ be an infinite cardinal. Then $\kappa < \kappa^{\operatorname{cof} \kappa}$.*

Proof. By the theorem on cofinality we proved last time, there are $\{A_\alpha\}_{\alpha\in\operatorname{cof}\kappa}$ with $\operatorname{Card} A_\alpha < \kappa$ for all α and $\cup_{\alpha\in\operatorname{cof}\kappa} A_\alpha = \kappa$. By König's Theorem,

$$\operatorname{Card}\left(\sum_{\alpha\in\operatorname{cof}\kappa} A_\alpha\right) < \operatorname{Card} \prod_{\alpha\in\operatorname{cof}\kappa} \kappa \Rightarrow \kappa < \kappa^{\operatorname{cof}\kappa}$$ □

Remark. $2^{\aleph_0} \neq \aleph_\omega$

Proof. Assume $2^{\aleph_0} = \aleph_\omega$. Then $\operatorname{cof} 2^{\aleph_0} = \operatorname{cof} \aleph_\omega = \aleph_0$. This last equality follows from the theorem we proved last time that for every limit ordinal α, $\operatorname{cof} \aleph_\alpha = \operatorname{cof} \alpha$, so in this particular case $\operatorname{cof} \aleph_\omega = \operatorname{cof} \omega = \aleph_0$. Hence $(2^{\aleph_0})^{\operatorname{cof}(2^{\aleph_0})} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0^2} = 2^{\aleph_0}$. But this contradicts the above corollary to König, which states that $(2^{\aleph_0})^{\operatorname{cof}(2^{\aleph_0})} > 2^{\aleph_0}$. □

6 Counting models

6.1 A first example: The number of dense linear orderings

Here, I introduce a new reference. *Marker* refers to Model Theory: An Introduction by David Marker.

Definition 6.1. *A dense linear ordering (DLO) is an ordering* $(A, <)$ *that is:*

linear/total: $\forall x, y\ (x < y \vee x = y \vee x > y)$

dense: $\forall x, y\ (x < y \rightarrow \exists z(x < z < y))$

without endpoints: $\forall x\ \exists y, z(y < x < z)$

See Marker pg. 15 for a slightly more formal definition. But really, it's not bad. Also, the "without endpoints" part of the definition technically is not necessary for a DLO, but it will make them easier to work with, so we will always mean a DLO without endpoints.

Example 6.2.

- $\mathbb{Q}$ (the motivating example)
- $\mathbb{R}$
- $\mathbb{Q} \cap [0, 1]$
- $\mathbb{Q} \coprod_{<} \mathbb{Q}$, which stands for $\mathbb{Q}$ followed by a copy of $\mathbb{Q}$ (sort of like $\omega 2$ is with ω).
- $\mathbb{Q} \coprod_{<} \{\infty\} \coprod_{<} \mathbb{Q}$, which is the same as above, except there is a point (∞) which is greater than all the points in the first copy of $\mathbb{Q}$ and less than all the points in the second copy of $\mathbb{Q}$. These two examples will be important in the next class.

Theorem 6.3. *DLO's are* $\aleph_0$*-categorical, i.e. any two countable DLOs are (order) isomorphic.*

Proof. This proof introduces an important method known as the Back-and-Forth Method. It is pretty much what it sounds like, as we'll see. The idea here is to build an isomorphism inductively, going back and forth between the domain and the range (i.e. the two DLOs). I have chosen to more-or-less recreate the proof in Marker (pg. 48) because although the idea came across in class, we got a little bogged down in the indices.

Let $(A, <)$ and $(B, <)$ be DLOs of cardinality $\aleph_0$. Thus we may enumerate their elements $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$. The base of our induction will be the function $f_0 = \emptyset$. Now, we proceed to the inductive step, in which we will assume that f_n maps n elements of A into B in such a way that preserves the order and is injective. We will call the domain of f_n A_n, and the range B_n.

Step $n+1$, $n+1 = 2m+1$ (i.e. $n+1$ odd). We want to make sure that a_m is included in A_{n+1}. If $a_m \in A_n$, we let $f_{n+1} = f_n$ and proceed. Otherwise, we need to find a suitable element in B for a_m to map to. We need to preserve the ordering. So, if $\forall a_k \in A_n$, $a_m < a_k$, we map a_m to an element in B less than all of the elements in B_n. Such an element exists since B is a DLO. Similarly if $a_m > a_k$ for all $a_k \in A_n$. The only other case is that a_m is between two elements in A_n (since A_n finite), say $\alpha < a_m < \beta$. Then we simply pick $b \in B \setminus B_n$ such that $f_n(\alpha) < b < f_n(\beta)$ (which exists since B is a DLO) and let $f_{n+1}(a_m) = b$. Let $f_{n+1}|_{A_n} = f_n$, and we know by our inductive assumption combined with our choice of b that f_{n+1} is injective and preserves order.

Step $n+1$, $n+1 = 2m$ (i.e. $n+1$ even). On this step, we want to make sure that b_m is included in B_{n+1}. If $b_m \in B_n$, we let $f_{n+1} = f_n$. Otherwise, we do the same thing as above, just backwards.

So, at the odd stages we have made sure that $\cup_n A_n = A$, and at the even stages we have made sure that $\cup_n B_n = B$. Since at each step f_n is injective, we have that $f = \cup_n f_n$ is an isomorphism between A and B. □

Definition 6.4. *The transcendence degree of an extension is the largest cardinal of an algebraically independent family.*

Remark. The algebraic degree of $\mathbb{C}/\mathbb{R} = 2$, but the transcendence degree of $\mathbb{C}/\mathbb{R} = 0$. When not otherwise specified, transcendence degree is the transcendence degree over the algebraic closure of the prime field.

Theorem 6.5. *Let $\mathbb{K}$, $\mathbb{L}$ be two algebraically closed fields of infinite transcendence degree. Assume there is $< \underline{a} > \cong < \underline{b} >$ where $\underline{a} \in \mathbb{K}$ and $\underline{b} \in \mathbb{L}$ are tuples of the same (finite) length and $f(a_i) = b_i$ for all $i < n$. (f is the isomorphism.) Then $\forall \alpha \in \mathbb{K}$, one can extend f to $< \underline{a}, \alpha >$.*

Proof. Write $k = < \underline{a} >$ and $l = < \underline{b} >$. We "place" α with respect to k. (This is the "forth" step.) If α is algebraic over k, let $\mu = \mathrm{Irr}_k^\alpha$, the minimum polynomial of α over k. Then $f(\mu) \in \mathbb{L}[x]$ has a solution β in $\mathbb{L}$. Map α to β. If α is transcendental over k, pick any β transcendental and it will work. The "back" step would be similar. □

This theorem is actually essential for a one line proof of Hilbert's Nullstellensatz, which roughly speaking says "Algebraically closed fields eliminate quantifiers". More formally:

Theorem 6.6 (Nullstellensatz)**.** *Let $\mathbb{K}$ be an algebraically closed field and $m \lhd \mathbb{K}[\underline{x}]$ be a maximal ideal. Then $\exists \underline{a} \in \mathbb{K}$ such that $\forall P \in m$, $P(\underline{a}) = 0$.*

We will see the proof in a few weeks.

End of Lecture 7.

Lecture 8.

Our goal for today is to construct non-isomorphic dense linear orderings (DLOs) of cardinality $\aleph_1$. I note that there were several pictures to go along with the extensions we are about to see; I may figure out how to put them in here later, but until then, imagine.

Definition 6.7. *Let A be a DLO. A rational right extension is a DLO* $B = A \coprod_{<} \{\infty\} \coprod_{<} \mathbb{Q}$.

Example 6.8.

- $\mathbb{Q} \subseteq \mathbb{Q} \coprod \{\infty\} \coprod \mathbb{Q}$
- $\mathbb{Q} \cap (-\infty, 0) \subset \mathbb{Q}$, where the subset is considered A, so $B = \mathbb{Q}$, since the positive rationals are isomorphic to all rationals, and 0 is not in either A nor the positive rationals, so ∞ takes the place of 0.

Definition 6.9. *An irrational extension of A is* $B = A \coprod \mathbb{Q}$.

Example 6.10.

- $\mathbb{Q} \subseteq \mathbb{Q} \coprod \mathbb{Q}$
- $\mathbb{Q} \cap (-\infty, \sqrt{2}) \subset \mathbb{Q}$. This creates something like a Dedekind cut at $\sqrt{2}$, and we don't put in a point as in the rational extension, which reflects the fact that $\sqrt{2}$ is not in the rationals.

We now construct to non-isomorphic DLOs. Consider the sets R_γ constructed by

$$\begin{aligned} R_\emptyset &= \mathbb{Q} \\ R_{\alpha+1} &= \text{the rational right extension of } R_\alpha \\ R_\alpha &= \bigcup_{\beta<\alpha} R_\beta \quad \text{if } \alpha \text{ is a limit ordinal} \end{aligned}$$

Similarly, consider the sets I_δ constructed by

$$\begin{aligned} I_\emptyset &= \mathbb{Q} \\ I_{\alpha+1} &= \text{the irrational right extension of } R_\alpha \\ I_\alpha &= \bigcup_{\beta<\alpha} R_\beta \quad \text{if } \alpha \text{ is a limit ordinal} \end{aligned}$$

A picture would be nice here. Essentially, the R_γs look like a bunch of copies of $\mathbb{Q}$ side by side with single points (the ∞s) separating them, while the I_δs look just like a bunch of copies of $\mathbb{Q}$ side by side. Both are DLOs.

Remark. For all $\alpha < \aleph_1$, $R_\alpha \cong I_\alpha \cong \mathbb{Q}$, as they are all countable DLOs, which we saw last time were all isomorphic.

If we look at R_2 and I_2 and say they're split up into two copies of the rationals (with a point between them in the case of R_2), we note that one of the rationals in R_2 (say the left, lesser one) can not map to a copy of the rationals in I_2. This is because the copy of the rationals in R_2 has a least upper bound (∞), but this is not true for either copy of the rationals in I_2. Obviously this type of situation occurs in higher indexed Rs and Is as well.

Claim: $R_{\aleph_1} \not\cong I_{\aleph_1}$

Proof. Assume there is $f\colon R_{\aleph_1} \to I_{\aleph_1}$. There can be no $\alpha < \aleph_1$ such that $f(R_\alpha) = I_\alpha$, as R_α has a least upper bound in $R_{\aleph_1}$, but I_α does not. So, let α_1 be minimal with $f(R_1) \subseteq I_{\alpha_1}$. Let α_2 be minimal with $f^{-1}(I_{\alpha_1}) \subseteq R_{\alpha_2}$. Continue this process in the obvious way. Let $\alpha = \sup_\omega \alpha_n < \aleph_1$. (We know this inequality holds from our results on cofinality and the fact that $\aleph_1$ is regular.) Then $f(R_\alpha) = f(\bigcup_{i<\omega} R_{\alpha_i}) \subseteq \bigcup_{i<\omega} I_{\alpha_{i+1}} = I_\alpha$. Similarly $f^{-1}(I_\alpha) \subseteq R_\alpha$. Hence $f(R_\alpha) = I_\alpha$, a contradiction. ☐

Definition 6.11. *Let κ be a regular cardinal. A club is a subset $C \subset K$ such that:*

C **is CLosed** *for all $\alpha < \kappa$, if $\{c_\beta\}_{\beta<\alpha} \subseteq C$, then* $\sup_{\beta<\alpha} c_\beta \in C$.

C **is UnBounded** *for all $\alpha \in \kappa$, $\exists \beta \in C$ such that $\beta > \alpha$.*

Example 6.12.

- $\kappa \subseteq \kappa$, obviously
- the even numbers as a subset of $\aleph_0$. Note that for the closed criterion one is only considering finite sequences.
- if κ is a regular cardinal, any club has cardinality κ (this is related to cofinality).
- the limit ordinals less than $\aleph_1$ are a club. Any "limit of limits" is a limit, and for any $\alpha < \aleph_1$ we know that $\alpha \leq \alpha + \omega$, a limit ordinal less than $\aleph_1$.
- In fact, limit ordinals less than κ with κ regular form a club.

Exercise: Find a subset of $\aleph_1$ of cardinality $\aleph_1$ that is not a club.

To be fair, I'm typing these up after the next class, where he told us the answer to this one. If you take the set of all ordinals less than $\aleph_1$ and remove the limit ordinals, you have an unbounded set (i.e. a cofinal set). (Given $\alpha < \aleph_1$, you always have $\alpha + 1$.) Since $\aleph_1$ is regular, this set must have cardinality $\aleph_1$. However, it clearly is not closed, and thus not a club.

Intuitively, clubs can be understood as "measure 1" subsets of κ.
In the proof of $R_{\aleph_1} \not\cong I_{\aleph_1}$, there was a club $\{\alpha < \aleph_1 | f(R_\alpha) = I_\alpha\}$. Work similar to the part in the proof where we let α be a limit ordinal shows this is closed. It's unbounded since rather than starting from 1 the proof could have started from any $\alpha < \aleph_1$.

Now we will generalize the construction from earlier to find more non-isomorphic DLOs.

For any $X \subset \aleph_1$, consider the sequence

$$\begin{aligned} D_\emptyset &= \mathbb{Q} \\ D_{\alpha+1} &= \text{the rational right extension of } D_\alpha \text{ if } \alpha \in X \\ D_{\alpha+1} &= \text{the irrational right extension of } D_\alpha \text{ if } \alpha \notin X \\ D_\alpha &= \bigcup_{\beta<\alpha} D_\beta \text{ if } \alpha \text{ is a limit ordinal} \\ D(X) &= D_{\aleph_1}(X) \end{aligned}$$

This construction gives $2^{\aleph_1}$ DLOs, since there is a DLO associated to every subset of $\aleph_1$. We'll prove that in fact this gives $2^{\aleph_1}$ nonisomorphic DLOs, although we will not finish this proof until the next class. This is maximal since there are at most $\mathcal{P}(\aleph_1 \times \aleph_1)$ binary relations on $\aleph_1$. We will show a necessary and sufficient condition for two of the above DLOs to be nonisomorphic today, and use this in the next class for our proof about the number of nonisomorphic DLOs.

Definition 6.13. *A subset $S \subseteq \kappa$ is stationary if for all clubs $C \subseteq \kappa$, $S \cap C \neq \emptyset$. "S misses no club's meetings."*

Stationary sets may be understood as "non-negligible" sets, or "sets of positive measure".

Example 6.14. Any club is stationary if $\kappa \geq \aleph_1$. Proof: Let C_1, C_2 be two clubs. Let $\alpha_1 \in C_1$. $\exists \alpha_2 \in C_2$ with $\alpha_2 > \alpha_1$ since C_2 is unbounded. $\exists \alpha_3 \in C_1$ with $\alpha_3 > \alpha_2$ since C_1 is unbounded. Et cetera. Let $\alpha = \sup_\omega \alpha$. By the closedness of C_1 and C_2, $\alpha \in C_1 \cap C_2$. □

Theorem 6.15. *$D(X) \not\cong D(Y) \Leftrightarrow X \triangle Y$ is stationary.*

Proof. ($\Leftarrow$) For this direction, the argument will only use the fact that $\aleph_1$ is regular, and so this direction holds if we generalize the above construction for regular κ.

Assume there is $f\colon D(X) \cong D(Y)$. We will show $X \triangle Y$ can't be stationary. Let $C \coloneqq \{\alpha < \aleph_1 | f(D_\alpha(X)) = D_\alpha(Y)\}$. We claim that C is a club. (We mentioned this earlier for our first construction.) If $X \triangle Y$ is stationary, then without loss of generality we may assume there is some $\alpha \in C \cap (X \setminus Y)$. Hence $f(D_\alpha(X)) = D_\alpha(Y)$ by the definition of C. $D_{\alpha+1}(X)$ is the rational extension of $D_\alpha(X)$ (since $\alpha \in X$), hence $D_\alpha(X)$ has a least upper bound in $D(X)$. However, $\alpha \notin Y$, so $D_{\alpha+1}(Y)$ is the irrational extension of $D_\alpha(Y)$, meaning $D_\alpha(Y)$ has no such least upper bound in $D(Y)$, a contradiction.

($\Rightarrow$) This direction explicitly relies on the fact that $\aleph_1 = \aleph_0^+$.

Assume that $X \triangle Y$ is not stationary. Then there is a club C that avoids $X \triangle Y$. We will use C as a "ladder" to work our way through the set. Enumerate $C = \{c_\beta\}_{\beta<\alpha}$.

We can begin by saying there is $f_0 \colon D_{c_0}(X) \cong D_{c_0}(Y)$ since both are countable. (This is where the fact that we're using $\aleph_1$ is important.) Now, assume that we have constructed $f_\beta \colon D_{c_\beta}(X) \cong D_{c_\beta}(Y)$. If $c_\beta \in X$, then by definition $D_{c_\beta+1}(X)$ is the rational extension of $D_{c_\beta}(X)$. Since $C \cap X \triangle Y = \emptyset$, we have $c_\beta \in Y$ as well. Hence $D_{c_\beta+1}(Y)$ is the rational extension of $D_{c_\beta}(X)$ as well. Therefore f_β extends to $f_{\beta+1} \colon D_{c_\beta+1}(X) \cong D_{c_\beta+1}(Y)$.
If $c_\beta \notin X$ then $c_\beta \notin Y$, and one has a similar situation to above except with the irrational extensions. In either case, f_β extends to $f_{\beta+1} \colon D_{c_\beta+1}(X) \cong D_{c_\beta+1}(Y)$. Note that we only know it must be an isomorphism by the $\aleph_0$-categoricity of DLOs, so we are again relying on the fact that we're in $\aleph_1$.

Finally, it's clear by our definition that we can extend f_β to limit ordinals as well. Repeating this step and again appealling to the definition, we get $f \colon D(X) \cong D(Y)$. □

Next time we will be counting stationary sets. We also noted (although this is related to an earlier class) that if $\operatorname{cof} \kappa > \aleph_0$, then it may be the case that $2^{\aleph_0} = \kappa$, though this requires forcing to show.

End of Lecture 8.

Lecture 9.

We begin by recalling the definition of closed for clubs: $\forall \alpha < \kappa$, and $\{c_\beta\}_{\beta<\alpha} \subseteq C$, then $\sup_\alpha c_\beta \in C$. We note that it actually suffices to show this for any $\alpha < \operatorname{cof} \kappa$. By the property of cofinality we proved near the end of Day 6, we know that in any club $\operatorname{cof} \kappa$ is in some sense the "number of pieces" that κ can be split into, so more "pieces" is a bit superfluous.

7 Counting stationary sets

Today we will prove that in regular κ, one can find κ disjoint stationary sets. This will allow us to finish our proof from the previous class that there are $2^{\aleph_1}$ non-isomorphic DLOs of cardinality $\aleph_1$. One step in this proof will be what's known as Fodor's Theorem. We will build up to this theorem using two lemmas.

κ is considered to be a regular cardinal throughout today's notes.

Lemma 7.1 ("Intersection"). *Let $\alpha < \kappa$, and $\{C_\beta\}_{\beta\in\alpha}$ be clubs. Then $\cap_{\beta<\alpha} C_\beta$ is a club.*

Proof. Let $C = \cap_{\beta<\alpha} C_\beta$. We will prove that C is a club.

First we would like to show that C is closed. Let $\{c_\gamma\}_{\gamma<\delta} \subseteq C$ for some $\delta < \kappa$. Then, let $c \coloneqq \sup_\delta(c_\delta)$. We need to prove that $c \in C$. For any $\beta < \alpha$, $\{c_\gamma\}_{\gamma<\delta} \subseteq C_\beta$ since the elements are all in the intersection, and hence by the closedness of each C_β, we have that $c \in C_\beta$. Hence $c \in C$.

Next, we show that C is unbounded. (There's a nice picture describing this proof, but as with last class, I currently will not be reproducing it here. Sorry.) Let $\gamma_0 \in C$. We want to find an element in C greater than γ_0. We will do so inductively.

Since C_0 is unbounded, $\exists \delta_{0,0} \in C_0$ with $\delta_{0,0} > \gamma_0$. (This is not necessarily in C though, so we press on.) Similarly, given $\delta_{0,\alpha}$, $C_{\alpha+1}$ is unbounded so we may find $\delta_{0,\alpha+1} > \delta_{0,\alpha}$. If ε is a limit ordinal less than α, we have $\sup_{\zeta\in\varepsilon} \delta_{0,\zeta} < \kappa$, hence $\exists \delta_{0,\varepsilon} \in C_\varepsilon$ with $\delta_{0,\varepsilon} > \sup_{\zeta\in\varepsilon} \delta_{0,\zeta}$.

Let $\gamma_1 \coloneqq \sup_{\beta\in\alpha} \delta_{0,\beta} \in \kappa$. Repeat the above process to find $\delta_{1,0} > \gamma_1$, $\delta_{1,1} > \delta_{1,0}$, etc. to get $\gamma_2 \coloneqq \sup_{\beta\in\alpha} \delta_{1,\beta}$. Continue until you have $\gamma_\omega = \sup_{n\in\omega} \gamma_n \in \kappa$. We check that $\gamma_\omega \in C$.

Let $\beta \in \alpha$. We will prove that $\gamma_\omega \in C_\beta$. By construction, $\gamma_\omega = \sup_{n\in\omega} \delta_{n,\beta}$, which is in C_β since C_β is closed. Hence γ_ω is in every C_β, whence $\gamma_\omega \in C$. Thus C is unbounded, since $\gamma_\omega > \gamma_0$. □

Corollary. *Let $S = \bigcup_{\beta<\alpha} X_\beta$ for some $\alpha < \kappa$. Then S is stationary iff some X_β is.*

Proof. The intersection of all clubs in κ is a club by the above lemma. It is also nonempty since every club is stationary (see last class). If S is stationary, then this club intersects it nontrivially, and by definition of union, it intersects some X_β. But this means that every club intersects this X_β, whence X_β is a club. The other direction is trivial. □

Lemma 7.2 ("Diagonal intersection"). *Let $\{C_\beta\}_{\beta<\kappa}$ be clubs. Then $\{\alpha \in \kappa \colon \forall\beta < \alpha,\ \alpha \in C_\beta\}$ (the set of ordinals α lying in all clubs of index less than α) is a club.*

Proof. Let $C = \{\alpha \in \kappa \colon \forall\beta < \alpha,\ \alpha \in C_\beta\}$. Similar to the last lemma, we will prove that C is closed and bounded.

First, we will show that C is closed. Let $\{c_\gamma\}_{\gamma<\delta} \subseteq C$. We may assume that $\{c_\gamma\}$ is increasing. Let $c \coloneqq \sup_\delta c_\delta$. We want to prove $c \in C$. To do so, we aim to prove that $\forall\beta < c$, $c \in C_\beta$.

Let $\beta < c$. There is γ_0 such that $c_{\gamma_0} > \beta$. Since $\{c_\gamma\}$ is increasing, $\gamma_0 < \gamma < \delta \Rightarrow c_\gamma > c_{\gamma_0} > \beta$. By our definition of C, $c_\gamma \in C_\beta$. In particular, using this observation, $c = \sup_{\gamma<\delta} c_\gamma \in C_\beta$, since C_β is closed.

Next, we prove that C is unbounded. Let $\gamma_0 \in \kappa$. We want to find an element of C greater than γ_0.

Let $\alpha \in \kappa$. By Lemma 1, $I_\alpha \coloneqq \cap_{\beta<\alpha} C_\beta$ is a club. Hence there is $f(\alpha) \in I_\alpha$ with $f(\alpha) > \alpha$. This defines a function $f\colon \kappa \to \kappa$ which, given an element of κ, takes this intersection and finds an element in the club greater than the index of the intersection. Let $\gamma = \sup_{n<\omega} f^n(\gamma_0)$, which is clearly greater than γ_0. We check that $\gamma \in C$.

Similar to before, we will let $\beta < \gamma$ and show that $\gamma \in C_\beta$. We note that $\beta < \gamma$ implies there is an $n \in \omega$ with $\beta < f^n(\gamma_0)$ just by the way that γ is defined. Now $f_{n+1}(\gamma_0) \in I_{f^n(\gamma_0)}$, by definition of f, and $I_{f^n(\gamma_0)} \subseteq C_\beta$ since it's an intersection of clubs including C_β. The same can be said for any $f^m(\gamma_0)$ where $m > n$, using the same reasoning. Hence the "tail" of $\gamma = \sup_{n<\omega} f^n(\gamma_0)$ is in C_β, and thus by closedness of C_β, $\gamma \in C_\beta$. Therefore, by the definition of C, $\gamma \in C$, whence C is unbounded. □

Note: The approach in the last portion of this proof is similar to what needs to be done for problem 2 in the exercises Deloro gave us.

Theorem 7.3 (Fodor)**.** *Let S be stationary in κ (regular). Let $f\colon S \to \kappa$ be regressive, i.e. $f(0) = 0$ and $\forall x \in S \setminus \{0\}$, $f(x) < x$. Then $\exists \beta \in \kappa$ such that $f^{-1}(\beta)$ (the inverse image/fiber) is stationary in κ.*

Remark. This is where the term stationary comes from. This theorem shows that stationary sets are those on which regressive functions are also stationary. In this sense, they haven't "moved". Also note that #4 from the exercises we were given concerns regressive functions.

Proof. Assume that no such β exists. Then $\forall \beta < \kappa$, $f^{-1}(\beta)$ is not stationary, so there is some club C_β such that $f^{-1}(\beta) \cap C_\beta = \emptyset$. Consider the diagonal intersection of the C_βs as in Lemma 2. We call it $C \coloneqq \{\alpha \in \kappa \colon \forall \beta < \alpha,\ \alpha \in C_\beta\}$. By Lemma 2, this is a club. In particular, $C \cap S \neq \emptyset$. So, let $x \in C \cap S$. Since f is regressive, $f(x) < x$. (We may ignore the 0 case, as C is unbounded.) Since $x \in C$, $x \in C_{f(x)}$. On the other hand, $x \in f^{-1}(f(x))$, a contradiction to $C_{f(x)} \cap f^{-1}(f(x)) = \emptyset$. □

Theorem 7.4. *If $\kappa \geq \aleph_1$ is regular, then κ contains κ disjoint stationary sets.*

Proof. Let $S = \{\alpha \in \kappa \colon \operatorname{cof} \alpha = \aleph_0\}$. S is stationary (since every club must be closed under ω-limits, which have cof $= \aleph_0$) (I think), and it is not a club if $\kappa > \aleph_1$. Further, by definition, for each $\alpha \in S$, there is an increasing ω sequence with limit α, which we will denote $\beta_{\alpha,n} \nearrow \alpha$.

Claim: $\exists n \in \omega$ such that $\forall \gamma < \kappa$, the set $S_\gamma \coloneqq \{\alpha \in S \colon \beta_{\alpha,n} \geq \gamma\}$ is stationary.

Proof: Assume not. Then for all $n < \omega$, $\exists \gamma_n < \kappa$ and a club C_n such that $S_{\gamma_n} \cap C_n = \emptyset$. Let $C = \cap_{n\in\omega} C_n$ (which is a club by Lemma 1) and $\gamma = \sup_{n<\omega} \gamma_n \in S$, since it has cofinality ω.
Pick any $\delta \in C \cap S$ (recall that S is stationary). Then $\forall n < \omega$, $\delta \in C_n \Rightarrow \delta \notin S_{\gamma_n}$, which means $\beta_{\delta,n} < \gamma_n \leq \gamma$. Hence $\delta = \sup_{n<\omega} \beta_{\delta,n} \leq \gamma$. Thus $C \cap S$ (a club)

is bounded in κ, a contradiction. $\square$

Consider the function $f\colon S \to \kappa$ given by $\alpha \mapsto \beta_n$, where n is the number provided by the above claim. Note that this is regressive, since β_n is part of a sequence that increases to α. Let $\gamma \in \kappa$. Then $S_\gamma (= \{\alpha \in S, \beta_{\alpha,n} > \gamma\})$ is stationary and contained in S. We apply Fodor's Theorem to $f|_{S_\gamma}\colon S_\gamma \to \kappa$. This tells us that there is $\beta_\gamma \in \kappa$ with $T_\gamma := S_\gamma \cap f^{-1}(\beta_\gamma)$ stationary.

Let $x \in T_\gamma$. $f(x) = \beta_{x,n} \geq \gamma$ by the definition of S_γ, but also $f(x) = \beta_\gamma$ because $x \in f^{-1}(\beta_\gamma)$. Hence $\beta_\gamma \geq \gamma$.

The T_γs are all disjoint, since they are subsets of the inverse images of distinct elements, and they are all stationary. It remains only to show that there are κ of them and we will have proven the theorem.

It suffices to check $\{B_\gamma\}_{\gamma\in\kappa}$, since there is a 1:1 correspondence between the β_γs and the T_γs. Define the function $g\colon \kappa \to \kappa$ by $\gamma \mapsto \beta_\gamma$. We saw before that $\beta_\gamma \geq \gamma$, so $g(\gamma) \geq \gamma$ (g is "progressive"). If $\operatorname{Card}\operatorname{im}(g) < \kappa$, then by the regularity of κ, it must be that $\operatorname{im}(g)$ is bounded. (Else there would be an unbounded map of cardinality less than the cofinality of κ, a contradiction.) But this can't be since g is progressive. Hence by contradiction we have that there are κ stationary sets. $\square$

We finally arrive at the result we have been driving at since last class, and it's a measly corollary.

Corollary. *There are 2^κ DLOs of cardinality κ.*

Proof. Write $\kappa = \coprod_{\alpha\in\kappa} S_\alpha$, where each S_α is stationary. (We can do this by the theorem we just proved.) For any $A \in \mathcal{P}(\kappa)$, consider $X_A = \coprod_{\alpha\in A} S_\alpha$ (note the indexing set!). This gives us 2^κ sets X_A, with $X_A \triangle X_B = \coprod_{\alpha\in A\triangle B} S_\alpha$ stationary. In particular, $\mathrm{DLO}(X_A) \ncong \mathrm{DLO}(X_B)$. $\square$

End of Lecture 9.

Lecture 10.

Today we will begin talking about Model Theory. First, let's talk a little bit about some fields.

$\text{Aut}_{\text{field}}(\mathbb{R}) = \{Id\}$, while $\text{Aut}_{\text{field}}(\mathbb{C}) = \{Id, \text{complex conjugation, and infinitely more}\}$. For example, there is an automorphism of $\mathbb{C}$ that sends π to e.
Proof: (informal, just getting the juices flowing) We begin by regarding $\mathbb{R}$. We note that $\alpha \in \text{Aut}_{\text{field}}(\mathbb{R})$ fixes $\mathbb{Q}$ pointwise, as this is the only way to preserve the field properties of $\mathbb{R}$. α happens to be continuous. Also, α happens to be increasing. It is a nice surprise that we can talk about $<$ in $\mathbb{R}$ as a field, but this is because we can define $<$ using only algebraic properties. Specifically, we say $x \geq 0 \leftrightarrow \exists y(y^2 = x)$. This is why α is so rigid: because we can define the order algebraically, the automorphism has to preserve this order. Further, since $\alpha(0) = 0$, it must be that α is the identity.

Now, for $\mathbb{C}$, we can not apply this same reasoning, because the field is not ordered in such a way. In particular, every complex number is a square, so the above formulation certainly won't work. This is a fundamental difference between the two fields, so it should not be completely shocking that there are more automorphisms on $\mathbb{C}$. To show there are infinitely many, one uses transcendence bases and the Axiom of Choice, and we won't be getting into that.

So why do we work with $\mathbb{C}$? First order theorems in $\mathbb{C}$ and first order theorems in $\overline{\mathbb{Q}}$ (the *algebraic* closure) are the same. In fact, a first order theorem holds in $\mathbb{C}$ (viewed as a field) iff it holds for all but finitely many $\overline{\mathbb{F}_p}$s (again, algebraic closure). We're going to be talking about what this sort of thing means.

8 Model Theory

Some good references for Model Theory:

- Model Theory by Chiang and Kiesler, which is too complete.
- Model Theory: an Introduction by David Marker, which is a nice introduction. (Note that we have already referred to this one in the past and will continue to do so.)
- A Course in Model Theory, by Bruno Poizat, which is wonderful.

8.1 Keisleriana

8.1.1 Syntax and the semantical connection

We call it first order logic when one quantifies over elements, and not over properties or sets of elements (unless we're in set theory, when the elements are sets, doh). This entire section is basically just definitions and examples, hence the strange title, since I believe Keisler is one of those people that obsesses over definitions.

Definition 8.1. *A first order language $\mathcal{L}$ consists of:*

- *a set of constant symbols $\mathcal{C}$*
- *a set of relation symbols $\mathcal{R}$*
- *a set of function symbols $\mathcal{F}$.*

We note that in the case of relations, "arity" refers to the number of items the relation is on, e.g. $R(x, y)$ has arity 2; $R(x_1, \ldots, x_n)$ has arity n. For functions it is similar, and "arity" refers to the number of "parameters" the function takes. This is slightly different because it means that a function with arity n actually involves $n + 1$ items (the $n + 1$-th item being the "output" of the function). We will always assume, *even when not explicitly stated*, that the "=" relation is in $\mathcal{R}$. We will also use variables when discussing languages. See Marker, pg. 8 for more or less this exact same definition.

Example 8.2.

- The language of orderings is $\mathcal{L}_{ord} = \{<\}$.
- The language of groups is $\mathcal{L}_{grp} = \{e, {}^{-1}, \cdot\}$. e is a constant, $^{-1}$ is a 1-ary function, and $\cdot$ is a binary function.
- The language of (unitary) rings is $\mathcal{L}_{ring} = \{0, 1, +, -, \cdot\}$.
- The language of ordered fields is $\mathcal{L}_{ordfields} = \{0, 1, +, -, \cdot, <\}$.
- There are other examples in Marker, pg. 8.

Definition 8.3. *We define $\mathcal{L}$-terms. (Just "terms" if $\mathcal{L}$ is understood.)*

- *If c is a constant symbol in $\mathcal{C}$, it is an $\mathcal{L}$-term.*
- *If x is a variable, it is an $\mathcal{L}$-term.*
- *If $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol in $\mathcal{F}$, $f(t_1, \ldots, t_n)$ is an $\mathcal{L}$-term.*

See Marker, pg. 9.

Example 8.4. $(x \cdot (x + 1) - 0 \cdot x) - x$ is a term in $\mathcal{L}_{ring}$. Technically the infix notation is incorrect, but we are only human, so this is what makes sense. If pressed, we know that (for example) $0 \cdot x = \cdot(0, x)$. Note that we can not simplify this expression yet because we have no rules for doing so, e.g. we don't know that $0 \cdot x = 0$ yet.

Definition 8.5. *We define $\mathcal{L}$-formulas.*

- *If $t_1, \dots, t_n$ are terms and R is an n-ary relation symbol, $R(t_1, \dots, t_n)$ is a formula. This is called an atomic formula.*
- *If ϕ, ψ are formulas, then $\neg\phi$, $\phi \vee \psi$, $\phi \to \psi$, $\phi \leftrightarrow \psi$ are formulas.*
- *If ϕ is a formula and x is a variable, $\exists x\phi$ and $\forall x\phi$ are formulas.*
- *A quantifier-free formula is a formula with no quantifiers. (Surprise!)*
- *An existential formula is of the form $\exists x_1 \exists x_2 \dots \exists x_n \phi$, where ϕ is quantifier-free and there are no $\neg$ symbols before any of the $\exists$ symbols. A universal formula is defined similarly except with $\forall$ symbols.*

See Marker pg. 10.

Definition 8.6. *We define the quantification rank of a formula. Assume that ϕ, ψ are formulas.*

- *If ϕ atomic, $qrk(\phi) = 0$.*
- $qrk(\neg\phi) = qrk(\phi)$
- $qrk(\phi \vee \psi) = \max(qrk(\phi), qrk(\psi))$
- $qrk(\exists x\phi) = qrk(\forall x\phi) = qrk(\phi) + 1$

Example 8.7 (of a formula). In $\mathcal{L}_{ring}$, $\exists x(x = 0 \to \exists y(z = 1))$ is a formula.

If syntax is important to you, you need to tweak the definitions above a little bit to include parentheses, and then one can prove a theorem of "unique readability". This is not too important (Marker doesn't even mention it), but it's easy enough to dig up such a proof if you wish.

A variable v occurs freely (is free) in a formula ϕ if it is not inside a $\exists v$ or $\forall v$ quantifier, otherwise we say it is bound. (Marker, pg. 10) A statement (or sentence) is a formula with no free variables.

Definition 8.8. *An $\mathcal{L}$-structure $\mathcal{M}$ is a set $M \neq \emptyset$ with the following things holding:*

- *For each constant symbol c, there is an element $c^{\mathcal{M}} \in M$.*
- *For each n-ary relation symbol R, there is a subset $R^{\mathcal{M}}$ of M. We note that the "=" relation is* always *interpreted as the diagonal on M, i.e. (x, x) for all $x \in M$.*
- *For each function symbol f of arity n, there is a function $f^{\mathcal{M}} : M^n \to M$.*

See Marker, pg. 8.

Example 8.9. The ring $\mathbb{Z}$ is the $\mathcal{L}_{ring}$ structure consisting of M=the set of integers equipped with $0^{\mathbb{Z}} = 0$, $1^{\mathbb{Z}} = 1$, $+^{\mathbb{Z}}$ as regular addition, $-^{\mathbb{Z}}$ as regular subtraction, and $\cdot^{\mathbb{Z}}$ as regular multiplication. This is in some sense a trivial example, so let us see a slightly less obvious one.

$\mathfrak{S}_n$ is the $\mathcal{L}_{grp}$ structure with M=the set of bijections on the set $\{1, \ldots, n\}$ with $e^{S_n} = id_{\{1,\ldots,n\}}$, ${-1}^{S_n}$=the function which maps a bijection to its inverse bijection, and $\cdot^{S_n} = \circ$, the composition of functions.

Definition 8.10. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let $\underline{x}$ be a tuple of variables of length n, $\underline{m}$ be a tuple of elements of M also of length n. We define the interpretation of an $\mathcal{L}$-term with parameters $\underline{m}$ (for $\underline{x}$) is:*

- *If $t = c$, t a term and c a constant symbol, then $t^{\mathcal{M}} = c^{\mathcal{M}}$.*
- *If y is a variable in $\underline{x}$, then $y^{\mathcal{M}}$ is the corresponding element of $\underline{m}$.*
- *If f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, the interpretation of $f(t_1, \ldots, t_n)$ is $f^{\mathcal{M}}(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}})$.*

See Marker pg. 9.

For example, the interpretation of $1 + 1$ in $\mathbb{Z}$ is 2 (hooray!).

Definition 8.11. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $\underline{x}$ a tuple of variables, $\underline{m}$ a tuple of parameters. We define satisfaction of a formula ϕ with parameters $\underline{m}$ in place of $\underline{x}$.*

- *if $\phi = R(t_1, \ldots, t_n)$, $\mathcal{M}$ satisfies ϕ (written $\mathcal{M} \models \phi$) iff $(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}}) \in R^{\mathcal{M}}$.*
- *If $\phi = \psi \vee \chi$, $\mathcal{M} \models \phi$ iff $\mathcal{M} \models \psi$ or $\mathcal{M} \models \chi$.*
- *If $\phi = \exists y \psi$, $\mathcal{M} \models \phi$ iff there is $\mu \in M$ such that $\mathcal{M} \models \psi$ with parameters $\underline{m}\mu$ for $\underline{x}y$.*
- *If $\phi = \neg\psi$, $\mathcal{M} \models \phi$ iff $\mathcal{M} \not\models \psi$, i.e. $\mathcal{M}$ does not satisfy ψ.*

See Marker, pg. 11. You may think that we should define it for all the other logical connectives, and while this would be nice, it's not strictly necessary since we can write all the other ones in terms of these three.

Definition 8.12. *Two formulas $\phi(\underline{x})$, $\psi(\underline{x})$ are equivalent if for any $\mathcal{L}$ structure $\mathcal{M}$, and any tuple $\underline{m} \in M$, $\mathcal{M} \models \phi(\underline{m})$ iff $\mathcal{M} \models \psi(\underline{m})$.*

Example 8.13. $\exists x \phi(x)$ and $\neg\forall x \neg\phi(x)$ are equivalent (as alluded to above). Also, ϕ and $\neg\neg\phi$ are equivalent (yes, we are reasoning using the Law of the Excluded Middle).

End of Lecture 10.

Lecture 11.

8.2 Theories and Models

Definition 8.14. *An $\mathcal{L}$-theory is a set T of $\mathcal{L}$-sentences such that there exists and $\mathcal{L}$ structure $\mathcal{M}$ such that for each $\phi \in T$, $\mathcal{M} \models \phi$. Such an $\mathcal{M}$ is called a model of T.*

Note that this is slightly different than the definition in Marker (pg. 14). Marker says any set of $\mathcal{L}$-sentences is a theory, and if a model exists for a theory (as in *our* definition) then it is called satisfiable.

Remark. By our definition, theories are consistent (since a model must exist for them). However, we'll often say "Let's prove that this theory is consistent/not consistent" when what we mean is "Let's prove that this set of axioms is/is not a theory."

Example 8.15.

- The theory of groups in $\mathcal{L}_{grp}$ is given by:
  - $\forall x(x \cdot e = e \cdot x = x)$
  - $\forall x(x \cdot x^{-1} = x^{-1} \cdot x = e)$
  - $\forall x, y, z((x \cdot y) \cdot z = x \cdot (y \cdot z))$

  A group is a model of T_{grp}. See Marker pg. 16 for this example, as well as some further axioms that describe special (e.g. abelian) groups.

- The theory of commutative rings in $\mathcal{L}_{ring}$ is given by:
  - $\forall x, y, z(x + (y + z) = (x + y) + z) \wedge (x \cdot y) \cdot z = x \cdot (y \cdot z))$
  - $\forall x, y(x + y = y + x \wedge x \cdot y = y \cdot x)$
  - $\forall x, y, z(x \cdot (y + z) = x \cdot y + x \cdot z)$
  - $\forall x(x + 0 = x \cdot 1 = x)$
  - $\forall x + (-x) = 0$

  See Marker pg. 17 for a slightly different (and honestly, more thorough, but I am too lazy to type it up) axiomatization of rings, to which one only need add the commutativity axiom to get a theory of commutative rings.

Exercise: Axiomatize dense linear orderings in $\mathcal{L}_{ord} = \{<\}$, fields in $\mathcal{L}_{ring}$, and algebraically closed fields in $\mathcal{L}_{ring}$.

- We sort of did this in a previous class (we didn't axiomatize what an ordering was), but here is another listing of the axioms. (See Marker, pg. 15.)
  - $\forall x(\neg(x < x))$
  - $\forall x, y, z(x < y \wedge y < z \rightarrow x < z)$
  - $\forall x, y, z(x < y \vee x = y \vee y < x)$
  - $\forall x, y(x < y \rightarrow \exists z(x < z < y))$
  - $\forall x(\exists y, z(y < x < z))$
- Use all of the theory of commutative rings (assume that we actually had a full listing), and add the axiom $\forall x(x \neq 0 \rightarrow \exists y(x \cdot y = 1))$. (See Marker, pg. 17. I'll admit before I looked I forgot to stipulate $x \neq 0$.)
- Use the theory of fields and add the axioms $\forall a_0 \forall a_1 ... \forall a_{n-1} \exists x(x^n + \sum_{i=0}^{n-1} a_i x^i = 0)$ for each $n \in \mathbb{N}$. It makes sense this theory requires an infinite number of axioms, because we are saying that every polynomial (of which there are infinitely many) has a root. Since this is first-order logic, we can't make statements about "every polynomial", so we just have to add an axiom for each length of a polynomial. (see Marker pg. 17-18.)

Definition 8.16. *A theory T is finitely axiomatizable if there is a finite theory T_0 that has exactly the same models.*

It is important to understand why algebraically closed fields are not finitely axiomatizable; see my above answer to the exercise on ACFs.

When $\mathcal{M}$ is a model of T, we write $\mathcal{M} \models T$. For example $S_n \models T_{grp}$.

Definition 8.17. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The theory of $\mathcal{M}$ is* $\text{Th}(\mathcal{M}) = \{\phi$ *an $\mathcal{L}$-sentence with* $\mathcal{M} \models \phi\}$.

Definition 8.18. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$ (the underlying set of $\mathcal{M}$). Let $\mathcal{L}_A$ be $\mathcal{L}$ together with new constant symbols for elements of A. Then*

$$\begin{aligned}\text{Th}(\mathcal{M}, A) &= \mathcal{L}_A\text{-}\textit{theory of } \mathcal{M} \\ &= \{\phi \in \mathcal{L}_A \mid \mathcal{M} \models \phi\} \\ &= \{\phi(\underline{a}) \mid \underline{a} \in A \vee \mathcal{M} \models \phi((\underline{a}))\}\end{aligned}$$

Hence $\text{Th}(\mathcal{M}) = \text{Th}(\mathcal{M}, \emptyset)$.

Example 8.19. "$\pi + e = 0$" is not in $\text{Th}_{ring}(\mathbb{R})$ since the symbols don't even exist there, but neither is it in $\text{Th}_{ring}(\mathbb{R}, \{e, \pi\})$ (in this case, because it's wrong).

Disclaimer: In what follows, we will be defining realized types only.

Definition 8.20. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$, and $\underline{m} \in M$. (If I haven't mentioned this earlier, this last thing is an abuse of notation declaring that $\underline{m}$ is a tuple of some length with all entries in M.) The type of $\underline{m}$ over A is $\{\phi(\underline{x}, \underline{a}) \mid \phi(\underline{x}, \underline{a})$ an $\mathcal{L}_A$ sentence and $\mathcal{M} \models \phi(\underline{x}, \underline{a})\}$. Denote this as $\mathrm{tp}_{\mathcal{M}}(\underline{m}/A)$. Alternately, $\mathrm{tp}_{\mathcal{M}}(\underline{m}/A) \coloneqq \{(\phi(\underline{x}, \underline{y}), \underline{a})$ with the length of $\underline{y}$ and $\underline{a} \in A$ equal and $\mathcal{M} \models \phi(\underline{x}, \underline{a})\}$. Note that $\mathrm{Th}(\mathcal{M}, A) = \mathrm{tp}_{\mathcal{M}}(\emptyset/A)$.*

Example 8.21. $\mathrm{tp}_{\mathbb{R}}(-1/\emptyset) \supseteq \{" - 1^2 = 1", "\forall x(x + 0 = x)", \ldots\}$.
$\mathrm{tp}_{\mathbb{R}}(\pi/\emptyset)$ has no minimal "essential" subset, since it is not algebraic.

Caution: Note that $\mathrm{tp}_{\mathbb{R}}(-1/\emptyset) \ni \forall x(x^2 \neq -1)$, while $\mathrm{tp}_{\mathbb{C}}(-1/\emptyset) \ni \exists x(x^2 = -1)$. This shows that the subscripts are important, and that the inclusion of one subscript in another does not guarantee that the type of one will we included in the other.

Definition 8.22. *Write $T \models \phi$ if whenever $\mathcal{M} \models T$, then $\mathcal{M} \models \phi$.*

Example 8.23. $ACF \models \exists x, y(x \neq y)$

Definition 8.24. *A theory T is complete if for each $\mathcal{L}$-sentence ϕ, either $T \models \phi$ or $T \models \neg\phi$.*

Example 8.25.

- T_{ord} is not complete. One does not know whether or not the ordering is linear, dense, or what have you, but $\mathcal{L}$-sentences can be written to describe these situations.
- T_{grp} is not complete, since one can have two finite groups of different order. These would both model T_{grp}, but there are sentences on which they disagree (such as the sentence which corresponds to "This group contains (blank) elements"). Thus we can not say that $T \models \phi$ or that $T \models \neg\phi$, since it has models which do both.
- In fact, using similar reasoning we see that a theory with two finite models of different cardinality can't be complete, or with one finite model and one infinite model.
- ACF is not complete, since there are ACFs with different characteristics. However, ACF_p (p prime or 0) is complete. (See Marker, pg. 18 for a description of the axioms of ACF_p.)
- DLO is complete.

Remark. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq \mathcal{M}$. Then $\mathrm{Th}(\mathcal{M}, A)$ is complete.

Proof. Let $\phi(\underline{(a)}) \notin \mathrm{Th}(\mathcal{M}, A)$. This means by definition $\mathcal{M} \not\models \phi(\underline{a})$. Hence $\mathcal{M} \models \neg\phi(\underline{a})$, and $\neg\phi(\underline{a}) \in \mathrm{Th}(\mathcal{M}, A)$. □

This result can be stated as "Models give rise to complete theories".

Definition 8.26. *Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures. $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent ($\mathcal{M} \equiv \mathcal{N}$) if* $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$. *In other words if* $\mathrm{tp}_{\mathcal{M}}(\emptyset/\emptyset) = \mathrm{tp}_{\mathcal{N}}(\emptyset/\emptyset)$.

Example 8.27.

- $\mathbb{R}_{ring} \not\equiv \mathbb{C}_{ring}$
- $(\mathbb{R}, <) \equiv (\mathbb{Q}, <)$
- Two finite $\mathcal{L}$-structures are elementarily equivalent iff they are isomorphic.

Remark. T is complete iff any two models of T are elementarily equivalent.

8.3 Elementary Inclusion

Note: "Elementary" is here being used in a different sense than for elementary equivalence.

Definition 8.28. *Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures. Write $\mathcal{M} \subseteq \mathcal{N}$ ($\mathcal{M}$ is a substrucure of $\mathcal{N}$) if $M \subseteq N$, $c^{\mathcal{M}} = c^{\mathcal{N}}$, $R^{\mathcal{N}}|_{M^n} = R^{\mathcal{M}}$, and $f^{\mathcal{N}}|_{M_n} = f^{\mathcal{M}}$. In other words, $\mathcal{M}$ has an interpretation induced by the inclusion map.*

See Marker, pg. 8.

Exercise: If α is an ordinal and $\{\mathcal{M}_i\}_{i<\alpha}$ are $\mathcal{L}$-structures with $i < j \Rightarrow \mathcal{M}_i \subseteq \mathcal{M}_j$, then $\mathcal{N} = \cup_{i<\alpha}\mathcal{M}_i$ is canonically an $\mathcal{L}$-structure.
We let M_i denote the underlying set of each $\mathcal{M}_i$. Then $N = \cup_{i<\alpha} M_i$ is a set, and we let it be the universe of $\mathcal{N}$. If $c \in C$, then its interpretation in each $\mathcal{M}_i$ must be the same (else one would not be a substructure of one above it), so we let this interpretation be $c^{\mathcal{N}}$. Similarly, the interpretation of a function $f \in \mathcal{F}$ may be done as follows. If f is n-ary and $\underline{m} \in N^n$, then each coordinate of $\underline{m}$ must appear in some M_i. Let j be the maximum subscript of these sets (n is finite, so this is no problem to pick), so that $\underline{m} \in M_j$. Then $f^{\mathcal{N}}(\underline{m}) = f^{\mathcal{M}_j}(\underline{m})$. Do essentially the same thing for relations $R \in \mathcal{R}$. □

The relevant notion:

Definition 8.29. *Let $\mathcal{M} \subseteq \mathcal{N}$ be $\mathcal{L}$-structures. The inclusion is elementary (or "$\mathcal{M}$ is an elementary substructure of $\mathcal{N}$" or $\mathcal{N}$ is an elementary extension of $\mathcal{M}$"), written $\mathcal{M} \preccurlyeq \mathcal{N}$, if for all $\mathcal{L}_M$ formulas $\phi(\underline{m})$, $\mathcal{M} \models \phi(\underline{m})$ iff $\mathcal{N} \models \phi(\underline{m})$.*

Example 8.30.

- $\mathbb{R}_{field} \not\preccurlyeq \mathbb{C}_{field}$
- $\mathbb{Q}_{field} \not\preccurlyeq \mathbb{R}_{field}$
- $(\mathbb{Q}, <) \preccurlyeq (\mathbb{R}, <)$

Exercise: Consider the theory of an infinite set without structure: $\{\exists x_1, x_2(x_1 \neq x_2), \exists x_1, x_2, x_3(x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3), \ldots\}$. Let $\mathcal{M} \subseteq \mathcal{N}$ be two models of this theory. Prove that $\mathcal{M} \preccurlyeq \mathcal{N}$.
Left for later.

Comment on the definition: $\mathcal{M} \preccurlyeq \mathcal{N} \Leftrightarrow$ for all $\underline{m} \in \mathcal{M}$, $\mathrm{tp}_{\mathcal{M}}(\underline{m}/\emptyset) = \mathrm{tp}_{\mathcal{N}}(\underline{m}/\emptyset) \Leftrightarrow \mathrm{Th}(\mathcal{M}, M) = \mathrm{Th}(\mathcal{N}, M)$. In particular, $\mathcal{M} \preccurlyeq \mathcal{N} \Rightarrow \mathcal{M} \equiv \mathcal{N}$.

The converse does not hold. Consider $(\mathbb{Z}, <) = \mathcal{N}$ and $(\mathbb{Z} \setminus \{0\}, <) = \mathcal{M}$. Then $\mathcal{M} \simeq \mathcal{N}$ (which we haven't defined yet), so $\mathcal{M} \equiv \mathcal{N}$, but $\mathrm{tp}_{\mathcal{M}}(\{-1, 1\}/\emptyset) \neq \mathrm{tp}_{\mathcal{N}}(\{-1, 1\}/\emptyset)$ (since one has an element between the two and the other doesn't).

Exercise: Let $\mathcal{M} \subseteq \mathcal{N}$ be $\mathcal{L}$-structures. Let $\phi(\underline{m})$ be an $\mathcal{L}_M$-formula. Prove:

- If $\phi(\underline{m})$ is quantifier free, $\mathcal{M} \models \phi(\underline{m})$ iff $\mathcal{N} \models \phi(\underline{m})$.
- If $\phi(\underline{m})$ is existential and $\mathcal{M} \models \phi(\underline{m})$ then $\mathcal{N} \models \phi(\underline{m})$.
- If $\phi(\underline{m})$ is universal and $\mathcal{N} \models \phi(\underline{m})$ then $\mathcal{M} \models \phi(\underline{m})$.

End of Lecture 11.

Lecture 12.

We began with a brief discussion of the difference between isomorphism and elementary equivalence. The important thing to keep in mind is that saying $\mathcal{M} \equiv \mathcal{N}$ is the same as saying that $\mathcal{M}$ and $\mathcal{N}$ have the same first order theorems, which is the relevant concept when we discuss model theory.

Proposition 8.31 (Tarski's criterion)**.** *Let $\mathcal{M} \subseteq \mathcal{N}$ be two $\mathcal{L}$-structures. Then $\mathcal{M} \preccurlyeq \mathcal{N}$ iff for every formula $\phi(\underline{m}x)$ with parameters in M if $\mathcal{N} \models \exists x(\phi(\underline{m}x))$, then $\mathcal{M} \models \exists x(\phi(\underline{m}x))$. Put more colloquially, $\mathcal{M} \preccurlyeq \mathcal{N}$ iff "$\exists$s go down".*

See Marker pg. 45 for a presentation of this proof.

Remark. This is not a statement about existential formulas, as ϕ is allowed to have $\forall$ symbols in it. For example, $(\mathbb{Q}, <) \subseteq (\mathbb{Q}, <) \coprod_{<} \{\infty\} =: \mathbb{Q}'$ satisfy the same existential formulas. These are not elementarily equivalent (and hence not $\preccurlyeq$) because $\mathbb{Q}' \models \exists x(\forall y(y \leq x))$, something which is not true of $(\mathbb{Q}, <)$.

Proof. ($\Rightarrow$) This holds by definition, since $\mathcal{M} \preccurlyeq \mathcal{N}$ means that all $\mathcal{L}_M$ formula hold in both structures.

($\Leftarrow$) We proceed by induction on $\psi(\underline{m})$) with parameters in $\underline{M}$. If ψ is an atomic formula (and in particular quantifier free), then by the definition of substructure ($\subseteq$), the claim holds. (This is part of the final exercise from last class; see also Marker pg. 11).

Now, assume the claim holds for ψ_1, ψ_2. Then if $\psi = \neg\psi_1$, $\mathcal{M} \models \psi \Leftrightarrow \mathcal{M} \not\models \psi_1$, which is true iff $\mathcal{N} \not\models \psi_1$ by assumption, which is in turn true iff $\mathcal{N} \models \psi$. Similarly $\mathcal{M} \models \psi_1 \vee \psi_2 \Leftrightarrow \mathcal{M} \models \psi_1$ or $\mathcal{M} \models \psi_2 \Leftrightarrow \mathcal{N} \models \psi_1$ or $\mathcal{N} \models \psi_2 \Leftrightarrow \mathcal{N} \models \psi_1 \vee \psi_2$.

Finally, assume that $\psi = \exists x(\phi(\underline{m}x))$, and that the claim holds for ϕ. If $\mathcal{M} \models \psi$, there is $\mu \in M$ with $\mathcal{M} \models \phi(\underline{m}\mu))$, and hence by our inductive assumption, $\mathcal{N} \models \phi(\underline{m}\mu)$, so $\mathcal{N} \models \psi$. This takes care of the "if" part of our claim. On the other hand, if $\mathcal{N} \models \exists x(\phi(\underline{m}x))$, then by our overall assumption (i.e. the $\Leftarrow$ assumption), $\mathcal{M} \models \exists x(\phi(\underline{m}x))$, which takes care of the "only if" part of our claim. □

Theorem 8.32 (Löwenheim-Skolem, downwards)**.** *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let $A \subseteq M$ and κ be a cardinal with* $\operatorname{Card} A + \operatorname{Card} \mathcal{L} + \aleph_0 \leq \kappa \leq \operatorname{Card} M$. *Then there is $\mathcal{M}_0 \preccurlyeq \mathcal{M}$ with $A \subseteq M_0$ and* $\operatorname{Card} M_0 = \kappa$.

Remark. Some people let $\operatorname{Card} \mathcal{L} = \operatorname{Card} \mathcal{L} + \aleph_0$ by definition to make this statement shorter. This makes sense because one can always produce $\aleph_0$ formulas for a given language, so this different definition is simply taking these formulas into account from the beginning. For example, $\operatorname{Card} \mathcal{L}_{grp} = \aleph_0$ under this scheme even though the language is finite, because what matters is the cardinality of the set of formulas. In general, if $\operatorname{Card} \mathcal{L} = \lambda$ an infinite cardinal, then the old and new definitions coincide, meaning the cardinal of an infinite signature (i.e. the symbols of the language) is the cardinal of the set of formulas. Also, see Marker pg. 46 for a presentation of the following proof.

Proof. From our above proof we know (roughly) that determining the models on statements beginning with $\exists x$ will be good enough to say that one is elementarily included in the other. This knowledge guides the rest of our proof.

We will begin by expanding the language. For each $\mathcal{L}$-formula $\phi(x\underline{y})$, add a new function symbol f_ϕ with arity the length of $\underline{y}$. This is called a Skolem function. Let $\mathcal{L}' := \mathcal{L} \cup \{f_\phi\colon \phi(x\underline{y})$ an $\mathcal{L}$-formula, $\underline{y}$ a tuple$\}$. Now, we will expand $\operatorname{Th}(\mathcal{M})$ into T' by adding the axioms $\forall y((\exists x(\phi(x\underline{y})) \to \phi(f_\phi(\underline{y}), y))$. What this is saying is that $f_\phi(\underline{y})$ is a witness to the statement $\exists x(\phi(x\underline{y}))$.

Claim: There is an $\mathcal{L}'$ structure on $\mathcal{M}$ making it a model of T'.

We prove this by giving a meaning to f_ϕ. Let $\underline{m} \in M$. If $\mathcal{M} \not\models \exists x(\phi(x\underline{m}))$, let $f_\phi(\underline{m})$:=whatever you want in M. If $\mathcal{M} \models \exists x(\phi(x\underline{m}))$, let $f_\phi(\underline{m})$ be a witness, i.e. a μ such that $\mathcal{M} \models \phi(\mu\underline{m})$. Then $\mathcal{M} \models_{\mathcal{L}'} T'$, since we have constructed f_ϕ that is a witness to all of the appropriate statements.

Now that the f_ϕs mean something, we start the construction. We may assume that $\mathcal{C}^{\mathcal{M}} \subseteq A$ (i.e. all the constants of $\mathcal{M}$ are in A) and $\operatorname{Card} A = \kappa$, adding elements if necessary. We close A under functions of $\mathcal{L}'$, meaning that we add $\forall f \in \mathcal{L}'$ and all $\underline{a} \in A$ the element $f^{\mathcal{M}}(\underline{a})$ to A, and then add f of those elements and so on ω times. We denote the resulting set M_0.

Let $\mathcal{M}_0$ be the $\mathcal{L}$-structure (not $\mathcal{L}'$-structure!) induced by $\mathcal{M}$ on M_0. This is okay since $\mathcal{C}^{\mathcal{M}} \subseteq M_0$ and M_0 is closed under functions of $\mathcal{L}$, so when we attempt to do so we won't run into any elements we don't "recognize". If $\phi(x\underline{m})$ is any $\mathcal{L}$-formula such that $\mathcal{M} \models \phi(x\underline{m})$. Because we have all our Skolem functions, we have $\mathcal{M} \models \phi(f_\phi(\underline{m})\underline{m})$, and since we closed M_0 under these functions as well, $f_\phi(\underline{m}) \in M_0$. Thus, if $\mathcal{M} \models \exists x(\phi(x, \underline{m}))$, we have that $\mathcal{M}_0 \models \exists x(\phi(x\underline{m}))$. The other direction is obvious, so by Tarski's criterion, $\mathcal{M}_0 \preccurlyeq \mathcal{M}$. We also know that $\operatorname{Card} M_0 = \kappa$ since $A \subseteq M_0$. □

Caution: $\mathcal{M}_0 \preccurlyeq_{\mathcal{L}} \mathcal{M}$ but **not** $\mathcal{M}_0 \preccurlyeq_{\mathcal{L}'} \mathcal{M}$ a priori. Fortunately we don't care, but it's still worth noting.

Example 8.33.

1. For $\mathbb{C}$ as an ACF, if we set $A = \emptyset$ and $\kappa = \aleph_0$ then although there is no uniqueness, we get a prime model (term to be defined later) $\bar{\mathbb{Q}}$.
2. For $(\mathbb{R}, <)$, if we set $A = \emptyset$ and $\kappa = \aleph_0$ then there is no uniqueness and no prime model.
3. Exercise: Find $\mathcal{M} \equiv \mathcal{N}$, $\mathcal{M}$ and $\mathcal{N}$ countable, with $\mathcal{M} \not\cong \mathcal{N}$. We decided that this was true for $\bar{\mathbb{Q}}$ and $\bar{\mathbb{Q}}(\pi)$.

Corollary. *DLO is complete.*

Proof. Let A, B be DLOs. Then use the above to find A_0, B_0 countable DLOs with $A_0 \preccurlyeq A$ and $B_0 \preccurlyeq B$. Then we have seen that $A_0 \cong B_0$, whence $A_0 \equiv B_0$, so $A \equiv A_0 \equiv B_0 \equiv B$. Thus any two models of DLO are elementarily equivalent. □

We end the class with a "paradox" due to Skolem. It is purposely vaguely stated to give one the opportunity to figure out from where the "paradox" arises. We will see the answer next class.

Assume ZF is consistent. Then so is ZFC. So start with a universe $U \models ZFC$. Apply Skolem's process to $\emptyset$ to get a universe U_0 of cardinality $\aleph_0$ containing ω. In U_0 there is the U_0 notion of $P(\omega)$, which is not countable, but which is in U_0, which is countable. This is Skolem's paradox, since taken at face value it means that ZF is inconsistent.

End of Lecture 12.

Lecture 13.

We start the day by revisiting Skolem's paradox from the end of last class. Recall that we began by assuming that ZF (and therefore ZFC) is consistent. Then we had a universe U which modeled ZFC. Using the Löwenheim Skolem Theorem, we can create a model $U_0 \models ZFC$ such that U_0 is countable. However,

U_0 countains a non-countable set, since such a set can be created from the ZFC axioms. So what happened?

We need to be very careful when discussing models. By construction U_0 is U-countable. And by deduction, U_0 contains a U_0-uncountable set. This is to say that there is no bijection in U_0 between the natural numbers (in U_0) and that set. However, we would be able to find such a bijection in U. More succinctly, a U_0-uncountable set is still U-countable. So there's no paradox.

At this point Deloro mentioned that Chapter 7 of the Poizat book on model theory is a great resource on Gödel and these sorts of paradoxes. I have been warned that it's not really introductory level, however.

9 Arrows and the back-and-forth method

9.1 Morphisms

Definition 9.1. *Let $\mathcal{M}$, $\mathcal{N}$ be $\mathcal{L}$-structures. We say $\sigma\colon \mathcal{M} \to \mathcal{N}$ is an $\mathcal{L}$ morphism if the following conditions hold:*

- *for any constant symbol c, $\sigma(c^{\mathcal{M}}) = c^{\mathcal{N}}$.*
- *for any relation symbol R, $\underline{m} \in R^{\mathcal{M}} \Leftrightarrow \sigma(\underline{m}) \in R^{\mathcal{N}}$. Note that this is an if and only if condition.*
- *for any function symbol f, $\underline{m} \in M$, we have $\sigma(f^{\mathcal{M}}(\underline{m})) = f^{\mathcal{N}}(\sigma(\underline{m}))$.*

Since "=" is always a relation in the language, our morphisms are always injective by the second condition. I believe this is the same concept as Marker's partial embeddings (pg. 52).

Some quick examples: Field embeddings and ordering morphisms are $\mathcal{L}$ morphisms in the appropriate languages.

We note that you could define a "weak $\mathcal{L}$-morphism" by imposing only $\underline{m} \in R^{\mathcal{M}} \Rightarrow \sigma(\underline{m}) \in R^{\mathcal{N}}$ for the second condition. These are not necessarily injective, and would work to generalize things like group homomorphisms. However, they are ill-suited for our purposes, and so we don't care about them.

Definition 9.2. *An $\mathcal{L}$-isomorphism is an $\mathcal{L}$-morphism that is bijective.*

Lemma 9.3. *Let $\sigma\colon \mathcal{M} \to \mathcal{N}$ be an $\mathcal{L}$-isomorphism. Then for any $\phi(\underline{m})$ (with $\underline{m} \in M$), $\mathcal{M} \models \phi(\underline{m}) \Leftrightarrow \mathcal{N} \models \phi(\sigma(\underline{m}))$.*

Proof. If ϕ is an atomic formula, this holds by the definition of a morphism. If ϕ is built using $\vee$ or $\neg$ and shorter formulas, then the same sort of inductive argument we have done several times at this point shows that the claim holds.

Let $\phi(\underline{m}) = \exists x(\psi(\underline{mx}))$, where the claim holds for ψ. Assume $\mathcal{M} \models \phi(\underline{m})$. Then there is $\mu \in M$ with $\mathcal{M} \models \psi(\underline{m}\mu)$, so by our inductive assumption, $\mathcal{N} \models \psi(\sigma(\underline{m}\mu))$, i.e. $\mathcal{N} \models \psi(\sigma(\underline{m})\sigma(\mu))$. We know $\sigma(\mu) \in N$, so $\mathcal{N} \models \phi(\sigma(\underline{m}))$.

Now suppose $\mathcal{N} \models \phi(\sigma(\underline{m}))$. Then there is $\nu \in N$ with $\mathcal{N} \models \psi(\sigma(\underline{m})\nu)$. Since σ is an $\mathcal{L}$-isomorphism, and in particular is surjective, there is a $\mu \in M$ with $\sigma(\mu) = \nu$. Thus $\mathcal{N} \models \psi(\sigma(\underline{m})\sigma(\mu))$, whence $\mathcal{M} \models \psi(\underline{m}\mu)$ and $\mathcal{M} \models \phi(\underline{m})$. □

Definition 9.4. *An $\mathcal{L}$-morphism $\sigma\colon \mathcal{M} \to \mathcal{N}$ is elementary iff for all $\underline{m} \in M$ and all formulas ϕ, $\mathcal{M} \models \phi(\underline{m}) \Leftrightarrow \mathcal{N} \models \phi(\sigma(\underline{m}))$. Hence σ is elementary iff for all $\underline{m} \in M$, $\mathrm{tp}_{\mathcal{M}}(\underline{m}/\emptyset) = \mathrm{tp}_{\mathcal{N}}(\sigma(\underline{m})/\emptyset)$.*

Clearly an $\mathcal{L}$-isomorphism is elementary.

Exercise: Let $\mathcal{M}$, $\mathcal{N}$ be $\mathcal{L}$-structures. Prove that there is an elementary $\sigma\colon \mathcal{M} \to \mathcal{N} \Leftrightarrow$ there is $\mathcal{M}' \preccurlyeq \mathcal{N}$ with $\mathcal{M}' \simeq \mathcal{M} \Leftrightarrow \mathcal{N}$ is a model of $\mathrm{Th}(\mathcal{M}, M)$.

Exercise′: There is $\sigma\colon \mathcal{M} \to \mathcal{N} \Leftrightarrow$ there is $\mathcal{M}' \subseteq \mathcal{N}$ with $\mathcal{M}' \simeq \mathcal{M} \Leftrightarrow \mathcal{N}$ is a model of $\mathrm{Th}_{qf}(\mathcal{M}, M) - \{\phi(\underline{m})\colon \phi$ is quantifier free and $\mathcal{M} \models \phi(\underline{m})\}$.

Remark. If there is $\sigma\colon \mathcal{M} \to \mathcal{N}$ elementary, then $\mathcal{M} \equiv \mathcal{N}$. (If this is not obvious at first, think about the types.)

9.2 Transfinite back-and-forth

If $\underline{a} \in M$, $< \underline{a} >$ denotes the $\mathcal{L}$-structure generated by $\underline{a}$. This is the $\mathcal{L}$-structure with base set the closure under $\mathcal{F}^{\mathcal{M}}$ of $\underline{a} \cup \mathcal{C}^{\mathcal{M}}$ given the induced interpretation.

Definition 9.5. *A 0-isomorphism (also known as a local isomorphism) from $\mathcal{M}$ to $\mathcal{N}$ is a bijection from a finite tuple $\underline{a} \in M$ to a tuple $\underline{b} \in N$ with $< \underline{a} > \simeq < \underline{b} >$.*

Two tuples $\underline{a} \in M$ and $\underline{b} \in N$ are 0-equivalent (written $\underline{a} \simeq_0 \underline{b}$) if there is $\sigma\colon \underline{a} \to \underline{b}$ a 0-isomorphism. Note that $\mathcal{M}$ and $\mathcal{N}$ are 0-equivalent if $\emptyset_{\mathcal{M}} \simeq_0 \emptyset_{\mathcal{N}}$.

Example 9.6. Look at the models $(\mathbb{Z}, 0, +, -)$ and $(\mathfrak{S}_n, e, \cdot, ^{-1})$. We know $\mathbb{Z} \not\simeq \mathfrak{S}_n$, but $\mathbb{Z} \simeq_0 \mathfrak{S}_n$ since the $\mathcal{L}$-structures generated by the empty set only contain statements about the respective identities, which are isomorphic.

Example 9.7. Similarly, we have the models $(\mathbb{Z}, 0, 1, +, -, \cdot)$ and $(\mathbb{Q}, 0, 1, +, -, \cdot)$. Again we know $\mathbb{Z} \not\simeq \mathbb{Q}$, but we have $< \emptyset_{\mathbb{Z}} >= \mathbb{Z} =< \emptyset_{\mathbb{Q}} >$, since fractions can only be expressed using quantified statements so we only get the integers (through repeated addition or subtraction of 1). Hence $\mathbb{Z} \simeq_0 \mathbb{Q}$.

Example 9.8. The rings $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$ are not 0-isomorphic when $n \neq m$ since $\mathbb{Z}/n\mathbb{Z} \models \underbrace{1 + \ldots + 1}_{n \text{ times}} = 0$, but the same is not true for $\mathbb{Z}/m\mathbb{Z}$. We remark that in a similar vein, but with different results, $\mathbb{F}_p \simeq_0 \mathbb{F}_{p^2} \simeq_0 \ldots \simeq_0 \bar{\mathbb{F}}_p \simeq_0 \mathbb{F}_p[x]$, as they all have the same characteristic.

At this point, we make up some notation that is not standard which we will use in the rest of the class. We will write $\mathrm{tp}_n(\underline{a}) \coloneqq \{\phi(\underline{a})$ that hold with $\mathrm{qrk}(\phi) \leq n\}$. For example, $\mathrm{Th}(\mathcal{M}) = \mathrm{tp}_\omega(\emptyset_{\mathcal{M}})$ and $\mathrm{Th}_{qf}(\mathcal{M}) = \mathrm{tp}_0(\emptyset_{\mathcal{M}})$.

Lemma 9.9. $\underline{a} \simeq_0 \underline{b} \Leftrightarrow \text{tp}_0(\underline{a}) = \text{tp}_0(\underline{b})$. *In particular,* $\mathcal{M} \simeq_0 \mathcal{N} \Rightarrow \text{Th}_{qf}(\mathcal{M}) = \text{Th}_{qf}(\mathcal{N})$.

Proof. This follows from the definitions. □

Definition 9.10. *A bijection* $\sigma\colon \underline{a} \to \underline{b}$ *with finite domain* $\underline{a} \in M$ *and image* $\underline{b} \in N$ *is an* $(n+1)$*-isomorphism if:*

Forth $\forall \alpha \in M\, \exists \beta \in N\; \tau\colon \underline{a}\alpha \simeq_n \underline{b}\beta$ *extending* σ.

Back $\forall \beta \in N\, \exists \alpha \in M\; \tau\colon \underline{a}\alpha \simeq_n \underline{b}\beta$ *extending* σ.

We write $\underline{a} \simeq_{n+1} \underline{b}$ *if there is* $\sigma\colon \underline{a} \to \underline{b}$ *that is an* $(n+1)$*-isomorphism, and* $\mathcal{M} \simeq_{n+1} \mathcal{N}$ *if* $\emptyset_{\mathcal{M}} \simeq_{n+1} \emptyset_{\mathcal{N}}$.

Example 9.11. Again, we look at $(\mathbb{Z}, 0, +, -)$ and $(\mathfrak{S}_n, e, \cdot, ^{-1})$. Are they 1-isomorphic? No they are not, since $\mathfrak{S}_n \models \exists x(x^2 = e)$ (every $\mathfrak{S}_n$ has even order and so has such an element), but $\mathbb{Z} \not\models \exists x(x^2 = e)$. This means that (using the notation from the definition above) if β is such an x then there is no α that extends the 0-isomorphism.

Definition 9.12. *A bijection* $\sigma\colon \underline{a} \to \underline{b}$ *is an* α*-isomorphism (where* α *is a limit ordinal) if* $\forall \beta < \alpha$, σ *is a* β*-isomorphism.*

Example 9.13. Look at the models $(\mathbb{Z}, <)$ and $(\mathbb{Z} \coprod_< \mathbb{Z}, <)$. For all n, $(\mathbb{Z}, <) \simeq_n (\mathbb{Z} \coprod_< \mathbb{Z}, <)$ since one can always extend any two orderings between the two (although there is no uniform strategy such as with DLOs). Thus $(\mathbb{Z}, <) \simeq_\omega (\mathbb{Z} \coprod_< \mathbb{Z}, <)$. However, it is not the case that $(\mathbb{Z}, <) \simeq_{\omega+1} (\mathbb{Z} \coprod_< \mathbb{Z}, <)$ since there are orderings in $(\mathbb{Z} \coprod_< \mathbb{Z}, <)$ where two elements have an infinite number of elements between them, but this can never happen in $(\mathbb{Z}, <)$, so there is no way to extend σ.

In parting, we were told to think about how isomorphic the models $(\mathbb{Z}, s)$ and $(\mathbb{Z} \coprod_< \mathbb{Z}, s)$ are (s is the successor function).

End of Lecture 13.

Lecture 14.

Recall from last class that a morphism σ is elementary if for any tuple $\underline{a} \in \text{dom}\,\sigma$, $\mathcal{M} \models \phi(\underline{a}) \Leftrightarrow \mathcal{N} \models \phi(\sigma(\underline{a}))$. Hence we can talk about partial and elementary morphisms.

A local isomorphism (0-isomorphism) is a bijection $\sigma\colon \underline{a} \to \underline{b}$ such that $< \underline{a} > \simeq < \underline{b} >$. For example, $\mathcal{M} \simeq_0 \mathcal{N}$ iff $\text{Th}_{qf}(\mathcal{M}) = \text{Th}_{qf}(\mathcal{N})$, Also, $\mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Also recall that for $\underline{a} \in M$ and $\underline{b} \in N$, $\sigma\colon \underline{a} \to \underline{b}$ is an $(n+1)$-isomorphism if $\forall \alpha \in M\, \exists \beta \in N\ \tau\colon \underline{a}\alpha \simeq_n \underline{b}\beta$ extending σ, and similarly $\forall \beta\, \exists \alpha$ etc.

Finally, $\sigma\colon \underline{a} \to \underline{b}$ is a γ-isomorphism (γ a limit ordinal) if $\forall \delta < \gamma$, σ is a δ-isomorphism.

All of the preceding was review from last class, so consult those notes for expanded definitions.

We began by saying $(\mathbb{Z}, <) \simeq_{\omega+1} (\mathbb{Z} \coprod_< \mathbb{Z}, <)$ but not $(\mathbb{Z}, <) \simeq_{\omega+2} (\mathbb{Z} \coprod_< \mathbb{Z}, <)$, and told to check and prove this. I believe it's incorrect, since we said last time that $(\mathbb{Z}, <) \simeq_\omega (\mathbb{Z} \coprod_< \mathbb{Z}, <)$ but not $(\mathbb{Z}, <) \simeq_{\omega+1} (\mathbb{Z} \coprod_< \mathbb{Z}, <)$. This is true basically because there are elements in one model with an infinite number of other elements between them, but not in the other, so we can't extend the σ when we deal with an infinite number of elements.

We went on to say $(\mathbb{Z}, s) \simeq_1 (\mathbb{Z} \coprod_< \mathbb{Z}, s)$ but not $(\mathbb{Z}, s) \simeq_2 (\mathbb{Z} \coprod_< \mathbb{Z}, s)$. One can do the back and forth once, since any element in either model has an infinite number of successors, so whichever you pick you can always pick a corresponding one that will satisfy the same formulas. However, given $\alpha \simeq \beta$ there is no α' such that $\alpha\alpha' \simeq_0 \beta\beta'$ since $\forall \alpha'$, either $\alpha' = s^n(\alpha)$ or vice versa, but you can choose a β' such that this is not true.

Now, look at $(\mathbb{Q}, \mathcal{L}_{ring})$ and $(\mathbb{C}, \mathcal{L}_{ring})$. They are 0-isomorphic since the empty tuple just generates the integers in both cases, much like with $\mathbb{Z}$ and $\mathbb{Q}$ last time. However, they are not 1-isomorphic, since $\mathbb{C} \models \exists x(x^2 + 1 = 0)$, but $\mathbb{Q}$ does not.

Example 9.14. Let $\underline{a} \in (\mathbb{Q}, <)$ and $\underline{b} \in (\mathbb{R}, <)$, with $\sigma\colon \underline{a} \to \underline{b}$ a local isomorphism.

Claim: σ is a 1-isomorphism. In particular, with DLOs, any local isomorphism (which can be seen as an increasing function with finite support) is an α-isomorphism for any ordinal α, since this will show that we can always extend the isomorphism "+1".

Proof. We proceed by induction on α. If $\alpha = 0$ we're done since we are assuming that σ is a 0-isomorphism. Now let $\alpha = \beta + 1$. Consider $\sigma\colon \underline{a} \to \underline{b}$ that is a β-isomorphism. We will show that it is $\beta + 1$. Let $a' \in \mathbb{Q}$. Then using the same proof that we did to show that two countable DLOs are isomorphic, we may find $b' \in \mathbb{R}$ such that $\tau\colon \underline{a}a' \simeq_0 \underline{b}b'$. By induction, every 0-isomorphism is a β-isomorphism, so τ is a β-isomorphism, whence σ is a $(\beta+1)$-isomorphism.

Finally, if α is a limit ordinal, then our induction assumption tells us that $\forall \beta < \alpha$, σ is a β-isomorphism. Thus by definition σ is an α-isomorphism. □

Lemma 9.15. *For any natural number n, if $\underline{a} \simeq_n \underline{b}$, then* $\mathrm{tp}_n(\underline{a}) = \mathrm{tp}_n(\underline{b})$.

Proof. If $n = 0$, this is true by definition. Now, assuming that it's true for n, we attempt to show it is true for $n + 1$. This was left as an exercise. I find these concepts tricky, and I am currently unable to come up with a proof I find satisfactory. We were given the hint that we should find a witness (I'm unclear for what) and do a back and forth argument. I hope to return to this later. $\square$

Counterexample to the converse: Look at 0 in $(\mathbb{Z}, s)$ and 0_1 in $(\mathbb{Z} \coprod \mathbb{Z}, s)$ (0_1 is the 0 in the "left" copy of $\mathbb{Z}$). Then $\mathrm{tp}(0) = \mathrm{tp}(0_1)$ because with a finite formula you can't talk about the "right" $\mathbb{Z}$ using only the given constants and language. However, it is not the case that $0 \simeq_1 0_1$ since you can choose an element from the "right" $\mathbb{Z}$ and you won't be able to do the back step for the isomorphism. The difference here is that we are now talking about two elements, while for type we can really only talk about the given element.

Corollary. *If $\mathcal{M} \simeq_\omega \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$. This follows since we have seen that* $\mathrm{Th}(\mathcal{M}) = \mathrm{tp}_\omega(\emptyset_\mathcal{M})$.

Corollary. *DLO is complete. This follows since we have seen that any two DLOs are ω-isomorphic and so must be equivalent by the above corollary.*

Remark. We still can't prove that ACF_p is complete. We will need more properties, namely ω-saturation and compactness.

The converse of the first corollary does not hold in general. A counterexample is provided by the fact that $(\mathbb{Z}, <) \simeq_\omega (\mathbb{Z} \coprod_< \mathbb{Z}, <)$ but **not** $(\mathbb{Z}, <) \equiv (\mathbb{Z} \coprod_< \mathbb{Z}, <)$.

Theorem 9.16. *If $\mathcal{L}$ has finite signature and has no functions, then indeed $\mathcal{M} \equiv \mathcal{N} \Rightarrow \mathcal{M} \simeq_\omega \mathcal{N}$.*

Proof. We prove by induction that for all $n < \omega$, $l < \omega$ there are finitely many formulas $\phi_1^{n,l}, \ldots, \phi_{k(n,l)}^{n,l}$ of qrk$\leq n$ that describe the n-isomorphism classes of l-tuples. Here $k(n,l)$ is a constant that depends on n and l.

Since $\mathcal{L}$ is finite and function-free, there is a finite number of atomic formulas. In particular, each $\mathrm{tp}_0(\underline{a})$ can be described by a single formula (the conjunction of all the relations that $\underline{a}$ satisfies), and there are only finitely many possibilities since the base set is finite and there is a maximum arity of the relations in $\mathcal{L}$.

Now for the inductive step. Let $\underline{a}$ be an l-tuple. Consider $I := \{1 \leq i \leq k(n, l{+}1)$ such that $\exists \alpha \in M$ with $\mathcal{M} \models \phi_i^{n,l+1}(\underline{a}\alpha)\}$ That is, I is the set of is. Let $\phi(\underline{x}) := \bigwedge_{i \in I} \exists y(\phi_i^{n,l+1}(\underline{x}y)) \wedge \bigwedge_{i \notin I} \neg\exists y(\phi_i^{n,l+1}(\underline{x}y))$. Note that qrk$\phi \leq n+1$. ϕ describes the $(n+1)$-isomorphism class of $\underline{a}$, and there are finitely many possibilities for ϕ when $\underline{a}$ varies. This proves the claim about the number of formulas.

Now, if $\mathcal{M} \equiv \mathcal{N}$, then for all natural numbers n, $\mathrm{tp}_n(\emptyset_\mathcal{M}) = \mathrm{tp}_n(\emptyset_\mathcal{N})$. In particular, $\emptyset_\mathcal{M} \simeq_n \emptyset_\mathcal{N}$ for all n. By definition, $\emptyset_\mathcal{M} \simeq_\omega \emptyset_\mathcal{N}$. $\square$

Definition 9.17. *Let $\mathcal{F} \neq \emptyset$ be a family of 0-isomorphisms such that $\forall \sigma \in \mathcal{F}$, $\forall \alpha \in \mathcal{M}\, \exists \beta \in N$ and $\tau \in \mathcal{F}$ with $\tau(\alpha) = \beta$ and $\tau|_{\mathrm{dom}\,\sigma} = \sigma$. (This is the forth step.) Also let the family satisfy the similar statement with $\forall \beta\, \exists \alpha$. Then there is a unique maximal such family (known as a Karp family) whose elements are called ∞-isomorphisms.*

This definition sets up a family of functions that can always be extended. It's a very powerful concept.

Example 9.18.

- For DLOs, increasing functions with finite support are ∞-isomorphisms.
- If $\mathcal{M} \simeq \mathcal{N}$, then $\mathcal{M} \simeq_\infty \mathcal{N}$. However, the converse does not hold, since $\mathbb{R}$ and $\mathbb{Q}$ as DLOs are ∞-isomorphic but not $\mathcal{L}$-isomorphic.
- Exercise: If $\sigma\colon \mathcal{M} \simeq_\infty \mathcal{N}$, then for any ordinal α, σ is an α-isomorphism.

Remark. $\mathcal{M}$ and $\mathcal{N}$ being fixed models, there is an α such that any α-isomorphism is an ∞-isomorphism. In other words, once you get an α big enough, you're set. This follows from the facts that any set of ordinals has a supremum, and if $\alpha > \beta$, then having an α-isomorphism implies that there is a β-isomorphism.

Lemma 9.19. *If $\mathcal{M} \simeq_\infty \mathcal{N}$ and both are countable, then $\mathcal{M} \simeq \mathcal{N}$.*

Proof. We may use the back-and-forth method and cover everything. □

Caution: The above lemma is not true in uncountable cardinals, since ∞-isomorphisms have finite domain. It's also not true if we consider ω-isomorphisms. Think about the orderings $\mathbb{Z}$ and $\mathbb{Z} \coprod \mathbb{Z}$.

Question: If $\mathcal{M} \simeq_{\aleph_1} \mathcal{N}$ and $\mathcal{M}$, $\mathcal{N}$ countable, do we have $\mathcal{M} \simeq \mathcal{N}$? The answer is yes; for a more general result along these lines see the back-and-forth exercises, #5.

So, we now know the big chain of implications $\mathcal{M} \simeq \mathcal{N} \Rightarrow^1 \mathcal{M} \simeq_\infty \mathcal{N} \Rightarrow^2 \mathcal{M} \simeq_\alpha \mathcal{N} \Rightarrow_{\alpha \text{infinite}} \mathcal{M} \simeq_\omega \mathcal{N} \Rightarrow^3 \mathcal{M} \equiv \mathcal{N}$. The converse holds for arrow 1 if both are countable. The converse holds for arrow 2 if α is "big enough". The converse holds for arrow 3 if $\mathcal{L}$ is finite and function-free.

For now we're done with back-and-forth, but we will come back to it after we discuss compactness.

End of Lecture 14.

Lecture 15.

10 Compactness

Definition 10.1. *A theory T is finitely consistent if every finite $T_0 \subseteq T$ is consistent, i.e. T_0 has a model.*

Theorem 10.2 (Compactness)**.** *A finitely consistent theory is consistent.*

Today's class will largely be concerned with proving this theorem. This may be done fairly easily if one has the Completeness Theorem (and this is how Marker establishes it), but we take a different approach. I note that this approach can be found as a series of exercises in Marker pgs. 63 and 64.

10.1 The Łoś structure, a first proof

Definition 10.3. *Let $X \neq \emptyset$ be a set. Then $\mathcal{F} \subseteq P(X)$ is a filter on X if the following conditions hold:*

- $\emptyset \notin \mathcal{F}$
- *if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$*
- *if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$*

Remark. Let $\mathcal{B}$ be a family such that if $B_1, \ldots, B_n \in \mathcal{B}$, then $B_1 \cap \ldots \cap B_n \neq \emptyset$. Then there is a filter $\mathcal{F}$ such that $\mathcal{B} \subseteq \mathcal{F}$. This is a simple matter of letting $\mathcal{F} = \{Y \subseteq X : \exists B_1, \ldots, B_n \in \mathcal{B}(B_1 \cap \ldots \cap B_n \subseteq Y)\}$, i.e. $\mathcal{F}$ is the smallest family of sets that contains every finite intersection of members of $\mathcal{B}$. This clearly meets the three conditions.

Definition 10.4. *A maximal filter on X is called an ultrafilter.*

Remark. Let $\mathcal{U}$ be a filter. Then $\mathcal{U}$ is an ultrafilter iff $\forall A \in P(X)$, $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

Proof. ($\Rightarrow$) If $\mathcal{U}$ is an ultrafilter and $A \notin \mathcal{U}$, then A contains no subsets in $\mathcal{U}$ by the second property. If there were a set B in U such that $A \cap B = \emptyset$, then $B \subseteq X \setminus A \Rightarrow X \setminus A \in \mathcal{U}$, and we'd be done. If we assume that $X \setminus A$ is not in $\mathcal{U}$, we are left with the case that every set in $\mathcal{U}$ has a nontrivial intersection with A (as well as with $X \setminus A$, else we would have a subset of A). I claim that in this case, $\mathcal{U}$ is not maximal, which is a contradiction.

I would like to find a filter that extends $\mathcal{U}$ which contains $X \setminus A$ (which is not empty, since A is not the whole set). My first step in creating such a filter is to add to $\mathcal{U}$ the family $\mathcal{U}' := \{Y \in P(X) : Y \supseteq X \setminus A\}$. This new family satisfies the first and second conditions to be a filter, but not necessarily the third. So add every set of the form $D \cap E$, where $D \in \mathcal{U}'$ and $E \in \mathcal{U}$. These intersections are never empty, since every set in $\mathcal{U}'$ contains $X \setminus A$ and every set in $\mathcal{U}$ has a nontrivial intersection with $X \setminus A$. Call this family (the sets in $\mathcal{U}$, $\mathcal{U}'$ and these intersections) $\mathcal{U}''$. I claim that $\mathcal{U}''$ satisfies the first and third conditions (although we may no longer satisfy the second).

Let $D, D' \in \mathcal{U}'$ and $E, E' \in \mathcal{U}$. Then $E' \cap (D \cap E) = D \cap (E \cap E') = D \cap E''$ for some $E'' \in \mathcal{U}$ (since $\mathcal{U}$ is a filter), which is in $\mathcal{U}''$. For $D' \cap (D \cap E) = (D' \cap D) \cap E$ we have a similar situation if we can show that $D \cap D' \in \mathcal{U}'$. But this is clear since $D \cap D' \supseteq X \setminus A$. Using these two results it is clear that $(D \cap E) \cap (D' \cap E') = (D \cap D') \cap (E \cap E')$ is in $\mathcal{U}''$. So the intersection of any two sets in our $\mathcal{U}''$ is in $\mathcal{U}''$.

As I said earlier, we don't know that we have the second condition any longer. So we do the same trick as before, and add the family (call it $\mathcal{V}$) of all sets that contain sets in $\mathcal{U}''$ (specifically this is adding sets that were not already in our family that contain the intersections we added). I claim that now we have a filter. It clearly satisfies the first and second conditions, so I need only check the third. Let $S, S' \in \mathcal{V}$ but not in $\mathcal{U}''$ and $T \in \mathcal{U}''$. Then $S \supseteq D \cap E$ for some

D, E as above. So $S \cap T \supseteq (D \cap E) \cap T \in \mathcal{U}''$ by above, and by construction this means $S \cap T$ is in our new family. Similarly $S \cap S' \supseteq (D \cap E) \cap (D' \cap E') \in \mathcal{U}''$, so $S \cap S'$ is in our new family. This covers all possible types of intersections. So our new family is a filter which contains $X \setminus A \notin \mathcal{U}$, contradicting maximality of $\mathcal{U}$.

($\Leftarrow$) If $\mathcal{U}$ is a filter and $\forall A \in P(X)$, $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$, then $\mathcal{U}$ is maximal. This is because adding any other set would necessarily be adding the complement of some set in the filter, and the intersection of a set and its complement is the empty set, which can not be in a filter, so we no longer have a filter. Therefore $\mathcal{U}$ is an ultrafilter. □

Definition 10.5. *Let $a \in X$. The principal ultrafilter on a is $\mathcal{P}_a = \{Y \subseteq X : a \in Y\}$. This shows that ultrafilters exist. Do non-principal ultrafilters exist? Assuming AC, yes, but AC is actually slightly stronger than necessary to show this. In fact, there is something called the Ultrafilter Lemma which is independent of ZF (and weaker than AC) that says every filter is a subset of an ultrafilter that would of course tell us that non-principal ultrafilters exist.*

Example 10.6. The Fréchet filter. This is given by the family $\mathcal{F}_{r\acute{e}} = \{Y \subseteq X : Y$ is cofinite in $X\}$.

Remark. Let $\mathcal{F}$ be a filter on X. Consider $I_{\mathcal{F}} = \{X \setminus Y : Y \in F\}$. Then $I_{\mathcal{F}}$ is an ideal of $(P(X), \triangle, \cap)$, a Boolean algebra. Then $\mathcal{F}$ is an ultrafilter iff $I_{\mathcal{F}}$ is a maximal ideal. Alternately, an ultrafilter can be viewed as a $\{0, 1\}$ measure on X (where a set has measure 1 if it is in the ultrafilter, 0 otherwise). This measure is not necessarily σ-additive, however, since for instance if we look at the Fréchet filter each singleton has measure 0 but their countable sum (i.e. all of $\mathbb{N}$) has measure 1.

Lemma 10.7 (requires AC)**.** *If $\mathcal{B}$ has the finite intersection property (i.e. the intersection of any finite number of sets in $\mathcal{B}$ is nonempty), then there is an ultrafilter $\mathcal{U}$ with $\mathcal{B} \subseteq \mathcal{U}$.*

Proof. This is a simple matter of extending $\mathcal{B}$ to a filter as described earlier, and then finding the maximal filter that this is in. AC certainly allows us to do this, but to be pedantic we only need the Ultrafilter Lemma to prove this lemma. □

Remark. There are $2^{2^{\text{Card } X}}$ filters on X. This theorem is due to Hausdorff and is nontrivial.

Definition 10.8. *Let $I \neq \emptyset$ be a set, $\mathcal{U}$ be an ultrafilter on I, and for each $i \in I$ let $\mathcal{M}_i$ be an $\mathcal{L}$-structure. We define the ultraproduct of the $\mathcal{M}_i$s (with respect to $\mathcal{U}$), which we call $\mathcal{M}^*$. We write $\mathcal{M}^* = \prod_I \mathcal{M}_i/\mathcal{U}$. The definition of $\mathcal{M}^*$ will be in several parts.*

The base of $\mathcal{M}^*$ is the set $\{(m_i)_{i \in I} \in \prod_I M_i\}$ modulo the equivalence relation $(m_i) \backsim (n_i)$ if $\{i \in I : m_i = n_i\} \in \mathcal{U}$. I will quickly show this is in fact an equivalence relation. First, $(m_i) \backsim (m_i)$ since $I \in \mathcal{U}$. The relation is clearly *symmetric. Finally, if $(m_i) \backsim (n_i)$ and $(n_i) \backsim (t_i)$, then $\{i \in I \colon m_i = t_i\} \supseteq \{i \in I \colon m_i = n_i\} \cap \{i \in I \colon n_i = t_i\}$, which is in $\mathcal{U}$ by the second and third filter properties. Thus $(m_i) \backsim (t_i)$, so the relation is transitive. Note that we did not care that $\mathcal{U}$ was maximal here. In what follows we use brackets to denote equivalence classes.*

We interpret constants in $\mathcal{L}$ by $c^{\mathcal{M}^*} := [(c^{\mathcal{M}_i})_{i \in I}]$. We interpret relations by saying $R^{\mathcal{M}^*}([(m_i^1)], \ldots, [(m_i^k)])$ if $\{i \in I \colon R^{\mathcal{M}_i}(m_i^1, \ldots, m_i^k)\} \in \mathcal{U}$. This is sort of like taking a majority vote of the sets to determine if the relation holds. Similarly for functions we say $f([(m_i^1)], \ldots, [(m_i^k)]) = [(m_i^{k+1})]$ if $\{i \in I \colon f^{\mathcal{M}_i}(m_i^1, \ldots, m_i^k) = m_i^{k+1}\} \in \mathcal{U}$. We now check these are well-defined. This is very similar to showing the transitivity of the equivalence relation: if we have two equivalent sequences, then they must agree on a set in $\mathcal{U}$, which means the functions or relations must agree on a set containing a set in $\mathcal{U}$ and so is in $\mathcal{U}$.

Theorem 10.9 (Łoś). *Let $\phi([(m_i^1)], \ldots, [(m_i^k)])$ (which we will hereafter denote $\phi([\underline{m}_i])$ in keeping with our previous tuple notation) be a formula with parameters. Then $\mathcal{M}^* \models \phi([\underline{m}_i])$ iff $\{i \in I \colon \mathcal{M}_i \models \phi(\underline{m}_i)\} \in \mathcal{U}$.*

Proof. We proceed by induction on ϕ. If ϕ is atomic then it is clear from our definition of $\mathcal{M}^*$ that the theorem holds. Now, assume $\phi = \neg\psi$. If $\mathcal{M}^* \models \neg\psi([\underline{m}_i])$, then $\mathcal{M}^* \not\models \psi([\underline{m}_i])$. By induction, this means that $\{i \in I \colon \mathcal{M}_i \models \psi(\underline{m}_i)\} \notin \mathcal{U}$, and since $\mathcal{U}$ is an ultrafilter (so it always contains a set or its complement), this means that $\{i \in I \colon \mathcal{M}_i \not\models \psi(\underline{m}_i)\} \in \mathcal{U}$, i.e. that $\{i \in I \colon \mathcal{M}_i \models \neg\psi(\underline{m}_i)\} \in \mathcal{U}$. All of these steps are reversible, so the iff holds.

Next assume that $\phi = \psi \wedge \theta$. Then

$$\begin{aligned} \mathcal{M}^* \models \psi \wedge \theta &\Leftrightarrow \mathcal{M}^* \models \psi \text{ and } \mathcal{M}^* \models \theta \\ &\Leftrightarrow \{i \in I \colon \mathcal{M}_i \models \psi\}, \{i \in I \colon \mathcal{M}_i \models \theta\} \in \mathcal{U} \\ &\Leftrightarrow \{i \in I \colon \mathcal{M}_i \models \psi\} \cap \{i \in I \colon \mathcal{M}_i \models \theta\} \in \mathcal{U} \\ &\Leftrightarrow \{i \in I \colon \mathcal{M}_i \models \psi \wedge \theta\} \in \mathcal{U} \end{aligned}$$

Finally, assume that $\phi([\underline{m}_i]) = \exists x(\psi([\underline{m}_i][x]))$. Then if $\mathcal{M}^* \models \exists x(\psi([\underline{m}_i][x]))$ then find a witness y so that $\mathcal{M}^* \models \psi([\underline{m}_i][y]) \Leftrightarrow \{i \in I \colon \mathcal{M}_i \models \psi(\underline{m}_i y)\} \in \mathcal{U} \Rightarrow \{i \in I \colon \mathcal{M}_i \models \exists x(\psi(\underline{m}_i x))\} \in U$. The opposite direction is essentially identical. □

Proof of Compactness. Let T be a finitely consistent theory. Let $I := \{T_i\}$, where each T_i is a finite subset of T. For each $i \in I$, there is a model $\mathcal{M}_i$ satisfying T_i by the definition of finite consistency. For each $\phi \in T$, let $A_\phi := \{i \in I \colon \mathcal{M}_i \models \phi\}$. Then the family $\{A_\phi\}_{\phi \in T}$ has the finite intersection property, since the intersection of a finite family corresponds to $A_{\phi_1 \wedge \phi_2 \wedge \ldots \wedge \phi_n}$, which must be nonempty since T is finitely consistent. So we may take $\mathcal{U}$ to be an ultrafilter extending $\{A_\phi\}_{\phi \in T}$. Let $\mathcal{M}^* := \prod_I \mathcal{M}_i/\mathcal{U}$ (i.e. let $\mathcal{M}^*$ be the ultraproduct of the $\mathcal{M}_i$s with respect to $\mathcal{U}$). Then let $\phi \in T$. We have (using Łoś and our definitions) $\mathcal{M}^* \models \phi \Leftrightarrow \{i \in I \colon \mathcal{M}_i \models \phi\} \in \mathcal{U} \Leftrightarrow A_\phi \in \mathcal{U}$, which we know is true by construction, and so $\mathcal{M}^* \models T$. □

This is nice, but we are not perfectly happy, since in this construction we really have no control over the size of $\mathcal{M}^*$. Here we are not just referring to cardinality but also other things.

Example 10.10. In $\mathcal{L} = \{<\}$ with a variable b, we let $T^* :- \mathrm{Th}(\mathbb{N}, \mathbb{N}) \cup \{b > n\}_{n \in \mathbb{N}}$. That is, we add a statement that $b > n$ for each element of $\mathbb{N}$. This theory is finitely consistent, and in fact for every finite fragment, $\mathbb{N}$ can be made into a model. So by compactness there is $\mathbb{N}^* \models T^*$. We now have that $\mathbb{N} \preccurlyeq \mathbb{N}^*$, and b "lies above" $\mathbb{N}$. However, since $\mathbb{N} \models \forall x (x \neq 0 \rightarrow \exists y (y < x \wedge \forall z (z \leq y \vee z \geq x)))$ (which says that every nonzero element has a predecessor), then $\mathbb{N}^*$ has this property as well, and in particular b must have a predecessor. So one should not think of b as ω. In fact, we have that $\mathbb{N} \cup \{\omega\} \not\models T^*$, but $\mathbb{N} \coprod_< \mathbb{Z} \models T^*$. It should be noted that this model is not the one given by the Łoś construction, for which we have $\mathrm{Card}\, \mathbb{N}^* \geq 2^{\aleph_0}$.

Example 10.11. In a similar vein, for $\mathcal{L} = \{<\}$ and a variable b we have $T^* := \mathrm{Th}(\mathbb{R}, \mathbb{R}) \cup \{0 < b < \varepsilon\}_{\varepsilon > 0}$ is consistent. So we get a model R^* in which b is an infinitesimal.

Exercise: A theory with arbitrarily large finite models has an infinite model.

End of Lecture 15.

Lecture 16.

10.2 "Big models": three constructions using compactness

We have already learned one Löwenheim-Skolem theorem (the "downwards" version) which tells us how to find models smaller than the one that we have. Today we will learn another Löwenheim-Skolem theorem which tells us how to find bigger models.

Theorem 10.12 (Löwenheim-Skolem)**.** *Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure, $A \subseteq M$, and $\kappa \geq \mathrm{Card}\, A + \mathrm{Card}\, \mathcal{L}$ be infinite. Then there exists $\mathcal{M}_0$ of cardinality κ containing A and such that either $\mathcal{M}_0 \preccurlyeq \mathcal{M}$ or $\mathcal{M} \preccurlyeq \mathcal{M}_0$. (The first option holds if* $\mathrm{Card}\, \mathcal{M} \geq \kappa$ *and is just the downwards version.)*

Proof. Clearly it suffices to construct $\mathcal{M}_0 \succcurlyeq \mathcal{M}$ of cardinality $\geq \kappa$, since once this is acheived we can apply the downwards Löwenheim-Skolem Theorem to find a model of cardinality equal to κ.

Consider $T := \mathrm{Th}(\mathcal{M}, M) \cup \{c_\alpha \neq c_\beta\}_{\alpha \neq \beta}$ for new constant symbols $\{c_\alpha\}_{\alpha < \kappa}$. Let T_0 be a finite fragment of this theory. Such a fragment is of the form $\phi(\underline{m}) \wedge c_{\alpha_1} \neq c_{\alpha_2} \wedge c_{\alpha_3} \neq c_{\alpha_4} \wedge \ldots \wedge c_{\alpha_{2n-1}} \neq c_{\alpha_{2n}}$, which is to say that you get some statement in our original theory and a finite number of our new statements. We interpret $\phi(\underline{m})$ in the canonical way in $\mathcal{M}$. For the rest, we simply find $2n$ distinct elements in M (which is possible since M is infinite), and these elements model the other statements. So T_0 is consistent, and thus by compactness, T is consistent.

Let $\mathcal{M}_0$ be some model of T. By definition $\mathcal{M} \preccurlyeq \mathcal{M}_0$. We know $\operatorname{Card} M_0 \geq \kappa$, since at the very least all those new constant symbols must be in M_0. Then we may apply downwards Löwenheim-Skolem to get a model with $\operatorname{Card} M_0 = \kappa$, and this model will still be an elementary extension of $\mathcal{M}$. $\square$

Definition 10.13. *Let κ be a cardinal. A theory T is κ-categorical if any two models of cardinality κ are isomorphic (and it has models of cardinality κ).*

Example 10.14. *DLO* is $\aleph_0$-categorical, but not $\aleph_1$-categorical, as we have seen in the past. ACF_p is $\aleph_1$-categorical (which we have not proven yet) but not $\aleph_0$-categorical (since for example $\mathbb{F}_p \not\cong \mathbb{F}_p(x)$).

Exercise: Let $\mathcal{L}_{\mathbb{Q}-vector\ space} = \{0, +, -, q, \cdot\}$ (where there is a q for each $q \in \mathbb{Q}$. Is $\mathrm{Th}_{\mathbb{Q}-v.s.}$ $\aleph_0$-categorical? Is it κ-categorical for $\kappa \geq \aleph_1$?
The answer to the first is no, since for instance $\mathbb{Q} \not\cong \mathbb{Q}^2$. The answer to the second is yes, since we are relatively limited in some sense with regards to the size; our variables are all in $\mathbb{Q}$. These answers were given in class; I apologize that the second is somewhat hazy.

Theorem 10.15. *If T is κ-categorical in some κ, then T is complete.*

Proof. Let $\mathcal{M}, \mathcal{N} \models T$. By Löwenheim-Skolem, there are $\mathcal{M}_0, \mathcal{N}_0$ of $\operatorname{Card} \kappa$ such that ($\mathcal{M}_0 \preccurlyeq \mathcal{M}$ or $\mathcal{M} \preccurlyeq \mathcal{M}_0$) and ($\mathcal{N}_0 \preccurlyeq \mathcal{N}$ or $\mathcal{N} \preccurlyeq \mathcal{N}_0$). In any case, by κ-categoricity $\mathcal{N}_0 \backsimeq \mathcal{M}_0$, so $\mathcal{M}_0 \equiv \mathcal{N}_0 \Rightarrow \mathcal{M} \equiv \mathcal{N}$. Thus any two models of T are equivalent, so it is complete. $\square$

Theorem 10.16 (Morley)**.** *If T is κ-categorical for some $\kappa \geq \aleph_1$, then it is categorical for any $\lambda \geq \aleph_1$.*

Deloro later warned me that this may be misstated; what Marker has as Morley's theorem is related but seemingly very different, and doesn't come until much later in the book. Regardless of all that, this is a non-trivial result, and we won't be proving it.

Now we would like to concern ourselves with getting common extensions for two models.

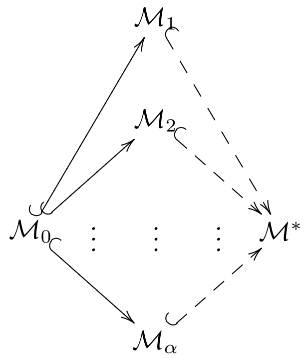
Theorem 10.17. *If $\mathcal{M}_1 \equiv \mathcal{M}_2$, then there is $\mathcal{M}^*$ with $\mathcal{M}_1 \hookrightarrow \mathcal{M}^*$ elementarily and $\mathcal{M}_2 \hookrightarrow \mathcal{M}^*$ elementarily.*

Proof. We will first prove the consistency of $T := \mathrm{Th}(\mathcal{M}_1, M_1) \cup \mathrm{Th}(\mathcal{M}_2, M_2)$. We will then show the model of this theory meets our criterion. In the proof, we'll show that $\mathcal{M}_1$ is a model of finite fragments of this, since then we will easily have $\mathcal{M}_1 \preccurlyeq \mathcal{M}^*$, which is even stronger than what we need. This means that we need to add constants c_{m_2} for elements $m_2 \in M_2$, since the two models may have different underlying sets.

A finite fragment T_0 of T is of the form $\{\phi(\underline{m_1})\} \cup \{\psi(\underline{c_{m_2}})\}$. We need to show this is consistent. We know how to interpret $\phi(\underline{m_1})$ in $\mathcal{M}_1$ already, so we need to figure out how to interpret $\underline{c_{m_2}}$. $\mathcal{M}_2 \models \psi(\underline{c_{m_2}})$, and so $\mathcal{M}_2 \models \exists \underline{x}(\psi(\underline{x}))$. Hence by elementary equivalence, $\mathcal{M}_1 \models \exists \underline{x}(\psi(\underline{x}))$, and so we can witness this statement in M_1. Thus T_0 is consistent, and by compactness so is T.

Let $\mathcal{M}^* \models T$. Then we have $\mathcal{M}_1 \preccurlyeq \mathcal{M}^*$ and $\mathcal{M}_2 \hookrightarrow \mathcal{M}^*$ elementarily via the map $m_2 \mapsto \underline{c_{m_2}}^{\mathcal{M}^*}$. □

Corollary.



If the normal arrows (which are elementary morphisms) exist, then the dotted ones do.

This is clear for a finite number of models, since if elementary morphisms exist between models they are equivalent, so we can iteratively apply the above result. By compactness this is good enough to imply the general result. Remark: This is not category theory in the following sense: there is no uniqueness up to isomorphism guaranteed. It's a nice picture nonetheless.

We now move on to talk about extending partial elementary isomorphisms. Recall that $\sigma\colon \mathcal{M} \to \mathcal{N}$ is a partial elementary morphism if $\operatorname{dom}\sigma \subseteq \mathcal{M}$ and $\forall A \in \operatorname{dom}\sigma$, $\mathcal{M} \models \phi(\underline{a})$ iff $\mathcal{N} \models \phi(\sigma(\underline{a}))$.

Theorem 10.18. *Let σ be a partial elementary automorphism of $\mathcal{M}$. Then there is $\mathcal{M}^* \succcurlyeq \mathcal{M}$ and τ extending σ a partial elementary automorphism of $\mathcal{M}^*$ with $M \subseteq \operatorname{dom}\tau$.*

Proof. Add constants c_m and d_m for each $m \in M$. Let $T \coloneqq \operatorname{Th}(\mathcal{M}, M) \cup \{\phi(c_{\sigma(\underline{m}_1)}, d_{\underline{m}_2})\colon \mathcal{M} \models \phi(\underline{c_{m_1}}, \underline{c_{m_2}})\}$. In $\operatorname{Th}(\mathcal{M}, M)$ we use constants c, and in the statement $M \models \phi(\underline{c_{m_1}}, \underline{c_{m_2}})$ we must have $\underline{c_{m_1}} \in \operatorname{dom}\sigma$. We claim that T is consistent.

Let $\psi(c_{\underline{m}_0}) \wedge \phi(c_{\sigma(\underline{m}_1)}, d_{\underline{m}_2})$ be a finite fragment of T, with $\mathcal{M} \models \phi(c_{\underline{m}_1}, c_{\underline{m}_2})$ and $\underline{m}_1 \in \operatorname{dom}\sigma$. Hence $\mathcal{M} \models \exists \underline{x}(\phi(\underline{m}_1, \underline{x}))$, so $\mathcal{M} \models \exists \underline{x}(\phi(\sigma(\underline{m}_1), \underline{x}))$ since σ is elementary. We interpret $d_{\underline{m}_2}$ to be a witness of that statement, and we interpret $\psi(c_{\underline{m}_0})$ in the canonical way. Thus T is finitely consistent, and so is consistent.

Let $\mathcal{M}^* \models T$. We know $\mathcal{M} \preccurlyeq \mathcal{M}^*$ (through constants c). Let $\tau\colon m \mapsto d_m^{\mathcal{M}^*}$, where $m \in M$. Then τ is elementary, extends σ, and is defined on (at least) M. □

There is a similar theorem about extending the image of a σ, which should follow from nearly the exact same proof since we have σ an elementary automorphism, so we talk about σ^{-1}.

Corollary. *Let σ be a partial elementary morphism from $\mathcal{M}$ to $\mathcal{M}$. Then there is $\mathcal{M}' \succcurlyeq \mathcal{M}$ and σ' extending σ a global elementary automorphism of $\mathcal{M}'$.*

Proof. Let $\mathcal{M}_0 = \mathcal{M}$ and $\tau_0 = \sigma$. Then we may apply the above theorem to find $\tau_1 : \mathcal{M}_1 \hookrightarrow \mathcal{M}_1$ with $M_0 \subseteq \operatorname{dom} \tau_1$. Then we can find $\tau_2 : \mathcal{M}_2 \hookrightarrow \mathcal{M}_2$ with $M_1 \subseteq \operatorname{im} \tau_2$. We proceed in this fashion, extending first the domain and then the image. Ultimately we end up with $\mathcal{M}' := \cup_\omega \mathcal{M}_i \succcurlyeq \mathcal{M}$, and $\sigma' := \cup_\omega \tau_i$ extending σ. By definition σ' is a bijection, so we're done. ☐

End of Lecture 16.

Lecture 17.

11 Saturated models

What follows is the main definition of model theory.

Definition 11.1. *Let $\underline{x}$ be a tuple of variables of length n and $A \subseteq M$ be a set drawn from some structure $\mathcal{M}$. A (partial) n-type over A (with parameters in A) is a consistent (with* $\operatorname{Th}(\mathcal{M}, A)$*) set $\pi(\underline{x})$ of formulas $\phi(\underline{xa})$ with $\underline{a}$ ranging over A.*

Remark. This generalizes $\operatorname{tp}(\underline{m}/A)$ from before. This is because those types required $\mathcal{M} \models \phi(\underline{xa})$, whereas for these types it is enough that there is some model $\mathcal{M}^* \models \phi(\underline{xa})$. As Deloro put it, it is possible that $\underline{x}$ is "not there yet". If π is a type over $A \subseteq M$, then there is $\mathcal{M}^* \succcurlyeq \mathcal{M}$ and $\underline{m} \in M^*$ satisfying π.

Example 11.2. $\{x > n\}_{n\in\mathbb{N}}$ is a 1-type in $(\mathbb{N}, <)$, as we have seen that there is a model for these statements. (This is a 1-type since there is only 1 variable.) However, there is clearly no element satisfying these statements in $\mathbb{N}$.

Definition 11.3. *If π is a type and $\underline{c} \in \mathcal{M}$ satisfies $\pi(\underline{x})$, we say that $\underline{c}$ is a realization of π (or that $\mathcal{M}$ realizes π). Otherwise, we say $\mathcal{M}$ omits π. Note that the types we used to work with were all realized types, which was briefly mentioned when we introduced them.*

Remark. If π is a (partial) type over $A \subseteq M$, then every finite fragment of π is realized in $\mathcal{M}$.

Proof. Let $\phi(\underline{xa})$ be a finite fragment of π. It is consistent to $\operatorname{Th}(\mathcal{M}, A)$, and hence a consequence, so $\mathcal{M} \models \exists \underline{x} \phi(\underline{xa})$. ☐

Example 11.4. Returning to our earlier example, note that every finite fragment of $\{x > n\}_{n\in\mathbb{N}}$ is realized in $\mathbb{N}$, though the whole thing is not.

Caution: "types over A" make sense only when $\mathrm{Th}(\mathcal{M}, A)$ is understood. An example explains what is meant by this. Is $\{x^2 = -1\}$ a type? In $\mathbb{R}$ the answer is no, since we have the sentence $\{\forall x(x^2 \neq -1)\}$, so the sentence is not consistent with the theory. In $\mathbb{C}$, the answer is yes. The point is that one has to fix a complete theory $\mathrm{Th}(\mathcal{M}, A)$ before talking about types over A. Hence a partial n-type over A is a consistent extension in $\mathcal{L}_{A\cup\{\underline{x}\}}$ of $\mathrm{Th}(\mathcal{M}, A)$.

Example 11.5. $\{P(x) \neq 0 \colon P \in \mathbb{Q}[t] \setminus \{0\}\}$ is the type of a non-algebraic number. Similarly, $\{P(x, y) \neq 0 \colon P \in \mathbb{C}[t, u] \setminus \{0\}\}$ with parameters in $\mathbb{C}$ is the type of an algebraically independent pair over $\mathbb{C}$. This type is realized in the algebraic closure of $\mathbb{C}(t, u)$. (We can't talk about $\mathbb{C}[t, u]$, it is no model of $\mathrm{Th}(\mathbb{C}, \mathbb{C})$.

Definition 11.6. *Let $\mathcal{M}$ be a structure, κ an infinite cardinal. $\mathcal{M}$ is κ-saturated if for any $A \subseteq M$ with* $\mathrm{Card}\, A < \kappa$ *and any* 1*-type π over A, π is realized* in $\mathcal{M}$.

Usage note: people say ω-saturated for $\aleph_0$-saturated, and sometimes ω_1-saturated for $\aleph_1$-saturated.

Example 11.7.

- $(\mathbb{Q}, <)$ is ω-saturated. It's certainly not ω_1-saturated. For example $\{x > q\}_{q\in\mathbb{Q}}$ is a 1-type which is not realized in $\mathbb{Q}$, and the size of this set is $\aleph_0$.
- $(\mathbb{R}, <)$ is ω-saturated. (As we have seen, all "pure" DLOs are ω-saturated.) But again we can show it's not ω_1-saturated, by the same sort of trick with $\{x > n\}_{n\in\mathbb{N}}$. For orderings, a countable cofinal subchain would certainly be enough to show that the ordering is not ω_1-saturated. However, there's more ways than that to do it, since for instance you could have a countable descending (and therefore bounded) sequence which would break things in a similar way.
- $(\mathbb{N}, 0, 1, +)$ is not ω-saturated. The trick will be to say something like $\{x > n\}_{n\in\mathbb{N}}$ without parameters. We can do this basically because we can talk about the ordering in here using only the operations. $\{\exists y(x = y + \underbrace{1 + \ldots + 1}_{n\ times})\}_{n\in\mathbb{N}}$ is a 1-type over $\emptyset$ that says "$\{x > n\}_{n\in\mathbb{N}}$" using the language we have available, and this type is not realized in $\mathbb{N}$.
- $(\mathbb{R}, 0, 1, +, \cdot)$ is not ω-saturated for the same reason.
- $(\mathbb{C}, 0, 1, +, \cdot)$ is ω-saturated, since basically all you can talk about is polynomials, and it's an algebraically closed field. It's also ω_1-saturated.

Understanding a theory is understanding its types.

Definition 11.8. *A model $\mathcal{M}$ is saturated if it is $|M|$-saturated.*

Remark. $\mathcal{M}$ can't be $|M|^+$-saturated, since $\{x \neq m\}_{m\in M}$ can't be realized in M.

Lemma 11.9. *Let κ be infinite. Then $\mathcal{M}$ is κ-saturated iff for all $A \subseteq M$ of cardinality $< \kappa$ and $n > 0$, all n-types over A are realized in $\mathcal{M}$.*

Proof. (⇐) Obvious from the definition, as in particular we know that all 1-types are realized in $\mathcal{M}$.

(⇒) We proceed by induction. By definition, if $\mathcal{M}$ is κ-saturated then every 1-type over A is realized in $\mathcal{M}$. Now, let A have cardinality $< \kappa$, $\rho(\underline{z})$ and $(n+1)$-type over A, and write $\underline{z} = \underline{x}y$. (In particular $\underline{x}$ has length n.) We want to show that ρ is realized in $\mathcal{M}$. Consider $\pi_1(x) := \{\exists y \phi(\underline{x}y) \colon \phi \in \rho\}$. (The parameters from A are implicit here.)
Claim: π_1 is an n-type over A.

Proof: Let $\exists y_1 \phi_1 \wedge \ldots \wedge \exists y_k \phi_k$ be a finite fragment of π_1. Then $\exists y(\phi_1(\underline{x}y) \wedge \ldots \wedge \phi_k(\underline{x}y))$ is consistent, since what's inside the parentheses is a consequence of ρ. Hence the finite fragment is consistent, so by Compactness π_1 is consistent. ▫
By induction, there is some $\underline{c} \in M$ realizing π_1. Let $\pi_2(y) := \{\phi(\underline{c}y) \colon \phi(\underline{x}y) \in \rho\}$.
Claim: π_2 is a 1-type over $A \cup \{\underline{c}\}$ (which has cardinality $< \kappa$ still).

Proof: Nearly identical to the above claim. Any finite fragment of π_2 is a conjunction $\phi_1(\underline{c}y) \wedge \ldots \wedge \phi_k(\underline{c}y)$ that is a consequence of ρ, so it's consistent. ▫
Again by induction, we know there is $d \in M$ that realizes π_2. Clearly $\underline{c}d$ is a realization of ρ in $\mathcal{M}$. □

Proposition 11.10. *Let $\mathcal{M}$ be a structure and κ and infinite cardinal. Then there is $\mathcal{M}^* \succcurlyeq \mathcal{M}$ that is κ-saturated.*

Proof. Count all subsets of $\mathcal{M}$ of cardinality $< \kappa$, and all 1-types over these: $\{\rho_\beta\}_{\beta<\alpha}$. For each $\beta < \alpha$, there is $\mathcal{M}_\beta \succcurlyeq \mathcal{M}$ realizing ρ_β. Then by a result from last class we know that there is $\mathcal{M}^1 \succcurlyeq \mathcal{M}_\beta$ for each $\beta < \alpha$. Hence any 1-type over some $A \subseteq M$ of cardinality $< \kappa$ is realized in $\mathcal{M}^1$ (not necessarily in $\mathcal{M}$!).

Let $\mathcal{M}^2$ be obtained similarly for $\mathcal{M}^1$, and iterate this process. Take unions at limit ordinals, i.e. if α is a limit ordinal then $\mathcal{M}^\alpha := \cup_{\beta<\alpha}\mathcal{M}^\beta$. Finally, let $\mathcal{M}^* := \cup_{\gamma<\kappa^+}\mathcal{M}^\gamma$.

Let $A \subseteq M^*$ with $\operatorname{Card} A < \kappa$. For each $a \in A$, let γ_a be the least ordinal such that $a \in M^\gamma$. The set $\{\gamma_a\}_{\operatorname{Card} A<\kappa<\kappa^+}$ can't be cofinal in κ^+ (which is regular), so there is $\Gamma < \kappa^+$ with $A \subseteq M^\Gamma$. Then every 1-type over A is realized in $\mathcal{M}^{\Gamma+1}$ by construction, and hence is realized in $\mathcal{M}^*$. □

At this point we had a digression on algebraically closed fields, wherein we created a similar construction.

Remark. Unfortunately, we have no good control over $\operatorname{Card} M^*$ unless we make set-theoretic assumptions (such as GCH or the existence of inaccessible cardinals) or we know that we have few types to worry about (for an appropriate notion of "few").

Exercise: How many types over $\emptyset$ are there in $(\mathbb{Z}, 0, 1, +, -, \cdot)$? We were told the result of this could be phrased "Arithmetic is not stable!"

End of Lecture 17.

Lecture 18.

A quick note for the exercises on formulas: Given a theory T, two formulas $\phi(\underline{x})$ and $\psi(\underline{x})$ are equivalent modulo T if $T \models \forall \underline{x}(\phi(\underline{x}) \leftrightarrow \psi(\underline{x}))$. For example, in T_{field}, the statements $\exists y(xy = 1)$ and $x \neq 0$ are equivalent.

11.1 Applications to algebraically closed fields

Definition 11.11. *A theory T is said to eliminate quantifiers if for all $n > 0$ and all $\phi(\underline{x})$ (where $\underline{x}$ has length n), there is a quantifier free $\psi(\underline{x})$ such that $T \models \forall \underline{x}(\phi(\underline{x}) \leftrightarrow \psi(\underline{x}))$. In other words, $\phi(\underline{x})$ is equivalent to a quantifier free statement modulo T.*

Caution: this strongly depends on the language.

Example 11.12. *DLO* eliminates quantifiers, since if two elements satisfy the same placement, they satisfy the same theorems. Check out Theorem 3.1.3 in Marker (pg. 72).

Theorem 11.13 (Elimination Theorem)**.** *T eliminates quantifiers iff for all $\mathcal{M}, \mathcal{N} \models T$ and all $\underline{m} \in \mathcal{M}$, $\underline{n} \in \mathcal{N}$ we have if $\mathrm{tp}_0^{\mathcal{M}} \underline{m} = \mathrm{tp}_0^{\mathcal{N}} \underline{n}$, then $\mathrm{tp}^{\mathcal{M}} \underline{m} = \mathrm{tp}^{\mathcal{N}} \underline{n}$.*

Proof. This is exercise 4 from the exercises on formulas. (It's stated slightly differently there; check it out if you find this unclear.) We will see another proof later. This is Corollary 3.1.6 in Marker (pg. 75); it's a corollary to something very similar to exercise 3 from the same exercises. □

We'll prove that ACF eliminates quantifiers.

Remark. Let $\mathbb{K}, \mathbb{L} \models ACF$. Then there is a local isomorphism $\mathbb{K} \simeq_0 \mathbb{L}$ iff char $\mathbb{K}$ =char $\mathbb{L}$. This is pretty clear; it's similar to the proof that (in the language of rings) $\mathbb{Z}/n\mathbb{Z}$ is not locally isomorphic to $\mathbb{Z}/m\mathbb{Z}$ if $m \neq n$ from Day 13.

Lemma 11.14. *Let $\mathbb{K} \models ACF$. Then $\mathbb{K}$ is ω-saturated iff $\mathbb{K}$ has infinite transcendence degree over its prime field.*

Proof. ($\Rightarrow$) Consider the n-type $\pi(\underline{x}) \coloneqq \{P(\underline{x}) \neq 0\}_{P \in A[\underline{T}] \setminus \{0\}}$, where A is the prime ring. We see that π is the type of an algebraically independent family over k the prime field. It uses no parameters, so by our assumption, this type is realized. Thus there is an algebraically independent family of cardinal n for all n. Hence tr $\deg(\mathbb{K}/k) \geq n$ for all n, and therefore is infinite.

($\Leftarrow$) Let $A \subseteq \mathbb{K}$ be finite, and π be a 1-type over A. If π contains some statement of the form $P(x) = 0$ (where $P \neq 0$), then x is algebraic over $\langle A \rangle$, so there is a solution in $\mathbb{K}$. If there is no such statement, then x is transcendental over $\langle A \rangle$. As tr $\deg(\mathbb{K}/k) \geq \aleph_0$, we have tr $\deg(\mathbb{K}/\langle A \rangle) \geq \aleph_0$, since $\langle A \rangle \subseteq k$. Therefore π is realized in $\mathbb{K}$, since we can just pick some transcendental element. Thus $\mathbb{K}$ is ω-saturated. □

Lemma 11.15. *Let $\mathbb{K}, \mathbb{L} \models ACF$ be ω-saturated. Then any local isomorphism is an ∞-isomorphism.*

Proof. First, notice that in the case char $\mathbb{K} \neq$ char $\mathbb{L}$ the statement is trivial, since there are no local isomorphisms. Otherwise, assume that there is a $\sigma\colon \langle a \rangle \simeq_0 \langle b \rangle$. We will just do the forth step to extend this; the back is the same. Let $\alpha \in \mathbb{K}$. We wish to find a suitable β to map it to. If α is algebraic over k (the prime field), let $\mu \coloneqq \mathrm{Irr}_k^\alpha$, the minimum polynomial of α over k. Consider $\sigma(\mu) \in l[T]$, solve it in $\mathbb{L}$, and let β be a solution. If instead α is transcendental, then simply let β be transcendental over l. □

Remark. Note that if the fields are not ω-saturated, and hence have infinite transcendence degree, then the above proof would not work. For instance, $\bar{\mathbb{Q}}$ is not ∞-isomorphic to $\bar{\mathbb{Q}(\pi)}$.

Key step: Understand the ω-saturated models. It is enough to establish back-and-forth between these.

Proposition 11.16. *Let T be a theory such that 0-isomorphisms between ω-saturated models are ∞-isomorphisms. Then T eliminates quantifiers.*

Proof. Let $\mathcal{M}, \mathcal{N} \models T$ and $\underline{m} \in \mathcal{M}$, $\underline{n} \in \mathcal{N}$. Assume that these tuples satisfy the same quantifier-free formulas. By the Elimination Theorem, it suffices to show that $\underline{m}$ and $\underline{n}$ satisfy the same formulas to prove the proposition.

Let $\mathcal{M}^* \succcurlyeq \mathcal{M}$ and $\mathcal{N}^* \succcurlyeq \mathcal{N}$ be ω-saturated extensions. We know that $\underline{m} \in \mathcal{M}^*$ and $\underline{n} \in \mathcal{N}^*$ are 0-isomorphic, so by assumption, they are ∞-isomorphic. Therefore $\underline{m}$ and $\underline{n}$ have the same types (in $\mathcal{M}^*$ and $\mathcal{N}^*$).

As $\mathcal{M}^* \succcurlyeq \mathcal{M}$, we have $\mathrm{Th}(\mathcal{M}^*, M) = \mathrm{Th}(\mathcal{M}, M)$, and similarly for $\mathcal{N}^*$. In particular, $\mathrm{Th}(\mathcal{M}^*, \underline{m}) = \mathrm{Th}(\mathcal{M}, \underline{m})$, which is the same thing as saying $\mathrm{tp}^{\mathcal{M}^*}(\underline{m}) = \mathrm{tp}^{\mathcal{M}}(\underline{m})$. Since we have a similar result for $\mathcal{N}^*$, and we knew the types were the same in the extensions, we get $\mathrm{tp}^{\mathcal{M}}(\underline{m}) = \mathrm{tp}^{\mathcal{N}}(\underline{n})$, which is the condition from the Elimination Theorem. □

Theorem 11.17. *ACF admits quantifier elimination. This is clear, using our second lemma and the above proposition.*

Remark. ACF is not complete.

Exercises:

- Let $\mathcal{M}$, $\mathcal{N}$ be ω-saturated. Prove $\mathcal{M} \equiv \mathcal{N}$ iff $\mathcal{M} \simeq_\infty \mathcal{N}$.
- Show T is complete iff for all ω-saturated $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M} \equiv \mathcal{N}$.
- Show if $\mathcal{M} \simeq_\infty \mathcal{N}$ and $\mathcal{M}$ is ω-saturated, then so is $\mathcal{N}$.
- If $\mathcal{M}, \mathcal{N}$ are saturated, of the same cardinality, and $\mathcal{M} \equiv \mathcal{N}$, then $\mathcal{M} \simeq \mathcal{N}$.

Definition 11.18. *A theory T is model-complete if for all $\mathcal{M}, \mathcal{N} \models T$ with $\mathcal{M} \subseteq \mathcal{N}$, one has $\mathcal{M} \preccurlyeq \mathcal{N}$.*

Example 11.19. If T eliminates quantifiers, then T is model-complete. This can be seen basically by taking any formula modeled by $\mathcal{M}$, replacing it with a quantifier-free formula, and recalling that these are preserved under substructures, so $\mathcal{N}$ models it as well. A slightly more fleshed-out version of this can be found in Marker (Prop. 3.1.14, pg. 78). Note that this does not mean that T is complete, since by this condition ACF is model-complete.

The converse of the above is not true. We will show an example. Consider the theory of $\mathbb{R}$ as a ring. In particular, there is no "$<$". (You may recall that this relation can be expressed anyways; herein lies the trick.) We admit that theory of Real Closed Fields eliminates quantifiers (this was left as an exercise). So, the theory in $\mathcal{L}_{ordering}$ is model-complete. As "$<$" is definable mod RCF in $\mathcal{L}_{ring}$, $\text{Th}_{ring}(\mathbb{R})$ is still model-complete. However, it does not eliminate quantifiers, since there is no way for us to get rid of the statement $\exists y(y^2 = x)$, which says x is positive and sets up our concept of $<$.

Theorem 11.20 (Hilbert's Nullstellensatz)**.** *Let $\mathbb{K}$ be an algebraically closed field. Let $I \lhd \mathbb{K}[\underline{x}]$ be a proper ideal. Then there is $\underline{a} \in \mathbb{K}^n$ such that $\forall P \in I$, $P(\underline{a}) = 0$.*

Proof. Let $I \subseteq m \lhd \mathbb{K}[\underline{x}]$ be a maximal ideal. (Note that since $\mathbb{K}[\underline{x}]$ is noetherian, finding such a maximal ideal does not require AC.) Let $\mathbb{L} := \overline{\mathbb{K}[\underline{x}]/m}^{alg}$. Then $\mathbb{L} \supseteq \mathbb{K}$, since m is a proper ideal, so the units (i.e. the elements of $\mathbb{K}$) must still be in $\mathbb{L}$. As $\mathbb{K}[\underline{x}]$ is noetherian (and thus ideals are finitely generated), there are $P_1, P_2, \ldots, P_n$ with $(P_1, P_2, \ldots, P_n) = m$. Now $\mathbb{L} \models \exists \underline{a}(P_1(\underline{a}) = 0 \wedge \ldots \wedge P_n(\underline{a}) = 0)$, since for instance the projection of $\underline{x}$ is a solution. By model-completeness, $\mathbb{K} \models \exists \underline{a}(P_1(\underline{a}) = 0 \wedge \ldots \wedge P_n(\underline{a}) = 0)$, and as any $P \in I \subseteq m$ is a combination of the P_is, $P(\underline{a}) = 0$. □

End of Lecture 18.

Lecture 19.

When trying to prove that a theory is "nice", going to ω-saturated models is enough. Today, we will see that ACF is very close to being complete.

Theorem 11.21. *ACF_p and ACF_0 are complete. Separately, if ϕ is a sentence in $\mathcal{L}_{ring}$, then $ACF_0 \models \phi$ iff $ACF_p \models \phi$ for all but finitely many ps.*

Proof. First, we will show that ACF_p (p prime or 0) is complete. Fix the characteristic p. Then $\emptyset$ (the empty arrow) is a local isomorphism. If $\mathbb{K}$, $\mathbb{L}$ are ω-saturated models of the same ACF_p, then $\mathbb{K} \simeq_0 \mathbb{L}$. Hence a result from last class says that $\mathbb{K} \simeq_\infty \mathbb{L}$. In this case $\mathbb{K} \equiv \mathbb{L}$. Now if $\mathbb{K}$, $\mathbb{L}$ are not necessarily ω-saturated (but of the same characteristic still), there exist $\mathbb{K} \preccurlyeq \mathbb{K}^*$ and $\mathbb{L} \preccurlyeq \mathbb{L}^*$, both ω-saturated. Since $\mathbb{K}^* \equiv \mathbb{L}^*$, we get that $\mathbb{K} \equiv \mathbb{L}$. Hence we have shown that any two models of ACF_p are elementarily equivalent, whence ACF_p is complete.

Now for the second statement. Assume that ϕ holds in all but finitely many prime characteristics. Consider $T := ACF_0 \cup \{\phi\}$. We will prove that T is consistent. Let T' be a finite fragment of T. This takes the form of some of the axioms of ACF, some statements like $2 \neq 0$, $3 \neq 0$, $5 \neq 0$, and perhaps ϕ. By assumption, there is p such that $\overline{\mathbb{F}_p} \models T'$, so T is finitely consistent, and thus by Compactness has a model. Since ACF_0 is complete (i.e. maximal as a theory), it must be that $ACF_0 \models \phi$ (else it would model $\neg\phi$ and T would not be consistent).

Conversely, assume $ACF_0 \models \phi$. Then $\neg\phi$ cannot hold in all but finitely many $\overline{\mathbb{F}_p}$s, since in this case by an argument similar to above we would have $ACF_0 \models \neg\phi$. But since each ACF_p is complete, this means that ϕ holds in all but finitely many $\overline{\mathbb{F}_p}$s. □

Definition 11.22. *A definable set in an $\mathcal{L}$-structure $\mathcal{M}$ is some $X \subseteq M^n$ such that there is $\phi(\underline{x}, \underline{a})$ (where $\underline{x}$ is of length n and $\underline{a}$ is from M) with for all $\underline{x} \in M^n$, $\mathcal{M} \models \phi(\underline{xa})$ iff $\underline{x} \in X$. When no parameters are required, X is called* 0*-definable (or $\emptyset$-definable). (See Marker, pg. 19.)*

Example 11.23. A definable (in $\mathcal{L}_{ring}$) set $X \subseteq \mathbb{C}$ is finite or cofinite, since you basically get to say X is the roots of a certain polynomial, or it's everything but the roots of a certain polynomial (or some combination). I think that's why, anyways. Think about the case of $\mathbb{R}$. Also, Marker has several examples.

Theorem 11.24 (Ax)**.** *Let $V \subseteq \mathbb{C}^n$ be an affine algebraic variety over $\mathbb{C}$, and $f\colon V \to V$ be an injective morphism of varieties. Then f is surjective.*

Remark. We didn't actually define what an affine variety is, beyond the intuitive notion that it consists of "things we can say with polynomials". If you really want to know, I'm pretty sure it's in Marker, and it's definitely a well-known concept, so it's easy to find a definition. Also, **caution**; $x \mapsto x^2$ is surjective from $\mathbb{C}$ to $\mathbb{C}$, but not injective.

Proof. We'll work in a finite set. V is defined by some formula $\phi(\underline{xa})$, while f is defined by some $\psi(\underline{xya})$. (We're assuming $\underline{a}$ is a big enough tuple to contain the relevant parameters for both.) The following statements are first order:

- ψ defines a functional relation from realizations of ϕ to realizations of ϕ.
- ψ is injective.
- ψ is not surjective. (This looks something like $\exists\underline{y}(\phi(\underline{ya}) \wedge \forall\underline{x}(\phi(\underline{xa}) \to \neg\psi(\underline{xya}))$.)

So if Ax's theorem does not hold, then $\mathbb{C} \models \exists\underline{a}(\psi$ defines an injective, but not surjective function $\phi(\underline{a}) \to \phi(\underline{a}))$. This is a first order statement, no worries. Hence by our previous result, there is a p prime with $\overline{\mathbb{F}_p} \models \exists\underline{a}(\psi$ defines...). This means there is $\underline{b} \in \overline{\mathbb{F}_p}$ with $\phi(\underline{xb}) = V' \subseteq \overline{\mathbb{F}_p}^n$ and a function $f'\colon V' \to V'$ that is injective and not surjective (defined by $\psi(\underline{xyb})$).

Let $\underline{c} \in V'$ such that $f'^{-1}(\{c\}) = \emptyset$. Recall that $\overline{\mathbb{F}_p} = \cup_k {\mathbb{F}_p}^k$, and let $q = p^k$ be big enough to have $\underline{bc} \in \mathbb{F}_q$. Consider $W = V' \cap {\mathbb{F}_q}^n$. Then we have $g := f'|_W^W$ a function that is injective but not surjective on a finite set, which is impossible. Thus we have reached a contradiction, so Ax's theorem holds. □

We do have a function $W \to W$ since f' is given by polynomials. It is certainly not surjective. It is injective since g is. (Note: I'm including this since it's in my notes, but it makes no sense to me now. I hope to clear it up later.)

Why is the theorem not true with "surjective implies injective"? Because the $\exists$s don't go down to arbitrary substructures, and we need those to preserve surjectivity.

Remark. Let U be a non-principal ultrafilter on the set of primes. Then $\mathcal{M}^* = \prod \overline{\mathbb{F}_p}/U \backsimeq \mathbb{C}$. Why is this? Well, it's clearly a field, since all the $\overline{\mathbb{F}_p}$s agree on that point. It's algebraically closed for the same reason. For each $p_0 \in P$ (P the set of primes), the set $\{p \colon \overline{\mathbb{F}_p} \models p_0 = 0\} \notin U$ since U is not principal. Hence $\mathcal{M}^*$ does not have characterstic p_0 for all p_0 prime. Thus $\mathcal{M}^* \models ACF_0$. We have $\operatorname{Card} \mathcal{M}^* = 2^{\aleph_0}$ (left as an exercise). It is the case that ACF_0 is $2^{\aleph_0}$ categorical, so $\mathcal{M}^* \backsimeq \mathbb{C}$.

Exercise: Show the theory of divisible, torsion-free abelian groups is complete but not $\aleph_0$-categorical.

End of Lecture 19.

Lecture 20.

11.2 Boolean algebras

Definition 11.25. *A Boolean algebra is a $(0, 1, \neg, \wedge, \vee)$ structure with the axioms one expects. These are sentences like $\neg 0 = 1$, $\forall x(\neg\neg x = x)$, $\forall x, y(\neg(x \wedge y) = \neg x \vee \neg y)$, etc.*

Example 11.26. $P(X)$ for X a set, with the structure $(\emptyset, X, {}^c, \cap, \cup)$. We could also talk about the set of formulas (equivalent modulo some theory T). The difference between the two is that in $P(X)$ you have minimal nonzero elements (the singletons), but not in the second case.

Definition 11.27. *We write $x \leq y$ if $x \wedge y = x$.*

Definition 11.28. *We say some $x \neq 0$ is an atom of $\mathcal{B}$ (a Boolean algebra) if $\mathcal{B} \models \forall y(y < x \to y = 0)$. The singletons of $P(X)$ provide a good example of this.*

Definition 11.29. *An algebra is atomic if $\forall x(\exists y(y \leq x$, and y is an atom$))$.*

We have two extensions of the theory of Boolean algebras, obtained by adding the appropriate axioms: atom-free Boolean algebras, and atomic Boolean algebras.

Lemma 11.30. *Let $\mathcal{B}$ be a Boolean algebra. Then there is some atom-free Boolean algebra $\mathcal{B}'$ such that $\mathcal{B} \subseteq \mathcal{B}'$.*

Proof. Consider $\mathcal{B}^2$ with the canonical structure (in particular, operations will be applied elementwise) and the diagonal embedding, i.e. $x \in \mathcal{B} \mapsto (x, x) \in \mathcal{B}^2$. If $x \in \mathcal{B}$ is an atom, then $(x, x) \in \mathcal{B}^2$ is not one, since $(x, x) > (x, 0) > (0, 0)$. However, we don't know if perhaps this has introduced new atoms, so we repeat the process by embedding $\mathcal{B}^2$ into $\mathcal{B}^4$, and then into $\mathcal{B}^8$, etc. Let $\mathcal{B}^\infty$ be the inductive limit of this process. Deloro did not go into this construction, but I believe that in the end, an element of $\mathcal{B}^\infty$ must have all but a finite number of coordinates equal to 1, making it similar to a direct sum. Otherwise I won't be able to show the next part. We get that $\mathcal{B}$ embeds into $\mathcal{B}^\infty$, and that $\mathcal{B}^\infty$ is atom-free. This is because an atom of $\mathcal{B}^\infty$ would have to be an atom of some $\mathcal{B}^{2^n}$ (else we easily have some element less than it) followed by all 1s. But by construction there is some element of $\mathcal{B}^{2^{n+1}}$ followed by all 1s less than the atom, contradiction. □

Definition 11.31. *Two theories T and T' are companions if for every $\mathcal{M} \models T$, there is $\mathcal{M} \subseteq \mathcal{M}' \models T'$ and conversely.*

See the exercises on formulas, #2, which also tells us that T and T' are companions iff $T_\forall = T'_\forall$.

The theory of Boolean algebras and the theory of atom-free Boolean algebras are companions. This follows more or less from the fact that you can really only model the same universal statements in each, except for the one stating that there are no atoms. But our lemma tells us that we can find an extension for our plain old Boolean algebra that is atom-free.

To show a similar fact for atomic Boolean algebras, we need a result.

Definition 11.32. *A topological space is called totally disconnected if there is a topological basis for it made of clopen sets. Note that this is not the same as saying all closed sets are open, since for example $\{x\}$ is closed. An example of a totally disconnected space is the Cantor space $P(\mathbb{N}) = 2^\mathbb{N}$ with the product topology.*

Theorem 11.33 (Stone)**.** *Every Boolean algebra $\mathcal{B}$ is isomorphic to the algebra of closed-open sets of some Hausdorff, compact, totally disconnected space which is unique up to homeomorphism. This is called the Stone space of $\mathcal{B}$.*

Proof. We will approach this by looking at the set $X := \{U \subseteq \mathcal{B} \colon U$ is an ultrafilter $\} = \{\mathcal{B}/m \colon m \in \mathrm{Spm}\mathcal{B}_{ring}\}$. (Spm is the set of maximal ideals; when we first introduced ultrafilters we noted this correspondence.)

A standard topology for X is generated by open sets of the form $O_b := \{U \in X \colon b \in U\}$, where $b \neq 0$ in $\mathcal{B}$. No O_b is empty, else we would have some b in no ultrafilters, which is impossible since there's always the principal ultrafilter. Further, we see that $O_b^c = O_{\neg b}$ since b and $\neg b$ can not be in the same filter, else

$b \wedge \neg b = 0$ is in the filter, and since ultrafilters are maximal, one or the other is in each ultrafilter. Thus the O_bs are clopen. In a similar vein, we see that $O_{b_1} \wedge \ldots \wedge O_{b_n} = O_{b_1 \wedge \ldots \wedge b_n}$. Thus the O_bs form a basis for the topology. So this space is totally disconnected.

Next we show X is Hausdorff. If $U_1 \neq U_2$, then there is some $b \in U_1 \setminus U_2$. Then $U_1 \in O_b$, and $U_2 \in O_{\neg b}$.

To show X is compact, we will appeal to the characterization of compactness in terms of the finite intersection property. That is, we will show that for any family $\{C_i\}_I$ of closed sets with the finite intersection property, we in fact have that $\cap_I C_i \neq 0$. For this, we may assume that the C_is are of the form O_{b_i}, since these form a basis and so each C_i would be a finite intersection of such sets anyway. So now we are proving that $\cap_I O_{b_i} \neq 0$.

Let $E = \{b_i\}_I \subseteq \mathcal{B}$. If E is in a filter, then it is certainly in some ultrafilter U, so $U \in \cap_I O_{b_i}$ and we are done. Otherwise we must have that there are $b_1, \ldots, b_n \in E$ with $b_1 \wedge \ldots \wedge b_n = 0$, which goes against the finite intersection property. This makes sense in light of the fact that if E has the finite intersection property, then the set of finite intersections of E is a filter. Thus we have shown that X is compact.

Now we wish to show that every clopen set of X is some O_b. If $O \subseteq X$ is clopen, consider $\cup_{O_b \subseteq O} O_b = O$ (the equality follows since O is open). Since O is closed and in a compact space, O is compact, whence we actually have $\cup_1^n O_{b_i} = O = O_{b_1 \vee \ldots \vee b_n}$.

Consider the mapping from $\mathcal{B}$ to the clopen algebra of X given by $b \mapsto O_b$. This is a morphism of algebras. We have just shown that it's surjective since all clopen sets are of the form O_b. Finally, it's injective, since if $O_b = \emptyset$, then $\{b\}$ is in no filter, which is only true for $b = 0$. This is the desired isomorphism.

We now prove that the Stone space is unique up to homeomorphism. Let X and Y be Hausdorff, compact, totally disconnected spaces such that the clopen algebra of X is isomorphic to the clopen algebra of Y (via the function f). We will prove that X is homeomorphic to Y. Let $x \in X$ and consider $U \coloneqq \{O$ clopen in $X \colon x \in O\}$. We see that U is an ultrafilter of the clopen algebra of X. Similarly, $V \coloneqq \{f(O) \colon O \in U\}$ is an ultrafilter of the clopen algebra of Y.

As Y is compact and V has the finite intersection property, we know that $\cap_{P \in V} P \neq \emptyset$. Let $y \in \cap V$. We get that $V = \{P$ clopen in $Y \colon y \in P\}$ since V is maximal as a filter. As Y is Hausdorff, $\cap V = \{y\}$, since there are open sets disjoint from $\{y\}$ surrounding every other element, and those open sets are not in V. So we may define a function $g \colon X \to Y$ by $g(x) = y$.

It remains to be seen that g is a homeomorphism. We can see that g is bijective by simply doing the same construction in the other direction. We also know that $g^{-1}(P)$ with P clopen is equal to $f^{-1}(P)$. Assume not. Then there is some $a \in g^{-1}(P) \setminus f^{-1}(P)$ (or vice versa). In this case, $g(a) = b$, where b is in V, the intersection of f(clopen sets containing a). In particular, $b \in P$, so we must have $f^{-1}(b) \neq a$. So since X is Hausdorff and totally disconnected, there are disjoint clopen sets around $f^{-1}(b)$ and a. But then there is a clopen set in V which does not contain b, which would mean that $g(a) \neq b$, a contradiction. Similar work shows it is not possible for there to be elements in $f^{-1}(P)$ not in $g^{-1}(P)$. Hence $g^{-1}(P)$ is clopen, as f is a homeomorphism and P is clopen. Therefore g is a continuous bijection of compact spaces, and is therefore a homeomorphism. □

As a consequence of this theorem, $\mathcal{B} \hookrightarrow P(\text{Stone}\mathcal{B})$, which is an atomic Boolean algebra. This proves that the theory of atomic Boolean algebras and the theory of Boolean algebras are companions in a way similar to the case of atom-free Boolean algebras.

We now direct our attention to the case of atomic Boolean algebras. From now on, $\mathcal{B}$ will always be considered to be atomic.

Remark. $x \in \mathcal{B}$ is determined by the collection of atoms $\leq x$.

Proof. Assume that x and y are greater than the same atoms. If there is an atom a such that $a \leq x \wedge \neg y$, then $a \leq x$, and by our assumption this means that $a \leq y$. But we have $a \leq \neg y$, so this would give us $a \leq y \wedge \neg y = 0$, a contradiction since atoms are nonzero. Thus by the atomicity of a we have that $x \wedge \neg y = 0 = \neg x \wedge y$. But then we have

$$\begin{aligned} x &= x \wedge 1 \\ &= (x \wedge y) \vee \underbrace{(x \wedge \neg y)}_{0} \\ &\leq y \end{aligned}$$

Similarly we see that $y \leq x$, whence $y = x$. □

Lemma 11.34. *Let $\mathcal{B}$ be an infinite atomic Boolean algebra. Then $\mathcal{B}$ is ω-saturated iff for all $a \in \mathcal{B}$ lying above infinitely many atoms, there is $b \in \mathcal{B}$ such that $a \wedge b$ and $a \wedge \neg b$ also lie above infinitely many atoms.*

Proof. ($\Rightarrow$) Let a be lying above infinitely many atoms, say $\{a_i\}_{i \in I}$. Consider $\pi(x) := \{x \wedge a \text{ lies above } n \text{ atoms}\}_{n \in \mathbb{N}} \cup \{\neg x \wedge a \text{ lies above } n \text{ atoms}\}_{n \in \mathbb{N}}$. This is a 1-type over $\{a\}$. I claim a finite fragment is satisfied by $x = a_1 \vee \ldots \vee a_n$. Clearly $x \wedge a = x$ lies above n atoms. Now $\neg x \wedge a \geq a_{n+1}$, since $a_{n+1} \leq a = (a \wedge x) \vee (a \wedge \neg x)$, and it's not less than the first part (which is x). The same argument holds for all a_i with $i > n$, an infinite amount. Therefore $\neg x \wedge a$ lies above n elements. Thus π is consistent. By ω-saturation, there is some $b \in \mathcal{B}$ satisfying π.

($\Leftarrow$) We will see this next class. □

End of Lecture 20.

Lecture 21.

You may recall that last time we were proving the following lemma:

Lemma 11.35. *Let $\mathcal{B}$ be an infinite Boolean algebra. Then $\mathcal{B}$ is ω-saturated $\Leftrightarrow$ for all $a \in \mathcal{B}$, if a lies above infinitely many atoms, there is $b \in \mathcal{B}$ such that $a \wedge b$ and $a \wedge \neg b$ do as well.*

Proof. ($\Rightarrow$) We did this direction in the last class.

($\Leftarrow$) Before we begin, we remark that we won't really be using this direction as a result. This is because in order to do back-and-forth, one needs to understand some properties of ω-saturated models, but it's not really necessary to characterize them.

Let $\underline{a} \in \mathcal{B}$ and $\pi(x)$ be a 1-type over $\underline{a}$. We will prove that π is realized in $\mathcal{B}$. We first talk about the sort of statements that π can make. It describes the isomorphism type of $< \underline{a}, x >$ with statements like $x \neq a_1 \wedge x \neq a_2$, etc. It also describes how many atoms lie under each conjunction of elements (or their negations) from the tuple $x\underline{a}$. Notice that the first kind of information is encoded in the second, as $c = 0$ iff c lies above no atoms (since $\mathcal{B}$ is atomic), so we may write $a = b$ if ($a \wedge \neg b = 0$ and $\neg a \wedge b = 0$).

Now, for each $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$, let $\underline{a}^{\underline{\epsilon}} = \epsilon_1 a_1 \wedge \dots \wedge \epsilon_n a_n$, where $\epsilon_i a_i = a_i$ if $\epsilon_i = 1$ and $\epsilon_i a_i = \neg a_i$ if $\epsilon_i = -1$. So by what we've said above, π is a collection of statements regarding the atoms lying under $\underline{a}^{\underline{\epsilon}} \wedge x$ and $\underline{a}^{\underline{\epsilon}} \wedge \neg x$.

Let $\mathrm{At}_{\underline{\epsilon}} := \{$ atoms of $\mathcal{B}$ lying under $\underline{a}^{\underline{\epsilon}}\}$. We notice that the set of atoms of $\mathcal{B}$ is $\coprod_{\underline{\epsilon} \in \{\pm 1\}^n} \mathrm{At}_{\underline{\epsilon}}$. This is clearly the case if $\underline{a}$ has length 1, since every atom would then be either under a or under $\neg a$. This is becuase if β is an atom, then

$$\begin{aligned} \beta &= \beta \wedge 1 \\ &= (\beta \wedge a) \vee (\beta \wedge \neg a) \end{aligned}$$

whence it must lie under one or the other (it can't lie under both, as it would then be 0). An induction argument takes care of larger tuples.

For each $\underline{\epsilon} \in \{\pm 1\}^n$, π says how many atoms lie under $\underline{a}^{\underline{\epsilon}} \wedge x$ and how many lie under $\underline{a}^{\underline{\epsilon}} \wedge \neg x$. We note that if $\underline{a}^{\underline{\epsilon}} \wedge x$ lies above infinitely many atoms, π says $(\exists \alpha_1, \dots, \alpha_k,\ \alpha_i$ atoms with $\alpha_i \leq \underline{a}^{\underline{\epsilon}} \wedge x)$ for each k. Otherwise it basically just lists the atoms that $\underline{a}^{\underline{\epsilon}} \wedge x$ lies above.

We fix some $\underline{\epsilon}$, with the aim of finding some $\underline{x}^{\underline{\epsilon}} \in \mathcal{B}$ witnessing the required properties of $\underline{a}^{\underline{\epsilon}} \wedge x_{\underline{\epsilon}}$ and $\underline{a}^{\underline{\epsilon}} \wedge \neg x_{\underline{\epsilon}}$. If π says "$\underline{a}^{\underline{\epsilon}} \wedge x$ is above exactly k atoms" and "$\underline{a}^{\underline{\epsilon}} \wedge \neg x$ is above exactly l atoms", then $\underline{a}^{\underline{\epsilon}}$ is above exactly $k + l$ atoms, say $\{\alpha_1, \dots, \alpha_{k+l}\}$. In this case, we let $\underline{x}_{\underline{\epsilon}} = \alpha_1 \vee \dots \vee \alpha_k$, and it is clearly a witness as desired. If π says "$\underline{a}^{\underline{\epsilon}} \wedge x$ is above exactly k atoms" and "$\underline{a}^{\underline{\epsilon}} \wedge \neg x$ is above infinitely many atoms", then we see that $\underline{a}^{\underline{\epsilon}}$ lies above infinitely many atoms. Otherwise it lies above N atoms for some finite N, and "$\underline{a}^{\underline{\epsilon}} \wedge \neg x$ is above at least $N + k + 1$ atoms" is in π but false, which contradicts the consistency of π. In this case, let $\{a_i\}_{i \in I}$ be the atoms under $\underline{a}^{\underline{\epsilon}}$. Then $x_{\underline{\epsilon}} := \alpha_1 \vee \dots \alpha_k$ will work, and a similar construction works for the statement "$\underline{a}^{\underline{\epsilon}} \wedge x$ is above infinitely many atoms" and "$\underline{a}^{\underline{\epsilon}} \wedge \neg x$ is above exactly k many atoms". The only remaining case is if π says "$\underline{a}^{\underline{\epsilon}} \wedge x$ and $\underline{a}^{\underline{\epsilon}} \wedge \neg x$ are both above infinitely many atoms". Then again we see that $\underline{a}^{\underline{\epsilon}}$ lies above infinitely many atoms, and so our assumption tells us that there is a suitable $\underline{x}_{\underline{\epsilon}}$.

Now let $b := \bigvee_{\epsilon \in \{\pm 1\}^n} \underline{x}_{\underline{\epsilon}} \in \mathcal{B}$. Since the sets $\mathrm{At}_{\underline{\epsilon}}$ are disjoint (we noted this above), we find that b realizes π. Thus for any type over a countable set of parameters, $\mathcal{B}$ contains some element that realizes it, and so $\mathcal{B}$ is ω-saturated. □

You may have understood that the theory of atomic Boolean algebras (which we'll call, say, Th_{BoolAt}) is complete; we will prove this next.

For each $k \in \mathbb{N}$, add a predicate $A_k(x)$ for "x lies above at least k atoms". Let $\mathcal{L}_{BoolAt} = \mathcal{L}_{Bool} \cup \{A_1, A_2, \ldots\}$.

Proposition 11.36. *The theory of infinite atomic Boolean algebras is complete; it eliminates quantifiers in $\mathcal{L}_{BoolAt}$.*

Proof. Before we proceed, a word of caution. Quantifier elimination is highly language-dependent, so we need to use the language that we set up immediately prior to this proposition. However, neither completeness nor ω-saturation are language-dependent.

We'll prove that local isomorphisms (which are language-dependent) between ω-saturated models are ∞-isomorphisms. We first remark that the empty morphism is a local $\mathcal{L}_{BoolAt}$-isomorphism between ω-saturated models, so we always have some local isomorphism.

Let $\mathcal{B}$, $\mathcal{B}'$ be infinite atomic Boolean algebras with $\underline{a} \simeq_0 \underline{b}$ in $\mathcal{L}_{BoolAt}$. Let $\alpha \in B$. The proof of the lemma from earlier today contains in the direction we showed last class an explanation of how to find β such that $\underline{a}\alpha \simeq_0 \underline{b}\beta$. This was done essentially by reflecting the realization of a given type. If you look at the proof, this would not be possible to do in the original language, since the type contains statements like "$x \wedge a$ lies above n atoms", which requires quantifiers to express without our new predicates. Since we can always extend any local isomorphism (with respect to $\mathcal{L}_{BoolAt}$), we see that they are all ∞-isomorphisms.

Thus any two ω-saturated models are ∞-isomorphic, and so by one of our exercises from Day 18 they are elementarily equivalent. Since every model elementarily extends to an ω-saturated model, this shows that any two models are elementarily equivalent. Thus the theory is complete. Quantifier elimination comes from a proposition we proved on Day 18 that said if local isomorphisms between ω-saturated models of a theory T are ∞-isomorphisms, then T eliminates quantifiers. □

Caution: If we looked instead at $\mathcal{L}_{Bool}$, the theory is not even model-complete. Take $\mathcal{B}$ atomic, and let $b \in \mathcal{B}$ be an atom. Let $\mathcal{B} \subseteq \mathcal{B}'$ atom-free, and $\mathcal{B}' \subseteq \mathcal{B}''$ atomic, which we recall we can do from last time. But by the way the construction works, $b \in \mathcal{B}''$ is not an atom. Thus it is clear that $(\mathcal{B}, b)$ is not elementarily equivalent to $(\mathcal{B}'', b)$, so $\mathcal{B}$ is not elementarily included in $\mathcal{B}''$, thus the theory is not model-complete.

Remark. $\mathcal{B} \subseteq \mathcal{B}''$ is not an inclusion of $\mathcal{L}_{BoolAt}$ structures.

Proposition 11.37. *The theory of atom-free Boolean algebras is complete and eliminates quantifiers in $\mathcal{L}_{Bool}$.*

Proof. Let $\mathcal{B}$, $\mathcal{B}'$ be two models of the theory. Let $\underline{a} \simeq_0 \underline{b}$. Then let $\alpha \in \mathcal{B}$. For each $\underline{\epsilon} \in \{\pm 1\}^n$, if $\underline{a}^{\underline{\epsilon}} \wedge \alpha = 0$, let $\beta_{\underline{\epsilon}} := 0$, and if $\underline{a}^{\underline{\epsilon}} \wedge \neg\alpha = 0$, let $\beta_{\underline{\epsilon}} := \underline{b}^{\underline{\epsilon}}$. We note that if $\underline{a}^{\underline{\epsilon}} \wedge \alpha = \underline{a}^{\underline{\epsilon}} \wedge \neg\alpha = 0$, then $\underline{a}^{\underline{\epsilon}} = 0$, and since $\underline{b} \simeq_0 \underline{a}$, one has $\underline{b}^{\underline{\epsilon}} = 0$, so the definition still makes sense. If $\underline{a}^{\underline{\epsilon}} \wedge \alpha \neq 0$ and $\underline{a}^{\underline{\epsilon}} \wedge \neg\alpha \neq 0$, let $\beta_{\underline{\epsilon}} :=$ some element such that $\underline{b}^{\underline{\epsilon}} \wedge \beta_{\underline{\epsilon}} \neq 0$ and $\underline{b}^{\underline{\epsilon}} \wedge \neg\beta_{\underline{\epsilon}} \neq 0$. Any element such that $0 < \beta_{\underline{\epsilon}} < \underline{b}^{\underline{\epsilon}}$ will do. This is because one must have $\underline{b}^{\underline{\epsilon}} \leq \neg\beta_{\underline{\epsilon}}$ unless $\underline{b}^{\underline{\epsilon}} = 1$, in which case $\beta_{\underline{\epsilon}}$ will still work fine. Such an element must exist since $\mathcal{B}'$ is atom-free.

Eventually, let $\beta := \bigvee_{\underline{\epsilon} \in \{\pm 1\}^n} \beta_{\underline{\epsilon}}$. Clearly $\underline{a}\alpha \simeq_0 \underline{b}\beta$. We have shown that every local isomorphism is an ∞-isomorphism, whence we get quantifier elimination. As the empty morphism is always local, we get that theory is complete as in our last proposition. □

Definition 11.38. *A model-companion of a theory is a companion which is model-complete. We will see shortly that if a theory has a model companion it is unique, so we may refer to "the" model companion.*

Example 11.39. The theory of atom-free Boolean algebras is a model-companion of the theory of (atomic) Boolean algebras.

Lemma 11.40. *A theory has at most one model-companion.*

Proof. Let T, T' be model-complete theories which are companions. We will prove that $T = T'$. Let $\mathcal{M}_0 \models T$. Since T, T' are companions, there is $\mathcal{M}_0 \subseteq \mathcal{M}_1' \models T'$, $\mathcal{M}_1' \subseteq \mathcal{M}_2 \models T$, etc. Let $\mathcal{M}^* := \cup_{i\ even} \mathcal{M}_i = \cup_{j\ odd} \mathcal{M}_j'$. By the model-completeness of the two theories, $\mathcal{M}_0 \preccurlyeq \mathcal{M}^*$ and $\mathcal{M}_1' \preccurlyeq \mathcal{M}^*$. Hence $\mathcal{M}^* \models T'$. As $\mathcal{M}_0 \equiv \mathcal{M}^*$, it follows that $\mathcal{M}_0 \models T'$. Hence $T \models T'$, and we can prove the other direction similarly. □

Counterexample: The theory of groups has no model-companion. See the exercises.

Example 11.41. DLO is the model companion of the theory of linear orderings. ACF is the model-companion of the theory of domains.

Exercises:

- Consider the theory T of discrete chains without endpoints. This is given by the axioms:
  - $<$ is a linear ordering.
  - $\forall x \exists x^+ (x^+ > x \wedge \forall y (y \leq x \vee y \geq x^+))$
  - $\forall x \exists X^- (x^- < x \wedge \forall y (y \leq x^- \vee y \geq x))$

  Prove that $T = \mathrm{Th}(\mathbb{Z}, <)$. Find a suitable language in which T eliminates quantifiers.

- Find the model-companion of the theory of abelian groups, if it exists.

End of Lecture 21.

Lecture 22.

Definition 11.42. *A Boolean ordering is an ordering satisfying the same properties as $\leq$ in a Boolean algebra. An example axiom would be $\forall x \exists y (\forall z (z \leq x \wedge z \leq y \rightarrow \forall t z \leq t) \wedge \forall y' (y'$ has the same property as $y \rightarrow y \leq y'))$. This axiom defines $\neg x$.*

Remark. Any poset embeds into a Boolean ordering by that map $f : (A, <) \rightarrow (P(A), \leq)$ with $f(a) = \{x \in a \colon x \leq a\}$.

Alas, the theory of atom-free Boolean orderings, though complete, is not model-complete.

Completeness: If P_1, P_2 are atom-free Boolean orderings, we may recover atom-free Boolean algebras $\mathcal{B}_1$, $\mathcal{B}_2$. As the theory of atom-free Boolean algebras is complete (see last class), $\mathcal{B}_1 \equiv \mathcal{B}_2$, hence $P_1 \equiv P_2$ as Boolean orderings.

Model-completeness (the lack thereof): Start with A an atom-free Boolean ordering. Denote its least element as 0. Do the following embeddings: $A \hookrightarrow P(A) \hookrightarrow A'$, where A' is an atom-free Boolean ordering. We can always do the second embedding since we know that the theory of atom-free Boolean algebras is a companion of the theory of Boolean algebras, and this carries over to Boolean orderings. We see that the image of 0 in A' is no longer the least element, since when we embed A into $P(A)$ we find that the empty set is the least element there. Hence A is not elementarily included in A', so the theory is not model-complete.

The theory of atom-free Boolean ordererings is a companion, but not the model-companion, of the theory of posets. Still, there is the model-companion for the theory of posets, called the random graph. Let T_{random} be the theory given by the following axioms:

- $<$ is an ordering.
- for each $\phi(\underline{x}y)$ (where the length of $\underline{x} = n$) and $\psi(\underline{x})$ its restriction to $\underline{x}$, add the axiom $\forall \underline{x}(\psi(\underline{x}) \rightarrow \exists y(\phi(\underline{x}y))$

See Marker, pg. 50.

Exercises:

- Prove that T_{random} is a companion of the theory of posets.
- Show T_{random} is complete and eliminates quantifiers.

12 Types

12.1 Stone spaces

Fix a language $\mathcal{L}$. Let S be the set of all complete $\mathcal{L}$-theories. For each $\mathcal{L}$-sentence ϕ, let $O_\phi = \{T \in S : T \models \phi\}$. This defines a topology on S, with O_ϕ open sets composing a base.

Lemma 12.1. *Under the above topology, S is Hausdorff, compact, totally disconnected, and every clopen set is of the form O_ϕ.*

Proof. If $T_1 \neq T_2$, then there is some ϕ such that $T_1 \models \phi$ and $T_2 \models \neg\phi$. Hence $T_1 \in O_\phi$ and $T_2 \in O_{\neg\phi}$. These are two disjoint open sets, and so S is Hausdorff. To see that S is compact, use the Compactness Theorem. See Marker, pg. 119. We see that S is totally disconnected since the base is made of clopen sets by construction. To see that every clopen set is of the form O_ϕ, consult the proof from when we showed the correspondence between Boolean algebras and Stone spaces. □

With this interpretation, a (not necessarily complete) theory T is a closed set of S, when we understand T to be the set of its complete extensions. This is because under this interpretation, T is the intersection of all the O_ϕs for which $T \models \phi$, which is closed, being the intersection of closed sets. The reason T is precisely this intersection is if there is some ψ for which neither $T \models \psi$ nor $T \models \neg\psi$, then T has a complete extension for either possibility. Thus the complete extensions all disagree on something outside of the ϕ that T models, but they obviously all agree on those ϕ.

You may also fix T_0 a partial (not necessarily complete) theory and work in $S(T_0) = \{T \text{ complete} : T \supseteq T_0\}$ with the induced topology.

Now, one may ask, what is an n-type $\pi(\underline{x})$ over A but a theory in $\mathcal{L}_A(\underline{x})$? So we may discuss types using this framework.

Definition 12.2. *A type $\pi(\underline{x})$ is complete if it is complete as an $\mathcal{L}(\underline{x})$-theory.*

We will use π to denote not necessarily complete types, and reserve p for complete types.

Remark. Let π be a type. Then π is complete iff π is maximal. For example, $\mathrm{tp}(\underline{a}/\emptyset)$ is always a complete type.

Definition 12.3 (Stone space)**.** *Let $n \geq 0$ and A be a set of parameters. We let $S_n(A) := \{$complete n-types over $A\}$ with the standard topology. Recall that when discussing types, A is actually $\mathrm{Th}_A = \mathrm{Th}(\mathcal{M}, A)$ with some $\mathcal{M} \supseteq A$. As before, $S_n(A)$ is Hausdorff, compact, and totally disconnected.*

If logicians were consistent, they would write $S_n(\emptyset)$ in place of what we have as $S_n(T)$. Unfortunately, they aren't.

Finally, we note that once A is given, things make sense only in models of $\mathrm{Th}(\mathcal{M}, A)$, and we may forget $\mathcal{M}$.

Theorem 12.4 (Elimination Theorem)**.** *Let E be a set of formulas, and T be a theory. Assume that whenever $\mathcal{M}, \mathcal{N} \models T$ and $\underline{m} \in \mathcal{M}$, $\underline{n} \in \mathcal{N}$ satisfy the same formulas from E, then they satisfy the same formulas. Then any formula is equivalent modulo T to some boolean combination of formulas of E.*

Proof. Let k be the length of the tuples that we're working with. We will work in $S_k(T)$. Let ϕ be a formula. We would like to show that ϕ is equivalent modulo T to some boolean combination of formulas of E. Note that we may assume that E is closed under boolean combination, since such formulas are obviously equivalent modulo T to some boolean combination of formulas of E.

Let $p, q \in S_k(T)$ with $p \in O_\phi$ and $q \in O_{\neg\phi}$. By our assumption, there is some $\psi_{p,q} \in E$ with $p \in O_{\psi_{p,q}}$ and $q \in O_{\neg\psi_{p,q}}$, since if this were not the case, then they would satisfy all the same formulas from E and hence all the same formulas; in particular they would both satisfy ϕ.

Let q vary in $O_{\neg\phi}$. Then $O_\phi \subseteq \cup_{q \in O_{\neg\phi}} O_{\neg\psi_{p,q}}$ by construction. This is an open cover, so by compactness there is a finite subcover, meaning there are $q_1, \ldots, q_a$ such that $O_{\neg\phi} \subseteq \cup_1^a O_{\neg\psi_{p,q_i}}$. Let $\psi_p := \bigwedge_1^a \psi_{p,q_i} \in E$. Note that $p \in O_{\psi_p}$ and $O_{\neg\phi} \subseteq O_{\neg\psi_p}$.

Now we let p vary in O_ϕ. We find that $O_\phi \subseteq \cup_{p \in O_\phi} O_{\psi_p}$. Again by compactness we find that there are $p_1, \ldots, p_b \in O_\phi$ such that $O_\phi \subseteq \cup_1^b O_{\psi_{p_i}}$. Let $\psi := \bigvee_1^b \psi_{p_i} \in E$. We have $O_\phi \subseteq O_\psi$, as well as $O_{\neg\phi} \subseteq O_{\neg\psi}$. It follows that $O_\phi = O_\psi$, which is the desired result. □

So we see that the topological methods can provide nice proofs. (If you've tried to show the Elimination Theorem as in the exercises and Marker, you know it's much more painful than above.) Now we will show another application of topological methods.

Proposition 12.5. *Let $A\underline{b}$ be drawn from $\mathcal{M}$ and $\phi(\underline{xb})$ be a formula. If for any $\mathcal{M}_1, \mathcal{M}_2 \succcurlyeq \mathcal{M}$ and $\underline{m}_1 \in M_1$, $\underline{m}_2 \in M_2$,*

$$\operatorname{tp}(\underline{m}_1/A) = \operatorname{tp}(\underline{m}_2/A) \Rightarrow \phi(\underline{m_1 b}) \leftrightarrow \phi(\underline{m_2 b})$$

then there is $\psi(\underline{xa})$ with $\underline{a} \in A$ such that $\phi(\underline{xb}) \leftrightarrow \psi(\underline{xa})$.

Proof. Let $\rho\colon S_n(A\underline{b}) \to S_n(A)$ be the restriction map. We see that ρ is continuous, as the preimage of a clopen set is clopen. Consider $\rho(O_\phi)$ and $\rho(O_{\neg\phi})$, which are in $S_n(A)$. They are disjoint by the assumption. Hence $S_n(A) = \rho(O_\phi) \coprod \rho(O_{\neg\phi})$. As ρ is a closed map, $\rho(O_\phi)$ and $\rho(O_{\neg\phi})$ are closed sets. Hence $\rho(O_\phi)$ is clopen in $S_n(A)$. Thus there is $\psi(\underline{xa})$ (with $\underline{a} \in A$) such that $\rho(O_\phi) = O_\psi$. This establishes the proposition. □

Remark. Fix $\mathcal{M} \models \mathrm{Th}_A$. Then the set of n-types over A realized in $\mathcal{M}$ is dense in $S_n(A)$.

Proof. Let O be a nonempty open set; we may assume $O = O_\phi$ for some $\phi(\underline{xa})$ (with $\underline{a} \in A$) since such sets form a base for the space. ϕ is consistent to Th_A, so $\mathrm{Th}_A \models \exists \underline{x} \phi(\underline{xa})$. Hence $\mathcal{M} \models \exists \underline{x} \phi(\underline{xa})$. Let $\underline{m}$ be a witness in $\mathcal{M}$ of this statement. Define $p := \operatorname{tp}(\underline{m}/A)$. Clearly $p \in O_\phi$. □

Remark. If $\mathcal{L}$ and A are countable, the $S_n(A)$ is metrizable (by Urysohn or Tietze, whichever is appropriate here). Deloro has yet to find an application of this fact, and invited us to do so.

End of Lecture 22.

Lecture 23.

In a previous class (although not recorded in these notes), there was some discussion regarding whether or not the theory of posets had a model-companion. It turns out that it does have one, called the random poset.

12.2 Algebraic and isolated types

Today we will be working with complete types. Also, fix some structure $\mathcal{M}$ and $A \subseteq \mathcal{M}$. We will work in models of $\mathrm{Th}_A \coloneqq \mathrm{Th}(\mathcal{M}, A)$.

Definition 12.6. *We say $p \in S_n(A)$ is algebraic if in any model (of Th_A), p has finitely many realizations.*

Example 12.7. Let $\mathbb{K}$ be a field (viewed as an $\mathcal{L}_{ring}$-structure), and $A \subseteq \mathbb{K}$. Then $x \in \mathbb{K}$ is "number-theoretically algebraic" over $< A >$ iff x is "model-theoretically algebraic" over A. That's reassuring, no?

Example 12.8. $\pi \in \mathbb{R}$ is not $\emptyset$-algebraic. Take $\mathbb{R} \preccurlyeq \mathbb{R}^*$ with infinitesimals. Then $\mathrm{tp}(\pi + \varepsilon/\emptyset) = \mathrm{tp}(\pi/\emptyset)$ for $0 < \varepsilon$ infinitesimal (of which there are an infinite number), as the two numbers define the same Dedekind cut.

Proposition 12.9. *A complete type $p \in S_n(A)$ is algebraic iff there are $\phi \in p$, $K \in \mathbb{N}$ such that in every model of Th_A every realization of ϕ is a realization of p (i.e. $\phi \models p$ modulo Th_A) and ϕ has exactly K realizations.*

Proof. ($\Leftarrow$) This direction is obvious.

($\Rightarrow$) We look for a suitable $\phi \in p$. Assume for each $\phi \in p$, $\forall K$ there is $\mathcal{M}_{\phi,K} \models \mathrm{Th}_A$ in which ϕ has $\geq K$ realizations. Let $(\underline{c}_i)_{i\in I}$ (I infinite) be new constants. Let $T' \coloneqq \mathrm{Th}_A \cup \{c_i \neq c_j\}_{i\neq j} \cup \{\phi(\underline{c}_i)\}_{i\in I,\phi\in p}$.

By compactness, T' is consistent, since any finite fragment has some $\mathcal{M}_{\phi,K}$ that satisfies it. If $\mathcal{M}' \models T'$, then in $\mathcal{M}'$, p has infinitely many realizations. This is a contradiction. Hence there is a $\phi \in p$ and $K \in \mathbb{N}$ such that in every $\mathcal{M} \models \mathrm{Th}_A$, ϕ has at most K realizations.

For each $\phi \in p$, let K_ϕ be the least $K \in \mathbb{N} \cup \{\infty\}$ such that in every model of Th_A, ϕ has $\leq K_\phi$ realizations. By the previous paragraph, there is some $\phi_0 \in p$ such that $K_{\phi_0} < \infty$ and is minimal. Let $\psi \in p$. We claim that $\phi_0 \models p$. Consider $\phi_0 \wedge \psi \in p$. Clearly $K_{\phi_0\wedge\psi} \leq K_\phi$, so equality must hold by minimality. Hence every realization in a model of Th_A of ϕ_0 is a realization of $\phi_0 \wedge \psi$; hence $\phi_0 \models \psi$.

We know that there is $\mathcal{M}^* \models \mathrm{Th}_A$ in which ϕ_0 has K_{ϕ_0} realizations. Thus $\mathrm{Th}_A \models \exists \underline{x}_1, \ldots, \underline{x}_{K_{\phi_0}}$ distinct satisfying ϕ_0, therefore every model of Th_A has this hold. □

"If it's finite everywhere, there is a bound."

Remark. If $\sigma \in \mathrm{Aut}\mathcal{M}$ fixing A pointwise, if $\sigma(\underline{m}) = \underline{\mu}$, then $\mathrm{tp}(\underline{m}/A) = \mathrm{tp}(\underline{\mu}/A)$.

Exercise: Let $A \subseteq M$, $\underline{b} \in M$. Then $\underline{b}$ is algebraic over A iff for every $\mathcal{M}^* \succcurlyeq \mathcal{M}$ and $\sigma \in \mathrm{Aut}(\mathcal{M}^*)$ fixing A pointwise, $\sigma(\underline{b}) \in M$.

Remark. Though $(\mathbb{R}, \mathcal{L}_{OrdRing})$ has only one automorphism, $\mathcal{R}^*$ has more. One may send $\pi \mapsto \pi + \varepsilon$, ε infinitesimal. So this exercise does not raise any contradictions, as π is not algebraic.

Definition 12.10. *Let $p \in S_n(A)$ be algebraic. Let $\deg p$ be the number of realizations of p in every model of Th_A.*

Lemma 12.11. *$\underline{m_1 m_2}/A$ is algebraic iff $\underline{m}_1/A$ and $\underline{m}_2/A$ are, in which case $\deg(\underline{m_1 m_2}/A) \leq \deg(\underline{m_1}/A)\deg(\underline{m_2}/A)$.*

Proof. ($\Rightarrow$) If $\phi(\underline{xya}) \in \mathrm{tp}(\underline{m_1 m_2}/A)$ has k realizations everywhere, let $\psi_1 := \exists \underline{y}\phi(\underline{xya}) \in \mathrm{tp}(\underline{m_1}/A)$. Thus ψ_1 has finitely many realizations everywhere. Hence $\underline{m_1}$ is algebraic over A, and similar work shows that so is $\underline{m_2}$.

($\Leftarrow$) If $\psi_1(\underline{xa}) \in \mathrm{tp}(\underline{m_1}/A)$ has k realizations everywhere and $\psi_2(\underline{ya}) \in \mathrm{tp}(\underline{m_2}/A)$ has l realizations everywhere, then $\phi(\underline{xy}) = \psi_1(\underline{xa}) \wedge \psi_2(\underline{ya}) \in \mathrm{tp}(\underline{m_1 m_2}/A$ has $\leq kl$ realizations everywhere. □

Example 12.12. $\deg(\underline{aa}/A) = \deg(\underline{a}/A) \leq (\deg(\underline{a}/A))^2$, and generally this is a strict inequality.

From the terminology, you may have wondered if there was a notion of model-theoretic algebraic closure. The answer is yes.

Definition 12.13. *Let $A \subseteq \mathcal{M}$. The algebraic closure of A in M is $\mathrm{acl}(A) := \{b \in M : b$ is algebraic over $A\}$.*

Remark. There is also the definable (or rational) closure $\mathrm{dcl}(A) := \{b \in M : b/A$ is rational$\}$, i.e. there is $\phi \in \mathrm{tp}(b/A)$ that has one realization everywhere. Note that $\mathrm{dcl}(A) \subseteq \mathrm{acl}(A)$.

Lemma 12.14. *The algebraic closure of A is an $\mathcal{L}$-structure. If $b \in M$ is algebraic over $\mathrm{acl}(A)$, it is in $\mathrm{acl}(A)$.*

Proof. The constants of the language are in $\mathrm{acl}(A)$. If $b = f(b_1, \ldots, b_n)$ with $b_i \in \mathrm{acl}(A)$ where b_i is algebraized by $\phi_i(\underline{x_i a})$, then b is by the formula $\exists y_1, \ldots, y_n(\bigwedge_1^n \phi_i(\underline{y_i a}) \wedge x = f(y_1, \ldots, y_n)$. This establishes the first sentence.

Let b be algebraic over $\mathrm{acl}(A)$. We will prove that b is algebraic over A. By assumption, there is $\phi(\underline{xc}) \in \mathrm{tp}(b/\mathrm{acl}(A))$ with finitely many realizations. We know $\underline{c} = c_1 \ldots c_n$, each in $\mathrm{acl}(A)$, so there are $\psi_i(y_i\underline{a})$ algebraizing the c_is over A. Then $\exists \underline{y} \bigwedge_1^n \psi_i(y_i\underline{a}) \wedge \phi(\underline{xy}) \in \mathrm{tp}(b/A)$ algebraizes $\underline{b}$ over A, as it has finitely many realizations everywhere. □

Caution: Though it does not depend on $\mathcal{M}(\models \mathrm{Th}_A)$, $\mathrm{acl}(A)$ is not necessarily a model of Th_A. For example, if we look at $\bigoplus_I \mathbb{Z}/2\mathbb{Z}$ and let $A = \{0\}$, we find that $\mathrm{acl}(A) = A = \{0\}$.

Definition 12.15. *A complete type $p \in S_n(A)$ is isolated (or principal) if there is $\phi \in p$ such that $\phi \models p$ (modulo Th_A).*

Example 12.16. Any algebraic type is principal. The converse does not hold: in $(\mathbb{Q}, (0, <))$, "$x > 0$" isolates a complete type, but it is certainly not algebraic.

Remark. Every isolated type is realized, since $\exists \underline{x}\phi(\underline{x}) \in \mathrm{Th}_A$.

Remark. $p \in S_n(A)$ is isolated as a type iff it is isolated as a point of $S_n(A)$.

Lemma 12.17. *$\underline{m_1 m_2}/A$ is isolated iff $\underline{m}_1/A$ and $\underline{m}_2/\underline{m}_1 A$ are.*

Proof. ($\Rightarrow$) If $\phi(\underline{xy})$ isolates $\underline{m_1 m_2}$ over A, then $\exists \underline{y}(\phi(\underline{xy})$ isolates $\underline{m}_1$ over A, and $\phi(\underline{m_1 y})$ isolates $\underline{m}_2$ over $\underline{m}_1 A$.

($\Leftarrow$) Assume $\psi_1(\underline{xa})$ isolates $\mathrm{tp}(\underline{m}_1/A)$, and $\psi_2(\underline{m_1 ya})$ isolates $\mathrm{tp}(\underline{m}_2/\underline{m}_1 A)$. Then $\phi(\underline{xya}) = \psi_1(\underline{xa}) \wedge \psi_2(\underline{xya})$ isolates $\mathrm{tp}(\underline{m_1 m_2/A})$. □

We note that the "transitivity" we see here is a little odd, but it has to be that way. Here is a counterexample to the proposition "$\underline{m}_1/A$ and $\underline{m}_2/A$ isolated $\Rightarrow \underline{m_1 m_2}/A$ is isolated". Add a predicate (on $\mathbb{C}$) for $\mathbb{Q}$-transcendentals. Then $\pi/\mathbb{Q}$ and $e/\mathbb{Q}$ are both isolated but $\pi, e/\mathbb{Q}$ is not isolated.

End of Lecture 23.

Lecture 24.

Exercise: $\mathrm{tp}(\underline{m}/A)$ is algebraic iff for all $B \supseteq A$, $\mathrm{tp}(\underline{m}/B)$ is isolated.

12.3 Omitting types

Caution: In what follows, the language *must* be countable.

Theorem 12.18. *Let $\mathcal{L}$ be countable, $A \subseteq \mathcal{M}$ be countable, and $p \in S_n(A)$ not a principal (i.e. isolated) type. Then there is a model omitting (i.e. not realizing) p.*

Caution: This theorem is false if the language is not countable. For a counterexample, let $\mathcal{L} = \{a_n\}_{n<\omega} \cup \{b_i\}_{i<\aleph_1}$, and $T := \{a_n \neq a_m\}_{n \neq m} \cup \{a_n \neq b_i\}_{n,i} \cup \{b_i \neq b_j\}_{i \neq j}$. The idea is that we essentially have an infinite set that is grouped into distinct countable and uncountable parts. Then $p(x) := \{x \neq a_n\}_{n \in \omega}$ is not a principal type, but it is realized in every model of T since one can always choose x to be some b_i.

To prove the above theorem, we will go through and prove Compactness again using a different construction. This construction will be used in a nearly unchanged form to prove the omitting types theorem.

Theorem 12.19 (Compactness)**.** *A finitely consistent theory T is consistent.*

Proof. We're going to build a complete, consistent theory extending T. Let $\lambda = \operatorname{Card} \mathcal{L} \leq \aleph_0$. Let $\mathcal{C}_H$ be a set of constants of size λ, which we will refer to as the Henkin constants. Enumerate by λ the $\mathcal{L}(\mathcal{C}_H x)$ formulas with one free variable x, of the form $\phi(x\underline{c})$ (with $\underline{c} \in \mathcal{C}_H$). Also enumerate by λ the $\mathcal{L}(\mathcal{C}_H)$-sentences, of the form $\psi(\underline{c})$.

We will build a λ-sequence of finitely consistent theories extending T. At odd steps, we'll take care of completeness, and at even steps, we'll make sure we can eventually get a model. To begin, let $T_0 \coloneqq T$. Now, at step $2\alpha + 1$, let $\psi(\underline{c})$ be the αth $\mathcal{L}(\mathcal{C}_H)$ sentence. We assume inductively that $T_{2\alpha}$ is finitely consistent. Then at least one of $T_{2\alpha} \cup \{\psi(\underline{c})\}$ or $T_{2\alpha} \cup \{\neg\psi(\underline{c})\}$ is as well. Add a consistent one to get $T_{2\alpha+1}$ finitely consistent.

At step $2\alpha + 2$, let $\phi(x\underline{c})$ be the αth $\mathcal{L}(\mathcal{C}_H x)$ formula with free variable x. Let $C_{\phi(x\underline{c})}$ be the first element of $\mathcal{C}_H$ not occuring in $T_{2\alpha+1}$. We call this the Henkin witness for $\phi(x\underline{c})$. Then $T_{2\alpha+2} \coloneqq T_{2\alpha+1} \cup \{\exists x\phi(x\underline{c}) \to \phi(c_{\phi(x\underline{c})}\underline{c})\}$.

At step α, where α is a limit ordinal, we do what we always do and take unions of the steps that came before. Eventually, we get $T' \coloneqq T_\lambda$ a finitely consistent, complete $\mathcal{L}(\mathcal{C}_H)$-theory. Let $\equiv$ be a new relation symbol. Write $a \equiv b$ $(a, b \in \mathcal{C}_H)$ if "$a = b$"$\in T_\lambda$. One can check that this is an equivalence relation. This depends on both the completeness and the finite consistency of the theory. So we may let $M \coloneqq \mathcal{C}_H/\equiv$. This being done, we interpret $\mathcal{L}(\mathcal{C}_H)$ in the obvious way. (See Marker pg. 36 if it's not so obvious to you.)

It remains to prove that $\mathcal{M} \models T$. We will actually prove that $\mathcal{M} \models T_\lambda$. We do this by proving that for all $\psi(\underline{c}) \in T_\lambda$, $\mathcal{M} \models \psi(\underline{c}) \Leftrightarrow \psi(\underline{c}) \in T_\lambda$. We will verify this by induction on formulas. If ψ is atomic, this is the definition of the interpretation. We may easily deal with the boolean operators $\wedge, \vee, \neg$ from their definitions, too. So we would like to see that this holds for $\psi(\underline{c}) = \exists x\phi(x\underline{c})$. First, assume that $\mathcal{M} \models \exists x\phi(x\underline{c})$. Then there is $d \in M$ such that $\mathcal{M} \models \phi(d\underline{c})$. By induction, $\phi(d\underline{c}) \in T_\lambda$. Then, by finite consistency and completeness, it must be that $\exists x\phi(x\underline{c}) \in T_\lambda$. Now, assume $\exists x\phi(x\underline{c}) \in T_\lambda$. Then as $\exists x\phi(x\underline{c}) \to \phi(c_{\phi(x\underline{c})}\underline{c}) \in T_\lambda$ by construction, we have (by modus ponens) $\phi(c_{\phi(x\underline{c})}\underline{c}) \in T_\lambda$. By induction, $\mathcal{M} \models \phi(c_{\phi(x\underline{c})}\underline{c})$, so $\mathcal{M} \models \exists x\phi(x\underline{c})$, and the claim is established. Thus $\mathcal{M} \models T_\lambda$, whence $\mathcal{M} \models T_0 = T$. □

Theorem 12.20 (Omitting types)**.** *Let $\mathcal{L}$, A be countable. If $p \in S_n(A)$ is not isolated, then there is $\mathcal{M}$ omitting p.*

Proof. First, we may assume that A is in the language, since we may add it to the language if necessary without affecting the cardinality. We will do essentially the same proof as the above proof for the Compactness Theorem, just adding a step to make sure that no tuple from the Henkin base can satisfy p. Enumerate n-tuples of $\mathcal{C}_H$, and let $T_0 \coloneqq T$. Then on step $3k + 1$, ensure completeness as above. On step $3k + 2$, deal with the Henkin witnesses as above. On step $3k + 3$, consider the kth tuple $\underline{c_k}$. We know by countability that T_{3k+2} is of the form $\phi(\underline{c_k d})$ modulo T, where $\underline{c_k}$ is the kth tuple and $\underline{d}$ is composed of the other constants previously added. By assumption, p is not principal, so

$\exists y(\phi(\underline{xy}) \not\models p(x)$. So there is some $\psi(\underline{x})$ inconsistent with p such that $\psi(\underline{c}_k)$ is consistent with T_{3k+2}. Let $T_{3k+3} \coloneqq T_{3k+2} \cup \{\psi(\underline{c}_k)\}$. Eventually, $\mathcal{M}$ constructed as in the Compactness Theorem omits p. $\square$

Remark. This generalizes to omitting $\aleph_0$ non-principal types simultaneously, since you still end up working in a countable language.

Exercise: Read the proof of this in Poizat. It uses the Baire category theorem!

What does it mean for $X \subseteq S_n(T)$ to be dense? For all ϕ, there is $p \in X$ such that $p \models \phi$. For example, p not principal $\Rightarrow S_n(T) \setminus \{p\}$ is a dense open set.

End of Lecture 24.

Lecture 25.

13 Countable models

As you may have guessed from the heading of this section, we will be dealing throughout with $\mathcal{L}$ countable.

13.1 ω-categoricity

Theorem 13.1 (Ryll-Nardzewski)**.** *Let T be a complete, countable theory (i.e. in a countable language). Then the following are equivalent:*

i. T is $\aleph_0$-categorical.

ii. Every $p \in S_n(T)$ is isolated.

iii. For each n, $S_n(T)$ is finite.

iv. For every $n \geq 0$, there are finitely many formulas $\phi(\underline{x})$ modulo T.

Proof. i)⇒ii) Assume that there is $p \in S_n(T)$ not isolated. There is $\mathcal{M}_1 \models T$ countable realizing p, since it is consistent and we may use Löwenheim-Skolem to get countability. By the omitting types theorem, there is also $\mathcal{M}_2 \models T$ omitting p. It follows that $\mathcal{M}_1 \not\cong \mathcal{M}_2$, hence T is not $\aleph_0$-categorical.

ii)⇒iii) In this case, $S_n(T)$ is made of isolated points (i.e. clopen singletons). Hence $S_n(T)$ is discrete. As it is a compact space, it must be finite.

iii)⇒iv) Assume that there are infinitely many non-equivalent formulas. We may assume that they are boolean independent, hence $S_n(T)$ is infinite. Or, done differently, as $S_n(T)$ is finite, so is the set of basic open sets, and every formula is equivalent modulo T to such a set.

iv)⇒i) Let $\mathcal{M}, \mathcal{N} \models T$ be countable. We will prove that $\mathcal{M} \cong \mathcal{N}$. Since $\mathcal{M}$ and $\mathcal{N}$ are countable, it suffices to prove that $\mathcal{M} \simeq_\infty \mathcal{N}$. Thus, we aim to prove that the family of partial elementary isomorphisms is a Karp family.

As T is complete, $\mathcal{M} \equiv \mathcal{N}$, so the empty morphism is an elementary (partial) isomorphism. (Given $\mathcal{M}$, $\mathcal{N}$, the empty morphism is a local isomorphism iff $\mathrm{Th}_{qf}(\mathcal{M}) = \mathrm{Th}_{qf}(\mathcal{N})$, and the empty morphism is elementary iff $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$. Both hold here.) This shows that the family of partial elementary isomorphisms is not empty. Now, assume $\sigma\colon \underline{a} \to \underline{b}$ is partial elementary isomorphism. Let $\alpha \in M$. We wish to extend σ into an elementary τ with $\alpha \in \mathrm{dom}\tau$.

Let $p := \{\phi(x\underline{b})\colon \mathcal{M} \models \phi(\alpha\underline{a})\}$. As there are finitely many formulas of length $n+1$, there is $\psi(x\underline{a})$ such that $\psi(x\underline{a}) \models \mathrm{tp}(\alpha/\underline{a})$ modulo $\mathrm{Th}_{\underline{a}}$ (if nothing else, the conjunction of all possible formulas will work, I believe). As $\mathcal{M} \models \exists x\psi(x\underline{a})$ (α is a witness) and σ is elementary, $\mathcal{N} \models \exists y\psi(y\underline{b})$. Let $\beta \in N$ be a realization of $\psi(y\underline{b})$. Then $\tau\colon \underline{a}\alpha \to \underline{b}\beta$ is elementary by construction (as $\mathrm{tp}(\underline{a}\alpha) = \mathrm{tp}(\underline{b}\beta)$). Hence the elementary isomorphisms form a Karp family, and so we have an ∞-isomorphism between the two models. Hence they are isomorphic. □

Remark. $\aleph_0$-categoricity is also equivalent to the statements

- For all $\underline{a} \in M \models T$, $S_n(\underline{a})$ is finite for every n.
- For all $\underline{a} \in M \models T$, $S_1(\underline{a})$ is finite.

Proof. Assume T is $\aleph_0$-categorical. Fix $\underline{a} \in M \models T$. By our previous theorem, $S_{n+1}(T)$ is finite, and two distinct $p, q \in S_1(\underline{a})$ give rise to distinct $p', q' \in S_{n+1}(\underline{a})$. Hence $S_1(\underline{a})$ is finite.

Now assume that for each $\underline{a} \in M \models T$, $S_1(\underline{a})$ is finite. Let $p \in S_n(\underline{a})$. Then $p = \mathrm{tp}(b_1 \dots b_n/\underline{a})$ is determined by $\mathrm{tp}(b_1 \dots b_{n-1}/b_n a)$, which has finitely many possible values by induction. If $S_n(\underline{a})$ is finite for every n and $\underline{a}$, then in particular so is $S_n(T)$ (which we recall would more accurately be written $S_n(\emptyset)$), hence T is $\aleph_0$-categorical by the previous theorem. □

13.2 Atomic and prime models

Definition 13.2. *$\mathcal{M}$ is atomic over $A \subseteq M$ if for each $\underline{m} \in M$, $\mathrm{tp}(\underline{m}/A)$ is isolated. $\mathcal{M}$ is atomic if it is atomic over $\emptyset$.*

Example 13.3. $\bar{\mathbb{Q}}$ is atomic over $\mathbb{Q}$, and atomic, as $\mathbb{Q}$ is atomic. Remark: Atomicity is transitive.

Terminology: $\mathcal{M}$ is atomic implies that {types realized in $\mathcal{M}$} is made of isolated points of $S_n(T)$. As this set is always dense, atomicity of $\mathcal{M}$ implies that the set of isolated points of $S_n(T)$ is dense. In the clopen algebra of $S_n(T)$, isolated points are atoms. Hence atomicity of some model of T implies atomicity (as a Boolean algebra) of the Boolean algebra $\mathrm{ClopAlg}(S_n(T))$.

Remark. If T is $\aleph_0$-categorical and $\mathcal{M}$ is countable, then $\mathcal{M}$ is atomic. This is because every $\text{tp}(\underline{m}/\emptyset)$ is an element of $S_n(T)$, but by Ryll-Nardzewski, all types of $S_n(T)$ are isolated.

Lemma 13.4. *If $\mathcal{M}$ is atomic, then for every $\underline{a} \in M$, $\mathcal{M}$ is atomic over $\underline{a}$.*

Proof. If $\text{tp}(\underline{a}\alpha/\emptyset)$ is isolated, then so is $\text{tp}(\alpha/\underline{a})$. □

Proposition 13.5. *If $\mathcal{M} \equiv \mathcal{N}$ are countable and atomic, then $\mathcal{M} \cong \mathcal{N}$.*

Proof. We prove that $\mathcal{M} \simeq_\infty \mathcal{N}$, and we do so by proving that the partial elementary isomorphisms form a Karp family. As $\mathcal{M} \equiv \mathcal{N}$, the empty morphism is elementary, so the family is nonempty. Let $\underline{a} \in M$, $\underline{b} \in N$, and $\sigma\colon \underline{a} \to \underline{b}$ be elementary. Let $\alpha \in M$, and $p := \text{tp}(\alpha/\underline{a})$ and $q := \{\phi(x\underline{b})\colon \phi(x\underline{a}) \in p\}$. We first check that q is consistent. If $\phi(x\underline{b}) \in q$, then $\phi(x\underline{a} \in p)$, so $\mathcal{M} \models \phi(\alpha\underline{a})$, i.e. $\mathcal{M} \models \exists x\phi(x\underline{a})$. As σ is elementary, $\mathcal{N} \models \phi(x\underline{b})$, so q is consistent.

Now we check that q is realized in $\mathcal{N}$. As $\mathcal{M}$ is atomic over $\emptyset$, it is atomic over $\underline{a}$ by the lemma. Let $\psi(x\underline{a})$ imply p modulo $\text{Th}_{\underline{a}}$ (this exists since p is isolated). Clearly $\psi(x\underline{b})$ isolates q. So a witness β for $\exists x\psi(x\underline{b})$ will do. □

What about the existence of countable atomic models?

Definition 13.6. *A formula $\phi(\underline{x})$ is complete if it isolates a complete type. It is completable if it belongs to a complete, isolated type.*

Definition 13.7. *T is atomic if every $\psi(\underline{x})$ consistent with T is completable, i.e. if for each n, the set of isolated n-types over $\emptyset$ is dense in $S_n(T)$, i.e. ClopAlg($S_n(T)$) is atomic for each n.*

Proposition 13.8. *Let T be a complete, countable theory. Then T is atomic iff T has a countable, atomic model.*

Proof. ($\Leftarrow$) This direction has been noted before.

($\Rightarrow$) We will construct a countable and atomic model. Let $\pi(\underline{x}) := \{\neg\psi(\underline{x})\colon \psi$ is a complete formula$\}$. We claim that π is not an isolated type. Otherwise there is $\phi \models \pi$, and T being atomic ϕ is completable. This would mean that there is a complete ψ such that $\psi \models \phi$, and hence $\psi \models \pi$, a contradiction.

Let $\mathcal{M} \models T$ be of cardinality $\aleph_0$, omitting $\pi(x_1, \dots, x_n)$ for each n. We claim that $\mathcal{M}$ is atomic. Let $\underline{m} \in M$. If $p := \text{tp}(\underline{m}/\emptyset)$ is not isolated, then there is no complete formula in p. Hence $p \models \pi$. Then $\underline{m}$ is a realization of π in $\mathcal{M}$, a contradiction. □

End of Lecture 25.

Lecture 26.

Definition 13.9. *Let $A \subseteq \mathcal{M}$. $\mathcal{M}$ is prime over A of for every $\mathcal{N} \models \text{Th}_A(:= \text{Th}(\mathcal{M}, A))$, there is $\sigma\colon \mathcal{M} \hookrightarrow \mathcal{N}$ elementary fixing A pointwise. We say $\mathcal{M}$ is prime if it is prime over $\emptyset$.*

Example 13.10. $\bar{\mathbb{Q}}$ is prime over $\mathbb{Q}$ in ACF and prime over $\emptyset$ in ACF_0. We note that ACF has no prime model, as the characteristic of the field is not specified. We also recall from last time that $\bar{\mathbb{Q}}$ is atomic in ACF.

Lemma 13.11. *Assume* Th_A *countable (i.e.* $\mathcal{L}$ *and* A *countable). Then a model is prime over* A *iff it is countable and atomic over* A*.*

Proof. ($\Rightarrow$) Let $\mathcal{P}$ be prime over A. By Löwenheim-Skolem, there is $\mathcal{C} \models \mathrm{Th}_A$ countable. We know $\mathcal{P} \hookrightarrow \mathcal{C}$ by primality, hence $\mathcal{P}$ is countable. Let $\underline{m} \in \mathcal{P}$. We prove that $q := \mathrm{tp}(\underline{m}/\emptyset)$ is isolated. If not, there is $O \models \mathrm{Th}_A$ omitting q, in which case there can be no $\mathcal{P} \hookrightarrow O$ elementary, contradicting the primality of $\mathcal{P}$.

($\Leftarrow$) Let $\mathcal{C}$ be countable and atomic over A, and let $\mathcal{M} \models \mathrm{Th}_A$. We prove there is $\mathcal{C} \hookrightarrow \mathcal{M}$ elementary fixing A pointwise. First, we map A to A. Now we may enumerate $\mathcal{C} \setminus A := \{m_1, m_2, \ldots\}_{i<\omega}$. We then construct σ_i of domain $A \cup \{m_1, m_2, \ldots\}$ inductively.

Assume σ_i is constructed and elementary. We extend it to σ_{i+1} with $m_{i+1} \in \mathrm{dom}\sigma_{i+1}$. As $\mathrm{tp}(m_1 \ldots m_{i+1}/A)$ is isolated, so is $q := \mathrm{tp}(m_{i+1}/Am_1 \ldots m_i)$. Say q is isolated by $\psi(x\underline{a}m_1 \ldots m_i)$. Consider $\psi(x\underline{a}\sigma_i(m_1 \ldots m_i))$. Since σ_i is elementary, there is $n_{i+1} \in \mathcal{M}$ such that $\mathcal{M} \models \psi(n_{i+1}\underline{a}n_1 \ldots n_i)$. Let $\sigma_{i+1}(m_{i+1}) = n_{i+1}$. Then σ_{i+1} is elementary. □

Corollary. *Let* T *be a countable and complete. Then* T *is atomic iff* T *has a prime model.*

Recall that ACF is not complete, so the above corollary does not cause in trouble, although ACF is atomic and has no prime model.

Corollary. *If* Th_A *is countable and a prime model exists, then it is unique up to isomorphism. This is because two countable atomic models are always isomorphic if they are elementarily equivalent.*

This corollary is not true if A is uncountable. See Poizat for a counterexample.

13.3 Small theories

Definition 13.12. T *is small if for each* n*,* $S_n(T)$ *is countable.*

Proposition 13.13. *Let* T *be countable and complete. Then* T *is small iff* T *has a countable* ω*-saturated model.*

Proof. ($\Leftarrow$) Let Ω be countable and ω-saturated. This means that Ω realizes every n-type over $\emptyset$, i.e. every type of $S_n(T)$. As Ω is countable, we must have $|S_n(T)| \leq \aleph_0$.

($\Rightarrow$) Assume T is small. Count 1-types over $\emptyset$. There are $a_1^1, a_1^2, \ldots$ realizing them. Let $A_1 := \{a_1^i\}_{i<\omega}$. Next, count 1-types over finite tuples of a_1^k. This is countable by assumption. Let $a_2^1, a_2^2, \ldots$ realize them, and let $A_2 := \{a_2^i\}_{i<\omega}$. Eventually let $A_\omega := \cup_{n<\omega} A_n$. This is a countable ω-saturated model of T. □

Lemma 13.14. *If T is countable, complete, and small, then it is atomic.*

Proof. We prove that the set of isolated n-types is dense in $S_n(T)$. Let p be non-isolated. $\{p\}$ is then not open, so $S_n(T) \setminus \{p\}$ is not closed. Hence $S_n(T)$ is a dense open set, since its closure must include $\{p\}$, and thus be the whole space. Now, let us look at $\cap_{p\, non-isolated} S_n(T) \setminus \{p\}$, which is obviously the set of isolated points. As T is small, the index set of the intersection is countable. By the Baire Category Theorem, the intersection is dense in $S_n(T)$. □

13.4 The number of countable models

Definition 13.15. *$I_T(\kappa)$ is the number of non-isomorphic models of T of cardinality κ.*

Example 13.16. We will construct a theory with 3 models. Let $\mathcal{L} = \{<\} \cup \{c_n\}_{n<\omega}$ and $T = DLO \cup \{c_i < c_{i+1}\}_{i<\omega}$. T is complete. There is a unique non-isolated 1-type, given by $p \coloneqq \{x > c_n\}_{n<\omega}$. The prime model omits p. (Think of this as "$c_n \to \infty$".) There are two ways to realize p, either by defining a rational cut (where there is a least realization of p) or an irrational cut (where there is not). One can think of these as "$c_n \to 0$" and "$c_n \to \sqrt{2}$", respectively. Deloro says this generalizes to n models for $n \geq 3$, although it's only immediately obvious how for odd n. So even n were left as an exercise.

Theorem 13.17. *Let T be countable and complete. Then $I(\aleph_0) \neq 2$.*

Proof. Assume T has at least two models, so it is not $\aleph_0$-categorical. We may assume that T is small, as otherwise $S_n(T)$ is uncountable, and one has many non-isolated types, hence many models. Under these assumptions, $S_n(T)$ is countable with a non-isolated point p.

As T is small, there is $\Omega \models T$ countable and ω-saturated, and there is $\mathcal{P} \models T$ prime. As p is realized in Ω (it being saturated) but not in $\mathcal{P}$ (it being prime), $\mathcal{P} \ncong \Omega$. Let $\underline{a} \in \Omega$ realize p. Consider $S_k(\underline{a})$. It is countable, as $S_{n+k}(T)$ is. Hence $\mathrm{Th}_{\underline{a}}$ is small, complete, and countable. But $S_n(\underline{a})$ is not finite, since $S_n(T)$ is infinite.

Let $\mathcal{P} \models \mathrm{Th}_{\underline{a}}$ be prime. By construction, p is realized in $\mathcal{P}_{\underline{a}}$, hence $\mathcal{P}_{\underline{a}} \ncong \mathcal{P}$. Also, $\mathcal{P}_{\underline{a}}$ is not ω-saturated, as $S_n(\underline{a})$ has a non-isolated point because it is infinite. Hence there is $q \in S_n(\underline{a})$ omitted somewhere, hence q is not realized in $\mathcal{P}_{\underline{a}}$ (recall that $\mathcal{P}_{\underline{a}}$ is prime for $\mathrm{Th}_{\underline{a}}$). As $\underline{a} \in \Omega$ and Ω is ω-saturated, q is realized in Ω. Hence $\Omega \ncong \mathcal{P}_{\underline{a}}$. Thus there are at least 3 distinct models. □

Proposition 13.18. *Let T be countable and complete. Assume $S(T)$ is uncountable. Then $|S(T)| = 2^{\aleph_0}$.*

Proof. Before we begin, note that $|S(T)| \leq 2^{\aleph_0}$ is obvious, since we're working from something that is countable. So, let n be such that $S_n(T)$ is not countable. I claim that there is ϕ such that O_ϕ and $O_{\neg\phi}$ are both uncountable. Suppose not, that is, for each ϕ, at least ϕ or $\neg\phi$ is in $\mathcal{C} \coloneqq \{\phi \colon O_\phi \text{ is countable}\}$. Let $\pi = \{\neg\phi \colon \phi \in \mathcal{C}\}$. We see that π, if consistent, is complete. If $\psi \notin \pi$, then $\neg\neg\psi \notin \pi$, hence $\neg\psi \notin \mathcal{C}$, so $\psi \in \mathcal{C}$, and $\neg\psi \in \pi$. Hence π defines at most a singleton of $S_n(T)$. Let $\mathcal{U} := \cup_{\phi \in \mathcal{C}} O_\phi$. Then $\mathcal{U}$ is countable, but $S_n(T) \backslash \mathcal{U} = \{\pi\}$, a contradiction to the uncountability of $S_n(T)$. This establishes the claim.

Now, apply the claim inside $S_n(T \cup \{\phi\})$ to get ψ_0 such that $O_{\phi \wedge \psi_0}, O_{\phi \wedge \neg\psi_0}$ are uncountable, and apply the claim inside $S_n(T \cup \{\neg\phi\})$ to get a similar ψ_1. Repeating the process, we get a 2-branching infinite tree of formulas. Each branch gives rise to a distinct $p \in S_n(T)$, and there are $2^{\aleph_0}$ branches. □

Proposition 13.19. *If* $|S(T)| = 2^{\aleph_0}$*, then* $I(\aleph_0) = 2^{\aleph_0}$*.*

Proof. Let $\kappa = I(\aleph_0)$, and fix countable models $\mathcal{M}_\alpha$ for each $\alpha < \kappa$. For $\alpha < \kappa$, let $R_\alpha := \{p \in S_n(T)$ realized in $\mathcal{M}_\alpha\}$. As every type is realized in some countable model, $S_n(T) = \cup_\kappa R_\alpha$, and then $|S_n(T)| = \kappa + \aleph_0 = \kappa = 2^{\aleph_0}$. □

Conjecture: (Vaught) If T is countable, complete, and $I(\aleph_0)$ is infinite, then $I(\aleph_0) = \aleph_0$ or $2^{\aleph_0}$.

Theorem 13.20 (Morley)**.** $I(\aleph_0) = \aleph_0$ *or* $\aleph_1$ *or* $2^{\aleph_0}$*.*

Thus Vaught's conjecture certainly holds if the Continuum Hypothesis does. If not, well, it *is* a conjecture.

End of Lecture 26.

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