MATH 566: AXIOMATIC SET THEORY

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Contents

List of Lectures

Lecture 1.

Remark. All exercises given in this class will be either true or independent.

1 Basic Set Theory

Definition (Axiom of Choice (AC))**.** *If $\mathcal{F}$ is a family of non-empty sets, then there's a function f such that $f(S) \in S$ for all $S \in \mathcal{F}$.*

We discussed a little how this relates to the example of $2^{\mathbb{N}}$, the collection of all binary sequences $(x_n \mid n \in \mathbb{N})$, and in particular the equivalence relation given by $(x_n) \backsim (y_n)$ iff $x_n = y_n$ for all but finitely many n.

Definition. *A linear order $\langle W, < \rangle$ is a well-ordering iff every $S \subseteq W$ such that $S \neq \emptyset$ has a $<$-least element.*

Theorem 1.1. *The following are equivalent:*

1. *The Axiom of Choice.*
2. *The statement "Every set can be well-ordered".*

1.1 The ordinals

The ordinals are "canonical representatives" of the isomorphism types of well-orderings. We know roughly what we want them to be:

$$\begin{aligned} 0 &= \emptyset \\ 1 &= \{\emptyset\} = \{0\} \\ 2 &= \{0, 1\} \\ &\cdots \\ n+1 &= \{0, 1, \ldots, n-1\} \\ &\cdots \\ \omega &= \{0, 1, \ldots, n, \ldots\} \\ \omega + 1 &= \omega \cup \{\omega\} \\ &\cdots \end{aligned}$$

Our problem, at least to start, is that we can't really say what we mean by "..." without using ordinal induction, which of course we don't have until we define what the ordinals even are. Luckily for us, von Neumann was a smart guy and found a definition of the ordinals that gets around this difficulty.

Definition. *A set x is transitive iff whenever $y \in x$ and $z \in y$, then $z \in x$, i.e. $\in \restriction x$ is a transitive relation.*

Definition. *A set x is an ordinal iff x is transitve and well-ordered by $\in$.*

Note that this means that when we speak of ordinals, $\alpha < \beta$ is the same as $\alpha \in \beta$.

Theorem 1.2. *If $\langle W, < \rangle$ is a well-ordering, then there exists a unique ordinal α such that $W \cong \alpha$.*

Definition. *If α is an ordinal, then its successor is $\alpha + 1 = \alpha \cup \{\alpha\}$. (This requires a little proving to see that it makes sense as a definition, but only a little.)*

This is the only ordinal arithmetic we will use. In particular, all exponentiation in this class will be **cardinal** exponentiation.

Definition. *$\boldsymbol{On}$ is the class of ordinals. V is the class of all sets.*

Definition. *Let $0 \neq \alpha \in \boldsymbol{On}$. α is called a successor ordinal iff there exists $\beta \in \boldsymbol{On}$ such that $\alpha = \beta + 1$. Otherwise, α is a limit ordinal.*

Definition. *The cumulative hierarchy is defined by transfinite recursion as follows:*

1. $V_0 = \emptyset$
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
3. $V_\lambda = \cup_{\alpha<\lambda} V_\alpha$ *if λ is a limit ordinal.*

In the future, if λ is a limit ordinal we will simply write $\lim \lambda$.

The following is proved by transfinite induction:

Theorem 1.3.

1. *V_α is a transitive set for all $\alpha \in \boldsymbol{On}$.*
2. *$V_\beta \subsetneq V_\alpha$ for all $\beta < \alpha \in \boldsymbol{On}$.*
3. *$V = \cup_{\alpha \in \boldsymbol{On}} V_\alpha$, which essentially means that ZFC has a model.*

Note that there is a little bit of an abuse of notation here, since we don't technically know how to take the union over a proper class, or what $\in$ means for a proper class.

This gives rise to a picture of the universe of sets, which I may reproduce here later.

1.2 Cardinality

Definition. *Two sets A and B are equinumerous iff there exists a bijection $f \colon A \to B$.*

Definition (requires AC)**.** *The cardinality of A is written $|A|$, and equals the least ordinal α such that A and α are equinumerous.*

Definition. *An ordinal* α *is a cardinal iff* $|\alpha| = \alpha$*. For example,* $0, 1, \dots, n, \dots, \omega$ *are cardinals;* $\omega + 1$ *is not, as its cardinality is* ω*.*

Remark. The first uncountable cardinal is $\omega_1 = \{\alpha \in \mathbf{On} \mid \alpha \text{ is countable } \}$.

Definition.

1. *If* κ *is a cardinal, then* κ^+ *denotes the least cardinal* λ *such that* $\kappa < \lambda$*.*
2. *A cardinal* κ *is a successor iff there exists a cardinal* λ *such that* $\kappa = \lambda^+$*.*
3. *Otherwise we call* κ *a limit cardinal.*

Definition. *The cardinal* $\aleph_\alpha$ *is defined recursively by*

1. $\aleph_0 = \omega$
2. $\aleph_{\alpha+1} = \aleph_\alpha^+$
3. $\aleph_\delta = \sup_{\alpha<\delta} \aleph_\alpha$ *if* $\lim \delta$*.*

We will usually write ω_α *instead of* $\aleph_\alpha$*.*

2 The Suslin Problem

Our starting point is the following classical theorem.

Theorem 2.1. *Suppose that the linear order* $\langle S, < \rangle$ *satisfies the following conditions:*

1. S *is dense without endpoints.*
2. S *is complete.*
3. S *is separable.*

Then $\langle S, < \rangle \cong \langle \mathbb{R}, < \rangle$*.*

Definition. $\langle S, < \rangle$ *is complete iff whenever* $\emptyset \neq A \subseteq S$ *is bounded above, then* A *has a supremum in* S*.*

Definition. *If* $\langle S, < \rangle$ *is a linear order, then an open interval is a subset of the form*

- $(x, y) = \{z \in S \mid x < z < y\}$
- $(-\infty, y) = \{z \in S \mid z < y\}$
- $(y, \infty) = \{z \in S \mid y < z\}$
- $(-\infty, \infty) = S$

Definition.

1. *A subset $D \subseteq S$ is dense iff $D \cap I \neq \emptyset$ for every nonempty open interval.*
2. *$\langle S, < \rangle$ is separable iff S contains a countable dense subset.*

Exercise: Prove that if $X \subseteq \mathbb{R}$, then $\langle X, < \rangle$ is separable. Note: I typed this up separately, and will continue to do so for exercises.

Lemma 2.2. *If $\langle C_1, < \rangle$ and $\langle C_2, < \rangle$ are countable dense linear orders without endpoints, then $\langle C_1, < \rangle \cong \langle C_2, < \rangle$.*

Proof. The proof is a back-and-forth argument. We saw this last year in the math logic class, it's also in Marker's model theory book, and I may come back and write it up nicely later. Essentially, you build up your map one step at a time, and since both orders are dense and countable you can always fit the mapping in where it needs to go. $\square$

Lemma 2.3. *Let $\langle P, < \rangle$ be a dense linear ordering without endpoints. Then there exists a complete dense linear order without endpoints $\langle C, \prec \rangle$ such that*

1. *$P \subseteq C$ and $<, \prec$ agree on P*
2. *P is dense in C.*

Moreover, if C_1, C_2 are two such orderings then there exists an isomorphism $\pi \colon C_1 \to C_2$ such that $\pi \restriction P = id_P$.

Proof. Professor Thomas just sketched this out; I will be filling in some of the details. The proof will proceed very similar to the way that one fashions the real numbers from the rationals using Dedekind cuts. A Dedekind cut in P is a pair (A, B) of disjoint nonempty subsets of P such that

1. $A \coprod B = P$
2. If $a \in A$, $b \in B$, then $a < b$.
3. If $\inf B$ exists, then $\inf B \in B$.

Let C be the set of all Dedekind cuts in P, ordered by $(A_1, B_1) \prec (A_2, B_2)$ iff $A_1 \subsetneq A_2$. We check that this is a complete dense linear ordering without endpoints.

First, we check that it is a linear ordering. It is clearly transitive, so we need only check that any two elements of C are comparable. Let $(A_1, B_1), (A_2, B_2) \in C$. Assume that $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$, since otherwise we are done. Then there exists $x \in A_1 \setminus A_2$ and $y \in A_2 \setminus A_1$. Because P is a linear order we have either $x < y$ or $y < x$. Assume that $x < y$. Then because $A_2 \coprod B_2 = P$, it must be that $x \in B_2$. But this contradicts the second part of the definition of Dedekind cut. There is a similar contradiction if $y < x$. So indeed we have that $\langle C, \prec \rangle$ is a linear ordering.

Clearly C has no endpoints, since P has no endpoints. To see that C is complete, we show that if $T = \{(A_i, B_i) \mid i \in I\}$ is bounded above, then

$\sup T = (\cup_{i \in I} A_i = X, \cap_{i \in I} B_i = Y)$. This is a Dedekind cut. For if we had $x \notin X \coprod Y$, then $x \notin A_i$ for all i, which would immediately imply $x \in B_i$ for all i, meaning $x \in Y$, a contradiction. Also, if we have $a \in X$, $b \in Y$, then $a \in A_j$ for some $j \in I$, and since $b \in B_j$ by definition of Y, it follows that $a < b$. Finally, if $\inf Y$ exists, then it must be in Y, since it is the greatest lower bound of elements in every B_i, which all have the property that they contain their infimums.

We must also check that (X, Y) is in fact the least upper bound of T. It is clearly an upper bound, since $A_i \subseteq X$ by definition. Now, suppose that $(X', Y') \prec (X, Y)$. Then $X' \subsetneq X$, so there is some $a \in X \setminus X'$, which means that there is some $A_k \ni a$, so by the definition of our ordering, $(X', Y') \prec (A_k, B_k)$. Thus (X, Y) is indeed the least upper bound.

Now, we can define an embedding $P \hookrightarrow C$ by $p \mapsto (\{x \in P \mid x < p\}, \{x \in P \mid p \leq x\})$. Obviously under this embedding the two orderings agree. Further, the image is dense. This establishes the first two parts of the theorem.

Finally, suppose $P \subseteq C_1$ and $P \subseteq C_2$. We can define an isomorphism $\pi \colon C_1 \to C_2$ by $\pi(x) = \sup\{p \in P \subseteq C_2 \mid p \leq x\}$. ☐

End of Lecture 1.

Lecture 2.

Proof of Theorem 2.1. Suppose S satisfies the conditions of the theorem. Let P be a countable dense subset. Then P is dense without endpoints since S is. Thus there exists an isomorphism $f \colon P \to \mathbb{Q}$. By Lemma 2.3, f can be extended to an isomorphism $\tilde{f} \colon S \to \mathbb{R}$. ☐

In 1920, Suslin asked whether $\langle \mathbb{R}, < \rangle$ is characterized by conditions 1 and 2 of Theorem 2.1 along with the countable chain condition (ccc).

Definition. *A linear order is said to satisfy the countable chain condition if every collection of pairwise disjoint nonempty open intervals is countable.*

Remark. Separable ⇒ ccc.

Proof. Suppose D is a countable dense subset. Suppose $\{I_j\}_{j \in J}$ is a collection of pairwise disjoint open intervals. Then for each $j \in J$, there exists $d_j \in D$ such that $d_j \in I_j$. If $j \neq k$, then $I_j \cap I_k = \emptyset$, and so $d_j \neq d_k$. Hence $|J| \leq |D| \leq \omega$. ☐

A positive answer to the above question (i.e. saying that the reals are characterized by Suslin's three conditions) is called the Suslin hypothesis.

Consider the following simpler statement.

SH: If the linear order S satisfies the ccc, then S is separable.

Theorem 2.4. *The following statements are equivalent:*

1. *The Suslin hypothesis*

2. SH

Clearly $SH \Rightarrow$ the Suslin hypothesis. To see the converse we will have to do more work.

Lemma 2.5. *Suppose $\langle Y, < \rangle$ is ccc, but not separable. Then there exists a linear order $\langle X, \prec \rangle$ which satisfies*

1. X is ccc.

2. X is dense.

3. No nonempty open interval of X is separable.

Proof of Theorem 2.4. Suppose $\langle X, < \rangle$ is a counterexample to SH. By Lemma 2.5 we can suppose that X is dense. After deleting the endpoints if necessary, we can suppose X has no endpoints.

Let S be the Dedekind completion of X. Clearly S is also ccc. Suppose that S is separable. Then $S \cong \mathbb{R}$, so without loss of generality $X \subseteq \mathbb{R}$, which means that X is separable by the exercise from last class. This is a contradiction. Thus S must not be separable, and so is a counterexample to the Suslin hypothesis. □

Proof of Lemma 2.5. Define a binary relation $\sim$ on Y by $x \sim y$ iff

1. $x = y$ or

2. $x < y$ and (x, y) is separable or

3. $y < x$ and (y, x) is separable.

Remark. If $x < y$ and $(x, y) = \emptyset$, then $x \sim y$ since (x, y) is trivially separable.

Obviously $\sim$ is an equivalence relation. Let X be the set of $\sim$-equivalence classes. It is easy to see that each $I \in X$ is convex, i.e. if $x, y \in I$ and $x < y$, then $(x, y) \subseteq I$. Hence we can define a linear order on X by $I < J$ iff $a < b$ for all/some $a \in I$ and $b \in J$.

Remark. Suppose $I \in X$ and $|I| \geq 3$. Then I contains a nonempty open interval of Y. Since Y is ccc, there exist only countably many $I \in X$ such that $|I| \geq 3$.

Claim. Each $I \in X$ is separable.

Proof. Let $\mathcal{M} = \{(x_n, y_n) \mid n < \lambda\}$ be a maximal collection of disjoint nonempty open intervals such that $x_n, y_n \in I$. Since Y is ccc, $\lambda \leq \omega$. Since each $x_n \sim y_n$, we can choose a countable dense $D_n \subseteq (x_n, y_n)$. Let $D = \cup_n D_n$. Then D is countable. Suppose that $x, y \in I$ and $(x, y) \neq \emptyset$. By the maximality of $\mathcal{M}$, there exists n such that $(x, y) \cap (x_n, y_n) \neq \emptyset$. Hence $D_n \cap (x, y) \neq \emptyset$. It follows that D, together with any endpoints of I, is a countable dense subset of I. □

Claim. X is dense.

Proof. Suppose there exists $I < J$ in X such that $(I, J) = \emptyset$. Clearly $I \coprod J$ is separable. Hence if $x \in I$ and $y \in J$, then $x \sim y$, contradiction. $\square$

Claim. X is ccc.

Proof. Suppose that $\{(I_\alpha, J_\alpha) \mid \alpha < \omega_1\}$ is a collection of disjoint nonempty open intervals. Choose $x_\alpha \in I_\alpha$ and $y_\alpha \in J_\alpha$ such that if $J_\alpha = I_\beta$ then $y_\alpha = x_\beta$. Then $\{(x_\alpha, y_\alpha) \mid \alpha < \omega_1\}$ are disjoint nonempty open subsets of Y, contradicting the fact that Y is ccc. $\square$

Claim. No nonempty open interval of X is separable.

Proof. Otherwise, there exist $I < J$ such that (I, J) is separable. Suppose that $\{K_n \mid n \in \mathbb{N}\}$ is a countable dense subset of (I, J). For each $n \in \mathbb{N}$, let $k_n \in K_n$. Also let $\mathcal{B} = \{K \in X \mid I \leq K \leq J \text{ and } |K| \geq 3\}$. For each $K \in \mathcal{B}$, let $D_K \subseteq K$ be a countable dense subset.

Consider $L = \cup\{K \mid I \leq K \leq J\} \subseteq Y$. We claim that $E = \{k_n \mid n \in \mathbb{N}\} \cup \{D_K \mid K \in \mathcal{B}\} \cup \{\text{endpoints of } L \text{ if any}\}$ is a countable dense subset of L, which is the final contradiction. It is enough to show that if $x, y \in L$ and $(x, y) \neq \emptyset$, then $(x, y) \cap E \neq \emptyset$.

First suppose there exists $I \leq K \leq K' \leq J$ such that $x \in K$ and $y \in K'$. Then there exists $n \in \mathbb{N}$ such that $K < K_n < K'$. So $x < k_n < y$. Otherwise, there exists $I \leq K \leq J$ such that $x, y \in K$. Clearly $K \in \mathcal{B}$ and hence $D_K \cap (x, y) \neq \emptyset$. Thus E is a countable dense subset of L. $\square$

$\square$

Historically nothing happens for 15 years, until a new way of looking at the problem is discovered.

2.1 Trees

Definition.

1. *A tree is a partially ordered set $\langle T, < \rangle$ such that for every $x \in T$, the set $\mathrm{pred}_T(x) = \{y \in T \mid y < x\}$ is well-ordered by $<$.*
2. *If $x \in T$, then its height $\mathrm{ht}_T(x)$ is the order type of $\mathrm{pred}_T(x)$.*
3. *If $\alpha \in \mathbf{On}$ then the αth level of T is $T_\alpha = \{x \in T \mid \mathrm{ht}_T(x) = \alpha\}$.*
4. *The height of T is $\mathrm{ht}(T) = \min\{\alpha \mid T_\alpha = \emptyset\}$.*
5. *A branch of T is a maximal linearly ordered subset.*
6. *An antichain of T is a subset $A \subseteq T$ such that if $x \neq y \in A$, then x, y are incomparable in T.*

Example. If T is a tree, then each level T_α is an antichain.

Example. Let T consist of all functions $f\colon \alpha \to 2$, where α is a countable ordinal, ordered by $f < g$ iff $f \subsetneq g$. Then T is a tree of height ω_1 and for each $\alpha < \omega_1$, $T_\alpha = \{f \mid f\colon \alpha \to 2\}$.

When discussing the number of nodes and brances, it was noted that $2^\omega \leq 2^{\omega_1}$ is obviously true, but it turns out that $2^\omega = 2^{\omega_1}$ is consistent, as we will see later in the course.

Definition. *If T is a tree, then $S \subseteq T$ is a subtree iff* $\mathrm{pred}_T(x) \subseteq S$ *for all* $x \in S$.

End of Lecture 2.

Lecture 3.

Definition. *A tree T is a Suslin tree iff*

i. $\mathrm{ht}(T) = \omega_1$

ii. T has no uncountable branches.

iii. T has no uncountable antichains.

Theorem 2.6 (Kurepa)**.** *The following are equivalent:*

a) SH

b) There does not exist a Suslin tree.

Proof. Delayed, for now. □

First we shall consider a natural weakening of the notion of a Suslin tree.

Definition. *The tree T is an Aronszajn tree iff*

i. $\mathrm{ht}(T) = \omega_1$

ii. T has no uncountable branches.

iii. $|T_\alpha| < \omega_1$ *for all* $\alpha < \omega_1$*, i.e. T has no uncountable levels.*

Remark. Clearly every Suslin tree is Aronszajn, since the levels of a tree are antichains.

Definition. *A tree T of height ω_1 is well-pruned iff $|T_0| = 1$ and if $\alpha < \beta < \omega_1$ and $x \in T_\alpha$, then there exists $y \in T_\beta$ such that $x < y$.*

Lemma 2.7. *If T is an Aronszajn tree, then T has a well-pruned Aronszajn subtree.*

Proof. Let T' consist of those $x \in T$ such that $\{y \in T \mid x < y\}$ is uncountable, i.e. x is beneath uncountably many elements. Clearly T' is a subtree of T.

Since T_0 is countable and $T \setminus T_0$ is uncountable, it follows that $T'_0 \neq \emptyset$. Next, we prove that T' satisfies the second part of the definition. In particular, this implies that $T'_\beta \neq \emptyset$ for all $\beta < \omega_1$, so $\mathrm{ht}(T') = \omega_1$.

Suppose that $x \in T'_\alpha$ and that $\alpha < \beta < \omega_1$. Since $\cup_{\gamma<\beta} T_\gamma$ is countable, there exists $z \in T$ with $\mathrm{ht}_T(z) \geq \beta$ and $x < z$. Hence $Y = \{y \in T_\beta \mid x < y\} \neq \emptyset$. Since Y is countable it follows that $T'_\beta \neq \emptyset$.

Finally, choose $x_0 \in T'_0$. Then $\{y \in T' \mid x_0 \leq y\}$ satisfies our requirements. □

Theorem 2.8. *There exists an Aronszajn tree.*

Proof. Let T consist of all functions $t\colon \alpha \to \omega$ such that

i. $\alpha < \omega_1$

ii. t is injective.

iii. $|\omega \setminus \mathrm{range}(t)| = \omega$

We define $t_1 < t_2$ iff $t_1 \subset t_2$. Clearly T is a tree of height ω_1.

Now, suppose that $B \subseteq T$ is an uncountable branch. Then $f = \cup B$ is an injection from ω_1 into ω, which is impossible. Thus T has no uncountable branches.

This tree is not Aronszajn, however, since $|T_\alpha| = 2^\omega > \omega$ for all $\omega \leq \alpha < \omega_1$. It is well-pruned, though. We shall find a suitable subtree $T^* \subset T$ which will be "thinner".

Some notation, before we proceed: we write ${}^B A$ for $\{f \mid f\colon B \to A\}$.

If $s, t \in {}^\alpha\omega$, define $s \sim t$ iff $|\{\beta < \alpha \mid s(\beta) \neq t(\beta)\}| < \omega$. Notice that if $\alpha < \omega_1$ and $t \in {}^\alpha\omega$, then there exist only countably many $s \in {}^\alpha\omega$ with $s \sim t$. We shall define inductively $s_\alpha \in T_\alpha$ for $\alpha < \omega_1$ such that if $\alpha < \beta < \omega_1$, then $s_\alpha \sim s_\beta \restriction \alpha$. Then we define $T^* = \cup_{\alpha<\omega_1} \{t \in {}^\alpha\omega \mid t \sim s_\alpha\}$.

Notice that if $t \in T^*_\beta$ and $\alpha < \beta$, then $t \restriction \alpha \sim s_\beta \restriction \alpha \sim s_\alpha$, and so $t \restriction \alpha \in T^*_\alpha$. Thus T^* is in fact a subtree of T. Clearly $\mathrm{ht}(T^*) = \omega_1$ and $|T^*_\alpha| = \omega$ for all $0 < \alpha < \omega_1$. Thus T^* is an Aronszajn subtree of T.

It only remains to define $s_\alpha \in T_\alpha$. First, let $s_0 = \emptyset$. Suppose inductively that s_α has been defined. Choose any $n \in \omega \setminus \mathrm{range}(s_\alpha)$ and put $s_{\alpha+1} = s_\alpha \cup \{(\alpha, n)\}$. Finally suppose $\gamma < \omega_1$ is a limit ordinal and that s_α has been defined for all $\alpha < \gamma$. Choose a sequence of ordinals

$$\alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots < \gamma$$

such that $\sup_n \alpha_n = \gamma$. Such a sequence exists since γ is countable. Let $t_0 = s_{\alpha_0}$ and inductively define $t_n\colon \alpha_n \to \omega$ such that

1. $t_n \sim s_{\alpha_n}$
2. $t_{n+1} \restriction \alpha_n = t_n$

3. the first $n+1$ elements of $\omega \setminus \mathrm{range}(t_n)$ are contained in $\omega \setminus \mathrm{range}(t_{n+1})$.

Then set $s_\gamma = \cup_n t_n$. The third condition is most important here, as it assures that $|\omega \setminus \mathrm{range}(s_\gamma)| = \omega$. This completes the construction. $\square$

It is interesting to note that we can never obtain a Suslin tree in this way.

Theorem 2.9. *Let* $T = \cup_{\alpha<\omega_1}\{t \mid t\colon \alpha \hookrightarrow \omega\}$*. If* $T^* \subseteq T$ *is a subtree, then* T^* *is not a Suslin tree.*

Proof. We can assume that $\mathrm{ht}(T^*) = \omega_1$. For each $n < \omega$, define $A_n = \{t \in T^* \mid \exists\alpha \text{ such that } \operatorname{dom} t = \alpha+1 \text{ and } t(\alpha) = n\}$. Clearly each A_n is an antichain. For each $\alpha < \omega_1$, choose $t_\alpha \in T^* \cap {}^{\alpha+1}\omega$. Then each $t_\alpha \in A_n$ for some $n < \omega$ and so some A_n is uncountable. $\square$

Definition. *An Aronszajn tree is special iff* T *is the union of countable many antichains.*

Remark. Clearly a special Aronszajn tree isn't Suslin.

Definition. *Let* T *be a tree and let* L *be a linear order. Then* T *is* L*-embeddable iff there exists* $f\colon T \to L$ *such if* $x < y \in T$*, then* $f(x) < f(y)$*.*

Exercise: Prove that an Aronszajn tree T is special iff T is $\mathbb{Q}$-embeddable. (Note that $\mathbb{Q}$-embeddable implies special is trivial.)

Exercise: Prove that there exists a special Aronszajn tree.

Exercise: Prove that there is a non-special Aronszajn tree. (We see later that this is independent.)

Exercise: Prove that every Aronszajn tree is $\mathbb{R}$-embeddable.

Next we return to the proof of Theorem 2.6. We shall make use of the following:

Lemma 2.10. *Suppose* T *is a well-pruned Aronszajn tree and* $x \in T_\alpha$ *for some* $\alpha < \omega_1$*. If* $n < \omega$*, then there exists* $\alpha < \beta < \omega_1$ *such that* $|\{y \in T_\beta \mid x < y\}| \geq n$*.*

Proof. We argue by induction on $n \geq 1$. When $n = 1$, we let $\beta = \alpha + 1$, since T is well-pruned. Suppose inductively that there exists $\alpha < \gamma < \omega_1$ and $y_1, \ldots, y_n \in T_\gamma$ with $x < y_i$ for $1 \leq i \leq n$. Since $\{z \in T \mid y_n < z\} \cap T_\beta \neq \emptyset$ for all $\gamma < \beta < \omega_1$ and $\{z \in T \mid y_n < z\}$ is not linearly ordered (else we would have an uncountable branch, since T is well-pruned), there exists $\gamma < \beta < \omega_1$ and $z_n, z_{n+1} \in T_\beta$ such that $y_n < z_n, z_{n+1}$. Choose $y_i < z_i \in T_\beta$ for $1 \leq i < n$. Then $z_1, \ldots, z_{n+1} \in T_\beta$ are distinct such that $x < z_1, \ldots, z_{n+1}$. $\square$

End of Lecture 3.

Lecture 4.

Proposition 2.11. *If there exists a Suslin tree, then SH is false.*

Proof. Let $\langle T, < \rangle$ be a Suslin tree. By Lemma 2.7 we can suppose that T is well-pruned. Let $L = \{C \subset T \mid C \text{ is a branch of } T\}$. For each $C \in L$, define $\mathrm{ht}(C) = \min\{\alpha < \omega_1 \mid C \cap T_\alpha = \emptyset\}$. This is well-defined since there is no uncountable branch. Since T is well-pruned, $\mathrm{ht}(C)$ is a limit ordinal for each $C \in L$.

For each $C \in L$ and $\alpha < \mathrm{ht}(C)$, define $C(\alpha) = C \cap T_\alpha$. We can define a linear order $\lhd$ on L as follows. First let $\prec$ be any linear ordering of the nodes of T. If $C \neq D \in L$, let $d(C, D) = \min\{\alpha \mid C(\alpha) \neq D(\alpha)\}$. Then we define $C \lhd D$ iff $C(d(C, D)) \prec D(d(C, D))$. It is easily checked that $\lhd$ is a linear order on L.

Claim. $\langle L, \lhd \rangle$ is ccc.

Proof. Suppose $\{(C_\xi, D_\xi) \mid \xi < \omega_1\}$ is a family of disjoint nonempty open intervals. For each $\xi < \omega_1$, choose $E_\xi \in (C_\xi, D_\xi)$ and a_ξ such that

$$\max\{d(C_\xi, E_\xi), d(E_\xi, D_\xi)\} < \alpha_\xi < \mathrm{ht}(E_\xi)$$

Notice that if $E_\xi(\alpha_\xi) \in F \in L$, then $F \in (C_\xi, D_\xi)$. Hence $E_\eta(\alpha_\eta) \neq E_\xi(\alpha_\xi)$ for all $\eta \neq \xi$.

Since T does not contain an uncountable antichain, there exists $\eta \neq \xi$ such that $E_\eta(\alpha_\eta) < E_\xi(\alpha_\xi)$. But then $E_\xi \in (C_\eta, D_\eta) \cap (C_\xi, D_\xi)$, contradiction. ☐

Claim. $\langle L, \lhd \rangle$ is not separable.

Proof. Suppose $\{C_n \mid n \in \mathbb{N}\}$ is a countable dense subset of L. Then there exists $\delta < \omega_1$ such that $ht(C_n) < \delta \ \forall n \in \mathbb{N}$. Let $x \in T_\delta$. By lemma 2.10, there exists $\delta < \alpha < \omega_1$ and $y, z, w \in T_\alpha$ such that $x < y, z, w$. Choose $D, E, F \in L$ such that $y \in D, z \in E, w \in F$. We can suppose that $D \lhd E \lhd F$. In particular, $(D, F) \neq \emptyset$. Clearly $C_n \cap (D, F) = \emptyset$ for all $n \in \mathbb{N}$, contradiction. ☐

Thus $\langle L, \lhd \rangle$ is a counterexample to SH. ☐

Proposition 2.12. *If SH is false, then there exists a Suslin tree.*

Proof. Suppose $\langle L, \lhd \rangle$ is a counterexample to SH. By Lemma 2.5, we can suppose that L is dense without endpoints, and no nonempty open interval of L is separable.

Let $\mathcal{I}$ be the set of all nonempty open intervals of L, partially ordered by $I < J$ iff $I \supsetneq J$. We shall find a subset $T \subseteq \mathcal{I}$ such that $\langle T, < \rangle$ is a Suslin tree. Suppose for each $\beta < \omega_1$ we can find a subset $\mathcal{I}_\beta \subseteq \mathcal{I}$ such that the following conditions are satisfied:

1. The elements of $\mathcal{I}_\beta$ are pairwise disjoint.
2. $\cup I_\beta$ is dense in L.
3. If $\alpha < \beta$ and $I \in \mathcal{I}_\alpha$, $J \in \mathcal{I}_\beta$, then either $I \cap J$ is empty or $J \subsetneq I$.

Let $T = \cup_{\beta<\omega_1}\mathcal{I}_\beta$. Clearly T is a tree such that $T_\beta = \mathcal{I}_\beta$. By condition 2, $\mathcal{I}_\beta \neq \emptyset\ \forall\beta < \omega_1$ and so $\mathrm{ht}(T) = \omega_1$.

Suppose that $A \subseteq T$ is an antichain. Then A is a collection of pairwise disjoint elements of $\mathcal{I}$. Since L is ccc, it follows that $|A| < \omega_1$. Thus T has no uncountable antichains.

Next, suppose that $\{I_\xi \mid \xi < \omega_1\}$ is an uncountable branch, where $I_\xi \in T_\xi$. If $\xi < \eta$, then $I_\xi \supsetneq I_\eta$. But this means that $\{I_\xi \setminus \bar{I}_{\xi+1} \mid \xi < \omega_1\}$ is a collection of disjoint nonempty open subsets, contradiction. Thus T is a Suslin tree.

Finally, we shall construct $\mathcal{I}_\beta$ by an induction on $\beta < \omega_1$. First, let $\mathcal{I}_0$ be any maximal collection of disjoint nonempty open intervals. Clearly 1-3 hold, 2 in particular holding because a failure to would contradict maximality.

Next, given $\mathcal{I}_\alpha$, we define $\mathcal{I}_{\alpha+1}$ as follows. For each $I \in \mathcal{I}_\alpha$, let K_I be a maximal family of pairwise disjoint nonempty open intervals J such that $J \subsetneq I$. Then $\mathcal{I}_{\alpha+1} = \cup_{I\in\mathcal{I}_\alpha}K_I$. Clearly 1-3 hold.

Finally, suppse γ is a limit ordinal and $\mathcal{I}_\alpha$ has been defined for all $\alpha < \gamma$. Let $\mathcal{K} = \{K \in \mathcal{I} \mid \text{For all } \alpha \in \gamma \text{ and } I \in \mathcal{I}_\alpha, \text{either } K \cap I = \emptyset \text{ or } K \subsetneq I\}$. Let $\mathcal{I}_\gamma$ be a maximal collection of pairwise disjoint elements of $\mathcal{K}$. Clearly 1 and 3 hold.

To see that 2 holds, it is enough to show that for each $J \in \mathcal{I}$, there exists $K \in K$ such that $K \subsetneq J$. Let E be the set of all endpoints of the intervals in $\cup_{\alpha<\gamma}\mathcal{I}_\alpha$. Since E is countable and J is non-separable, there exists $K_1 \in \mathcal{I}$ such that $K_1 \subseteq J$ and $K_1 \cap E = \emptyset$. Hence if $I \in \cup_{\alpha\in\gamma}I_\alpha$, then either $I \cap K_1 = \emptyset$ or $K_1 \subseteq I$. Choose $K \in \mathcal{I}$ such that $K \subsetneq K_1$. Then $K \subsetneq J$ and $K \in \mathcal{K}$. This completes the induction. ☐

We next unsuccessfully attempt to construct a Suslin tree in ZFC.

Definition. *A tree $\langle T, <\rangle$ is ever-branching iff for all $x \in T$, the set $\{y \in T \mid x < y\}$ isn't linearly ordered.*

Lemma 2.13. *Suppose the tree $\langle T, <\rangle$ satisfies*

i. *T is ever-branching.*

ii. $\mathrm{ht}(T) = \omega_1$

iii. *Every maximal antichain of T is countable.*

Then T is a Suslin tree.

Proof. Suppose B is an uncountable branch. Then for each $x \in B$ there exists $f(x) \in T \setminus B$ such that $x < f(x)$. Define inductively $x_\alpha \in B$ for $\alpha < \omega_1$ such that $\mathrm{ht}_T(x_\alpha) > \sup\{\mathrm{ht}_T(f(x_\beta)) \mid \beta < \alpha\}$. Then $\{f(x_\alpha) \mid \alpha < \omega_1\}$ is an uncountable antichain, contradiction. ☐

Definition. *A tree T is an ω_1-tree iff $\mathrm{ht}(T) = \omega_1$ and $|T_\alpha| < \omega_1$ for each $\alpha < \omega_1$.*

Now imagine trying to construct a Suslin tree $T = \cup_{\alpha<\omega_1}T_\alpha$ by defining T_α by induction such that the following conditions hold:

1. $|T_\alpha| < \omega_1$ for each $\alpha < \omega_1$.
2. T is ever-branching.
3. For each $\beta < \alpha$ and $x \in T_\beta$ there exists $y \in T_\alpha$ such that $x < y$.

Then we need only worry about uncountable antichains. You might try the following. At the $\alpha = 0$ stage set $T_0 = \{r\}$. If $\alpha = \beta + 1$ and $T_\beta = \{t_n \mid n \in \mathbb{N}\}$, then let $T_{\beta+1} = \{a_n, b_n \mid n \in \mathbb{N}\}$ and define $t_n < a_n, b_n$. We will see next time what happens at the limit stages.

End of Lecture 4.

Lecture 5.

At limit stages of the construction we started last time, we attempt to "kill off" potential uncountable antichains.

Suppose we fail, and let $A \subseteq T = \cup_{\alpha<\omega_1} T_\alpha$ be a maximal uncountable antichain.

Lemma 2.14 (Reflection)**.** *Suppose T is an ω_1-tree and that $A \subseteq T$ is an uncountable maximal antichain. Then for all $\alpha < \omega_1$, there exists $\alpha < \beta < \omega_1$ such that*

a) β is a limit ordinal.

b) $A \cap \cup_{\gamma<\beta} T_\gamma$ is a maximal antichain of $\cup_{\gamma<\beta} T_\gamma$.

Proof. For each $x \in T$, let $f(x) \in A$ be such that $x, f(x)$ are comparable. This is always possible since A is maximal. Define inductively an increasing sequence

$$\alpha < \beta_0 < \beta_1 < \ldots < \beta_n < \ldots \quad n \in \mathbb{N}$$

as follows. Since $\cup_{\xi<\alpha} T_\xi$ is countable, there exists $\alpha < \beta_0 < \omega_1$ such that $f[\cup_{\xi<\alpha} T_\xi] \subseteq \cup_{\gamma<\beta_0} T_\gamma$. Similarly there exists $\beta_0 < \beta_1 < \omega_1$ such that
$f[\cup_{\xi<\beta_0} T_\xi] \subseteq \cup_{\gamma<\beta_1} T_\gamma$. Continuing in this fashion, let
$f[\cup_{\xi<\beta_n} T_\xi] \subseteq \cup_{\gamma<\beta_{n+1}} T_\gamma$. Then $\beta = \lim_n \beta_n$ satisfies our requirements. ☐

Now we could have killed A as follows. Suppose that α is a limit ordinal and $D = A \cap \cup_{\gamma<\alpha} T_\gamma$ is a maximal antichain of $\cup_{\gamma<\alpha} T_\gamma$. We can prevent D from growing as follows.

For each $x \in \cup_{\gamma<\alpha} T_\gamma$, let $dx \in A$ be comparable with x. Then we can inductively define distinct branches B_x such that $x, dx \in B_x$ and $B_x \cap T_\gamma \neq \emptyset$ for all $\gamma < \alpha$. Let $T_\alpha = \{b_x \mid x \in \cup_{\gamma<\alpha} T_\gamma\}$ and set $z < b_x$ iff $z \in B_x$. Notice that every element of T_α is comparable with some element of D. Thus D remains maximal and doesn't grow into an uncountable antichain. Thus we have an uncountable number of opportunities to kill any uncountable antichain.

The problem is, how do we guess which antichain to kill at a given step of the construction?

2.2 Cofinality

Definition.

a) The map $f\colon \alpha \to \beta$ is cofinal iff $\operatorname{range} f$ *is unbounded in β, i.e. $\forall \eta < \beta$, there exists $\tau < \alpha$ such that $\eta \leq f(\tau)$.*

b) The cofinality of β, written $\operatorname{cf}(\beta)$*, is the least ordinal α such that there exists a cofinal map $f\colon \alpha \to \beta$.*

Example.

1. Suppose that $\beta = \gamma + 1 = \gamma \cup \{\gamma\}$. Then $f\colon 1 \to \beta$ where $f(0) = \gamma$ is cofinal.
2. $\operatorname{cf}(\omega) = \omega$
3. $\operatorname{cf}(\beta) \leq \beta$ for all $\beta \in \mathbf{On}$.

Definition. *A limit ordinal β is regular iff* $\operatorname{cf}(\beta) = \beta$.

Remark. A regular ordinal is necessarily a cardinal.

Definition. *A cardinal κ is singular iff* $\operatorname{cf}(\kappa) < \kappa$.

Example. $\aleph_\omega$ is singular. The map $m \mapsto \aleph_m$ is cofinal. Thus $\operatorname{cf}(\aleph_\omega) = \omega$.

Example. ω is regular.

Theorem 2.15. *If κ is an infinite cardinal, then κ^+ is regular.*

Proof. Supose $\alpha < \kappa^+$ and $f\colon \alpha \to \kappa^+$ is cofinal. Then

$$\begin{aligned} \kappa^+ &= \sup\{f(\beta) \mid \beta < \alpha\} \\ &= \cup_{\beta<\alpha} f(\beta) \end{aligned}$$

Now $|\alpha| \leq \kappa$ and each $|f(\beta)| \leq \kappa$, so $|\cup_{\beta<\alpha} f(\beta)| \leq \kappa \cdot \kappa = \kappa$, which is a contradiction. □

Definition. *An uncountable regular limit cardinal is said to be weakly inaccessible. The existence of such cardinals cannot be proved in ZFC.*

Theorem 2.16 (König)**.** *If κ is an infinite cardinal and $\lambda \geq \operatorname{cf}(\kappa)$, then $\kappa^\lambda > \kappa$.*

Corollary. *If λ is an infinite cardinal, then* $\operatorname{cf}(2^\lambda) > \lambda$.

Proof. Let $\kappa = 2^\lambda$. Then $\kappa^\lambda = (2^\lambda)^\lambda = 2^{\lambda\cdot\lambda} = 2^\lambda = \kappa$. Thus $\lambda < \operatorname{cf}(\kappa) = \operatorname{cf}(2^\lambda)$. □

Theorem 2.17. $2^{\aleph_0} \neq \aleph_\omega$

Proof. We will give a self-contained proof which does not rely on the above corollary to König, although that makes it pretty easy. Suppose that $2^{\aleph_0} = \aleph_\omega$. Then $|\mathcal{P}(\omega)| = \aleph_\omega$. Hence we can express $\mathcal{P}(\omega) = \cup_{n\in\omega} S_n$ where each $|S_n| = \aleph_n$. Express $\omega = X_0 \coprod X_1 \coprod \ldots \coprod X_n \coprod \ldots$, where each $|X_n| = \omega$. For each $n \in \omega$, let $P_n = \{A \cap X_n \mid A \in S_n\}$. Then $|P_n| \leq \aleph_n < \aleph_\omega = |\mathcal{P}(X_n)|$. Hence there exists $B_n \in \mathcal{P}(X_n) \setminus P_n$. But then $\cup_n B_n \notin \cup_n S_n$, contradiction. □

2.3 Clubs, stationary sets, and $\diamondsuit$

Definition. *Let κ be a regular uncountable cardinal.*

a) *A subset $C \subseteq \kappa$ is unbounded iff $\forall \alpha < \kappa$, $\exists \beta < \kappa$ such that $\alpha < \beta \in C$.*

b) *A subset $C \subseteq \kappa$ is closed if for every limit ordinal $\delta < \kappa$ and every increasing δ-sequence $\alpha_0 < \alpha_1 < \ldots < \alpha_\xi < \ldots$, $\xi < \delta$, where each $\alpha_\xi \in C$, we have $\sup_{\xi<\delta} \alpha_\xi \in C$.*

c) *A subset $C \subseteq \kappa$ is a club iff C is closed and unbounded.*

Example. $\{\alpha < \kappa \mid \lim \alpha\}$ is a club.

Remark. It is useful to think of clubs as "large" subsets, or "measure 1" subsets.

Definition.

1. *f is a finitary function on A iff $f\colon A^n \to A$ for some $n \in \omega$.*
2. *If $f\colon A^n \to A$ and $B \subseteq A$, then B is closed under f iff $f[B^n] \subseteq B$.*

Theorem 2.18. *Let κ be a regular uncountable cardinal and let $f\colon \kappa^n \to \kappa$. Then $C = \{\alpha < \kappa \mid \alpha \text{ is closed under } f\}$ is a club.*

Proof. Clearly C is closed. To see that C is unbounded, fix some $\beta < \kappa$. Since $|\beta^n| = |B| < \kappa$ and κ is regular, there exists $\beta < \alpha_0 < \kappa$ such that $f[\beta^n] \subseteq \alpha_0$. Continuing in this fashion, we can find $\beta < \alpha_0 < \alpha_1 < \ldots < \alpha_l < \ldots$, $n \in \mathbb{N}$ such that $f[\alpha_l^n] \subseteq \alpha_{l+1}$. Clearly $\alpha = \sup_n \alpha_n \in C$. ☐

In the next class we will prove a lemma that is the analog of the result that in a probability space, the intersection of countably many measure 1 sets has measure 1.

I also have written down a little from the end of class, where we were discussing clubs. I have that the set $\{G = \langle\omega_1, *, e\rangle \text{is a simple group}\}$ is a club, but $\{G = \langle\omega_1, *, e\rangle \text{is a group such that } \operatorname{Aut}(G) = \operatorname{Inn}(G)\}$ is not. It was left as an exercise to figure out why.

End of Lecture 5.

Lecture 6.

Lemma 2.19. *Let κ be a regular uncountable cardinal and let $\lambda < \kappa$. If C_α is a club of κ for each $\alpha < \lambda$, then $C = \cap_{\alpha<\lambda} C_\alpha$ is also a club.*

Proof. As usual, it is easily checked that C is closed. To see that C is unbounded, for each $\alpha < \lambda$, define $f_\alpha\colon \kappa \to \kappa$ by $f_\alpha(\beta) =$the least $\gamma \in C_\alpha$ such that $\beta < \gamma$.

Next define $g\colon \kappa \to \kappa$ by $g(\beta) = \sup_{\alpha<\lambda} f_\alpha(\beta)$. Note that since $\lambda < \kappa$ and κ is regular, $g(\beta) < \kappa$ and so g is well-defined. Next, for each $1 \leq n < \omega$, let $g^n = g \circ g \circ \ldots \circ g$ (n times), and define $g^\omega\colon \kappa \to \kappa$ by $g^\omega(\beta) = \sup_n g^n(\beta) < \kappa$.

Clearly $\beta < g^\omega(\beta) < \kappa$, and we claim that $g^\omega(\beta) \in C = \cap_{\alpha<\lambda} C_\alpha$. To see this, fix some $\alpha < \lambda$. Then $f_\alpha(\beta) \leq g(\beta) < f_\alpha(g(\beta)) \leq g^2(\beta) < \ldots$, and so $g^\omega(\beta) = \sup_n f_\alpha(g^n(\beta)) \in C_\alpha$ since each $f_\alpha(g^n(\beta)) \in C_\alpha$. ☐

Definition. *Let κ be regular and uncountable. A subset $S \subseteq \kappa$ is stationary iff $S \cap C \neq \emptyset$ for each club $C \subseteq \kappa$.*

Remark. Suppose that $T \subseteq \kappa$ is not stationary. Then there exists a club $C \subseteq \kappa$ such that $T \subseteq \kappa \setminus C$. Thus, nonstationary corresponds to "measure 0", and stationary corresponds to "positive measure". In other words, stationary means significant, or unavoidable.

Exercise: If S is stationary and C is a club, then $S \cap C$ is also stationary. (Very easy.)

To get some understanding of this notion, we see how it arises naturally in the construction of structures of size ω_1. We will construct "many" nonisomorphic DLOs (dense linear orderings without endpoints) of size ω_1. (Contrast this to the case of DLOs of size ω, which we have seen are all isomorphic.) For each $A \subseteq \omega_1$, we shall construct $D^A = \cup_{\alpha<\omega_1} D^A_\alpha$ as a smooth increasing union of countable DLOs. (Smooth means that if $\delta < \omega_1$ is a limit, then $D^A_\delta = \cup_{\alpha<\delta} D^A_\alpha$.)

Suppose $D = \langle \omega_1, \lhd \rangle$ is a DLO. Then $\{\alpha < \omega_1 \mid \langle \alpha, \lhd \restriction \alpha \rangle \text{ is a DLO}\}$ is a club. One sees this by seeing that it is the intersection of the sets of closure of three functions. We define $f_u(\alpha) \rhd \alpha$, $f_d(\alpha) \lhd \alpha$, and $\alpha \lhd f_b(\alpha, \beta) \lhd \beta$. f_u and f_d make sure that there are no endpoints, and f_b makes sure that between any two elements there is another.

Definition. *An embedding of countable DLOs $D_1 \subset D_2$ is a rational embedding if*

1. *D_1 is an initial segment of D_2, and*
2. *D_1 has a supremum in D_2.*

i.e. $D_1 \subset D_2 \cong (-\infty, 1) \cap \mathbb{Q} \subset \mathbb{Q}$.

Definition. *An embedding of countable DLOs $D_1 \subset D_2$ is an irrational embedding if*

1. *D_1 is an initial segment of D_2, and*
2. *D_1 has no supremum in D_2.*

i.e. $D_1 \subset D_2 \cong (-\infty, \sqrt{2}) \cap \mathbb{Q} \subset \mathbb{Q}$.

For each $A \subseteq \omega_1$, we construct a smooth union of countable DLOs $D^A = \sup_{\alpha<\omega_1} D^A_\alpha$ such that $D_\alpha \subset D_{\alpha+1}$ is rational if $\alpha \in A$, and is irrational if $\alpha \notin A$. It is reasonable to expect that if A, B are "sufficiently different", then $D^A \ncong D^B$.

Theorem 2.20. *$D^A \ncong D^B$ iff $A \triangle B$ is stationary.*

Proof. ($\Leftarrow$) First suppose that $A \triangle B$ is stationary. Suppose that $f \colon D^A \to D^B$ is an isomorphism. By the usual argument (creating an increasing sequence and taking the supremum), $\{\alpha < \omega_1 \mid f[D^A_\alpha] = D^B_\alpha\}$ is a club. Since $S = A \triangle B$ is stationary, there exists $\alpha \in S \cap C$. Without loss of generality $\alpha \in A \setminus B$. But then D^A_α has a supremum in D^A, while $f[D^A_\alpha] = D^B_\alpha$ doesn't have a supremum in D^B, contradiction.

($\Rightarrow$) Next, suppose that $A \triangle B$ isn't stationary. Then there exists a club C such that $C \cap (A \triangle B) = \emptyset$. Let $C = \{\alpha_\xi \mid \xi < \omega_1\}$ be the increasing enumeration of C. Then for each $\xi < \omega_1$, $D^A_{\alpha_\xi}$ has a supremum in $D^A_{\alpha_{\xi+1}}$ iff $D^B_{\alpha_\xi}$ has a supremum in $D^B_{\alpha_{\xi+1}}$. So we can inductively define isomorphisms $f_\xi \colon D^A_{\alpha_\xi} \to D^B_{\alpha_\xi}$ such that $f_\xi \subset f_\eta$ whenever $\xi < \eta$. Then $f = \cup_{\xi<\omega_1} f_\xi$ is an isomorphism from D^A to D^B. ☐

Exercise: For each $X \subseteq \omega_1$, the game G_x is defined as follows: we have an increasing sequence $\alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \ldots < \omega_1$, for $n \in \omega$. Player I chooses each α_n and Player II chooses each β_n. The players move alternately. Player I wins iff $\sup \alpha_n \in X$.

a) Determine for which $X \subseteq \omega_1$ at least one of the players has a winning strategy.

b) Are there any $X \subseteq \omega_1$ such that neither player has a winning strategy?

We are finally ready to define our first extra set-theoretic axiom, $\diamondsuit$.

Definition (The Diamond Principle $\diamondsuit$)**.** *There exists a sequence $\langle A_\alpha \mid \alpha < \omega_1\rangle$ such that*

a) $A_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$, and

b) for all $X \subseteq \omega_1$, the set $\{\alpha < \omega_1 \mid X \cap \alpha = A_\alpha\}$ is stationary.

Remark. $\langle A_\alpha \mid \alpha < \omega_1\rangle$ is called a $\diamondsuit$-sequence.

Proposition 2.21. *$\diamondsuit$ implies CH.*

Proof. Let $X \subseteq \omega \subseteq \omega_1$. Then $S = \{\alpha < \omega_1 \mid X \cap \alpha = A_\alpha\}$ is stationary. Since $C = \{\alpha < \omega_1 \mid \omega \leq \alpha\}$ is a club, there exists $\alpha \in S \cap C$. Then $X = X \cap \alpha = A_\alpha$. Thus $\mathcal{P}(\omega) \subseteq \{A_\alpha \mid \alpha < \omega_1\}$. ☐

$\diamondsuit$ also tells us there are 2^{ω_1} stationary sets in ω_1.

Notation: From now on, if T is an ω_1-tree and $\alpha < \omega_1$, then $T \restriction \alpha = \cup_{\beta<\alpha} T_\beta$.

Lemma 2.22. *Suppose that $\langle \omega_1, \lhd\rangle$ is an ω_1-tree. Then*

a) $C = \{\alpha < \omega_1 \mid \lim \alpha$ and $T \restriction \alpha = \alpha\}$ is a club.

b) If $A \subseteq \omega_1$ is a maximal antichain of T, then $D = \{\alpha < \omega_1 \mid A \cap T \restriction \alpha$ is a maximal antichain of $T \restriction \alpha\}$ is a club.

Proof. (a) Clearly C is closed. To see C is unbounded, let $\beta < \omega_1$. Then we can inductively define a strictly increasing sequence $\beta = \alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots < \omega_1$ such that $T \restriction \alpha_0 \subseteq \alpha_1 \subseteq T \restriction \alpha_1 \subseteq \alpha_2 \subseteq \ldots$ Then $\alpha = \sup_n \alpha_n \in C$.

(b) Clearly D is closed. By lemma 2.14, D is unbounded. ☐

Exercise: Suppose that T is an ω_1-tree such that T is a subtree of $\cup_{\alpha<\omega_1}\{t \mid t\colon \alpha \hookrightarrow \omega$ for some $\alpha < \omega_1\}$. Then T is $\mathbb{R}$-embeddable.

End of Lecture 6.

Lecture 7.

Lemma 2.23. *Let $\langle A_\alpha \mid \alpha < \omega_1\rangle$ be a $\diamondsuit$-sequence. Suppose $T = \langle \omega_1, \triangleleft\rangle$ is an ever-branching ω_1-tree and that the following holds:*

> *(*) If $\alpha < \omega_1$ is a limit ordinal such that $T \restriction \alpha = \alpha$ and A_α is a maximal antichain of $T \restriction \alpha$, then for all $x \in T_\alpha$, there exists $y \in A_\alpha$ such that $y \triangleleft x$.*

Then T is a Suslin tree.

Proof. Let A be a maximal antichain of T. By lemma 2.22, there exists a club E such that for all $\alpha \in E$,

i. $\lim \alpha$

ii. $T \restriction \alpha = \alpha$

iii. $A \cap T \restriction \alpha$ is a maximal antichain of $T \restriction \alpha$.

Since $S = \{\alpha < \omega_1 \mid A \cap \alpha = A_\alpha\}$ is stationary, there exists $\alpha \in E \cap S$. Thus $A_\alpha = A \cap \alpha = A \cap T \restriction \alpha$ is a maximal antichain of $T \restriction \alpha$. Since (*) holds, if $z \in T$ with $\mathrm{ht}_T(z) \geq \alpha$, then there exists $y \in A_\alpha$ such that $y \triangleleft z$. Hence A_α is a maximal antichain of T and so $A = A_\alpha$ is countable. ☐

Theorem 2.24 ($\diamondsuit$)**.** *There exists a Suslin tree.*

Proof. Let $\langle A_\alpha \mid \alpha < \omega_1\rangle$ be a $\diamondsuit$-sequence. Let $\langle \lambda_\alpha \mid 1 \leq \alpha < \omega_1\rangle$ be the increasing enumeration of the limit ordinals $\delta < \omega_1$. Let $\lambda_0 = 0$. We shall construct a tree $T = \langle \omega_1, \triangleleft\rangle$ by defining the levels T_α inductively so the following conditions hold:

1. $T_\alpha = \{\lambda_\alpha + n \mid n < \omega\}$
2. For each $n < \omega$, $\lambda_\alpha + n \triangleleft \lambda_{\alpha+1} + 2n, \lambda_{\alpha+1} + 2n + 1$.
3. If $\beta < \alpha$ and $x \in T_\beta$, there exists $y \in T_\alpha$ such that $x \triangleleft y$.
4. If $\lim \alpha$ and $T \restriction \alpha = \alpha$ and A_α is a maximal antichain of $T \restriction \alpha$, then for all $x \in T_\alpha$, there exists $y \in A_\alpha$ such that $y \triangleleft x$. (This is (*) from above.)

Notice that 2 ensures that T is ever-branching. Hence 4 implies that T is Suslin.

Clearly there is no problem defining T_0. Also, 2 completely defines $T_{\alpha+1}$ for each successor $\alpha + 1$. So suppose that $\lim \alpha$ and $T \restriction \alpha$ has already been defined.

In the first case, we suppose that $T \restriction \alpha = \alpha$ and A_α is a maximal antichain of $T \restriction \alpha$. For each $x \in T \restriction \alpha$, there exists $y \in A_\alpha$ such that x, y are comparable.

Choose $z_0 \in T \restriction \alpha$ such that $x, y \lhd z_0$ and let $\gamma_0 = \mathrm{ht}_{T\restriction\alpha}(z_0)$. Choose an increasing sequence $\gamma_0 < \gamma_1 < \ldots < \gamma_n < \ldots < \alpha$ such that $\lim_n \gamma_n = \alpha$, and inductively choose $z_n \in T_{\gamma_n}$ such that $z_n \lhd z_{n+1}$. Define $B(x) = \{y \in T \restriction \alpha \mid (\exists n) y \lhd z_n\}$. Then $B(x)$ is a branch through $T \restriction \alpha$. Now let $T \restriction \alpha = \{x_n \mid n \in \omega\}$ and for each $t \in T \restriction \alpha$, put $t \lhd \lambda_\alpha + n \Leftrightarrow t \in B(x_n)$. Then our conditions hold.

Otherwise, we choose any maximal antichain and kill it as in the first case. (Kunen doesn't worry about antichains in this case, he just creates branches in order to define the next level.) ☐

The following variants of $\diamondsuit$ are often useful.

Theorem 2.25 ($\diamondsuit$).

1. *There exists a sequence $\langle R_\alpha \mid \alpha < \omega_1\rangle$ such that*
   - *(a) $R_\alpha \subseteq \alpha \times \alpha$ for all $\alpha < \omega_1$.*
   - *(b) For all $R \subseteq \omega_1 \times \omega_1$, the set $\{\alpha \in \omega_1 \mid R \cap (\alpha \times \alpha) = R_\alpha\}$ is stationary.*
2. *There exists a sequence $\langle f_\alpha \mid \alpha < \omega_1\rangle$ such that*
   - *(a) $f_\alpha \colon \alpha \to \alpha$ for all $\alpha < \omega_1$.*
   - *(b) For all $f \colon \omega_1 \to \omega_1$, the set $\{\alpha < \omega_1 \mid f \restriction \alpha = f_\alpha\}$ is stationary.*

Proof. (a) Let $\langle A_\alpha \mid \alpha < \omega_1\rangle$ be a $\diamondsuit$-sequence. Fix some bijection $\phi \colon \omega_1 \to \omega_1 \times \omega_1$. Then define

$$R_\alpha = \begin{cases} \phi[A_\alpha] & \text{if } \phi[A_\alpha] \subseteq \alpha \times \alpha \\ \emptyset & \text{otherwise} \end{cases}$$

Let $R \subseteq \omega_1 \times \omega_1$ be any binary relation. Let $A = \phi^{-1}(R)$. Then $S = \{\alpha < \omega_1 \mid A \cap \alpha = A_\alpha\}$ is stationary. Also, it is easily checked that $C = \{\alpha < \omega_1 \mid \phi[\alpha] = \alpha \times \alpha\}$ is a club. Thus $T = C \cap S$ is stationary and $R \cap (\alpha \times \alpha) = R_\alpha$ for all $\alpha \in T$.

(b) Let $\langle R_\alpha \mid \alpha < \omega_1\rangle$ be as in a). Define $f_\alpha \colon \alpha \to \alpha$ by checking if $R_\alpha = \mathrm{graph}(\psi)$ for some $\psi \colon \alpha \to \alpha$. If so, then set $f_\alpha = \psi$. Otherwise, set $f_\alpha = id_\alpha$. Then argue as above. ☐

Exercise($\diamondsuit$) Construct a rigid Suslin tree. (Rigid means the only automorphism is the identity.) (The idea is to kill automorphisms as they arise, as we did with antichains.)

Exercise: If the ω_1-tree T is $\mathbb{R}$-embeddable, then T is an Aronszajn tree but T is not Suslin. Hint: Given an embedding $f \colon T \to \mathbb{R}$, construct another embedding $h \colon T \to \mathbb{R}$ such that $h(t) \leq f(t)$ for all $t \in T$, and if $\mathrm{ht}_T(t)$ is a successor ordinal, then $h(t) \in \mathbb{Q}$.

Theorem 2.26 ($\diamondsuit$). *There exists an ω_1-tree T such that*

i. T is a subtree of $\cup_{\alpha<\omega_1}\{t \colon \alpha \hookrightarrow \omega \mid |\omega \setminus \mathrm{range}(t)| = \omega\}$

ii. T is not $\mathbb{Q}$-embeddable.

In particular, T is $\mathbb{R}$-embeddable but not $\mathbb{Q}$-embeddable.

Idea of proof: We shall construct $T = \cup_{\alpha<\omega_1} T_\alpha$ by induction on $\alpha < \omega_1$. Suppose we fail and that $f \colon T \to \mathbb{Q}$ is an embedding. Then for each $q \in \mathbb{Q}$, $B_q = \{t \in T \mid f(t) = q\}$ is an antichain. Extend each B_q to a maximal antichain.

Step one: If we fail, we can express $T = \cup_{n<\omega} A_n$, where each A_n is a maximal antichain.

Step two: Let C consist of those $\alpha < \omega_1$ such that $\lim \alpha$ and $A_n \cap T \restriction \alpha$ is a maximal antichain of $T \restriction \alpha$ for every $n < \omega$. Then C is a club. So use $\diamondsuit$ to guess the sequence $\langle A_n \cap T \restriction \alpha \mid n < \omega\rangle$.

Question: Can we do anything about the embedding even with our guesses?

Step three: Suppose $\lim \alpha$, $T \restriction \alpha$ is well-pruned and $\langle D_n \mid n < \omega\rangle$ is a sequence of maximal antichains of $T \restriction \alpha$ such that $T \restriction \alpha = \cup_n D_n$. We now inductively construct a branch $\mathcal{B}$ through $T \restriction \alpha$ such that $\mathcal{B} \cap D_n \neq \emptyset$ for each $n < \omega$. We want to adjoin an element $t_{\mathcal{B}} \in T_\alpha$ such that $\mathrm{pred}_T(t_{\mathcal{B}}) = \mathcal{B}$. This will ensure that we cannot have $T = \cup_n A_n$ with $A_n \cap T \restriction \alpha = D_n$ for all $n < \omega$. This is because $t_{\mathcal{B}} \in A_n$ for some n and yet $t_{\mathcal{B}}$ lies above an element of D_n.

Slight problem: In order that $t_{\mathcal{B}} = \cup \mathcal{B}$ can be adjoined, we need $|\omega \setminus \mathrm{range}(t_{\mathcal{B}})| = \omega$.

Definition. *Let $T^* = \{t \colon \alpha \hookrightarrow \omega \mid \alpha < \omega_1, |\omega \setminus \mathrm{range}(t)| = \omega\}$. For each $n \in \omega$, define the partial order $\leq_n$ by $s \leq_n t$ iff $s \leq t$ and the first n elements of $\omega \setminus \mathrm{range}(s)$ are contained in $\omega \setminus \mathrm{range}(t)$.*

Lemma 2.27. *Suppose that $\{t_n \mid n < \omega\} \subseteq T^*$ and that $t_0 \leq_0 t_1 \leq_1 t_2 \leq_2 \ldots$ Then $t = \cup_n t_n \in T^*$ and $t_n \leq_n t$ for all $n \in \omega$.*

Now we slightly change our strategy. We inductively construct $t_0 \leq_0 t_1 \leq_1 t_2 \leq_2 \ldots$ through $T \restriction \alpha$ such that $t = \cup_n t_n \in {}^\alpha\omega \cap T^*$. The sequence is chosen so that if possible there exists $a_n \in D_n$ such that $a_n \leq t_{n+1}$. Again, we put $t = \cup_n t_n \in T_\alpha$.

Suppose $t \in A_n$. Then there doesn't exist $s \in T \restriction \alpha$ and $a \in A_n \cap T \restriction \alpha$ such that $t_n \leq_n s$ and $a \leq s$. But $t \in A_n$ and $t_n \leq_n t$. Thus this aspect isn't faithfully reflected in $T \restriction \alpha$. So we should make our stand on a club where this is reflected too. Next time we will actually go through the proof and see how this is all done. $\square$

End of Lecture 7.

Lecture 8.

Note: As I was absent for part of this lecture, the notes from before we discuss Martin's Axiom were originally taken by Susan Durst.

Theorem 2.26 ($\diamondsuit$)**.** *There exists an ω_1-tree T such that* *i. T is a subtree of $\cup_{\alpha<\omega_1}\{t\colon \alpha \hookrightarrow \omega \mid |\omega \setminus \mathrm{range}(t)| = \omega\}$*

ii. T is not $\mathbb{Q}$-embeddable.

In particular, T is $\mathbb{R}$-embeddable but not $\mathbb{Q}$-embeddable.

Proof. Let $\langle R_\alpha \mid \alpha < \omega_1\rangle$ be a sequence such that

i. $R_\alpha \subseteq \alpha \times \alpha$ for each $\alpha < \omega_1$, and

ii. for all $R \subseteq \omega_1 \times \omega_1$, the set $\{\alpha < \omega_1 \mid R \cap (\alpha \times \alpha) = R_\alpha\}$ is stationary.

Note this is just a sequence of the sort we know we have from theorem 2.25. Let $\langle \lambda_\alpha \mid 1 \leq \alpha \leq \omega_1\rangle$ be the increasing enumeration of limit ordinals $\delta < \omega_1$, and let $\lambda_0 = 0$.

We shall inductively construct a subtree $T \subseteq T^*$, $T = \{t_\xi \mid \xi < \omega_1\}$ such that the following conditions are satsfied.

1. $T_0 = \{t_0\} = \{\emptyset\}, T_1 = \{t_\xi \mid 0 < \xi < \lambda_2\}$
2. If $\alpha \geq 2$, then $T_\alpha = \{t_\xi \mid \lambda_\alpha \leq \xi < \lambda_{\alpha+1}\}$
3. If $\beta < \alpha$ and $t \in T_\beta$, then for all $n \in \omega$, there exists $s \in T_\alpha$ with $t \leq_n s$.
4. Suppose α is a limit ordinal such that $T \restriction \alpha = \{t_\xi \mid \xi < \alpha\}$. Suppose further that $R_\alpha \subseteq \alpha \times \omega$ and that

   (a) For each $n < \omega$, $D_n = \{t_\xi \mid \langle \xi, n\rangle \in R_\alpha\}$ is a maximal antichain of $T \restriction \alpha$, and

   (b) $T \restriction \alpha = \cup_n D_n$

   Then there exists an element $t \in T_\alpha$ such that for each $n < \omega$ there exists $t_n \in T \restriction \alpha$ such that

   (a) $t_n \leq_n t$

   (b) If there exists $s \in T \restriction \alpha$ and $d \in D_n$ such that $t_n \leq_n s$ and $d \leq s$, then there exists $d' \in D_n \cap \mathrm{pred}_T(t)$.

First we will check that the resulting tree satisfies our requirements. The only nontrivial point is that T is not $\mathbb{Q}$-embeddable. Suppose T is $\mathbb{Q}$-embeddable. Then we can express $T = \cup_n A_n$, where each A_n is a maximal antichain.

Definition. *For each $t \in T$, let $S(t) = \{n \in \omega \mid$ there exists $s \in T, a \in A_n$ such that $t \leq_n s, a \leq s\}$.*

Claim. Let $C \subseteq \omega_1$ be the set of those α for which the following conditions hold.

1. α a limit and $T \restriction \alpha = \{t_\xi \mid \xi < \alpha\}$.
2. For each $n < \omega$, $A_n \cap T \restriction \alpha$ is a maximal antichain of $T \restriction \alpha$.
3. For each $t \in T \restriction \alpha$ and $n \in S(t)$, there exists $s \in T \restriction \alpha$ and $a \in A_n \cap T \restriction \alpha$ such that $t \leq_n s$ and $a \leq s$.

Then C is a club.

Proof. We have more or less seen the proof that the sets defined by conditions 1 and 2 are clubs, so we just need to see that the set defined by condition 3 is a club. This is just a "catch your tail" type argument. $\square$

Let $R = \{\langle \xi, n\rangle \in \omega_1 \times \omega_1 \mid t_\xi \in A_n\}$. Since $\{\alpha < \omega_1 \mid R \cap (\alpha \times \alpha) = R_\alpha\}$ is stationary, there exists $\alpha \in C$ such that $R \cap (\alpha \times \alpha) = R_\alpha$.

Then at stage α of the construction, we have that $D_n = A_n \cap T \restriction \alpha$ for each $n < \omega$. Let $t \in T_\alpha$ be the element given by condition 4, and let $t \in A_n$. Let $t_n \in T \restriction \alpha$ be the element given by 4. Then t witnesses that $n \in S(t_n)$, so there exists $s \in T \restriction \alpha$ and $d \in D_n$ such that $t_n \leq_n s$ and $d \leq s$. But then there exists $d' \in D_n \cap \mathrm{pred}_T(t)$, contradiction.

Thus it only remains to show that we can actually carry out the construction. We start with $T_0 = \{\emptyset\}$. For $\alpha = \beta + 1$, we let $T_{\beta+1} = \{t \in T^*_{\beta+1} \mid t \restriction \beta \in T_\beta\}$.

If α is a limit, we suppose the hypotheses of 4 hold. Otherwise choose any collection $\{D_n \mid n < \omega\}$ of maximal antichains of $T \restriction \alpha$ such that $T \restriction \alpha = \cup_n D_n$. We construct T_α in two stages.

Stage 1: Argue as in the construction of an Aronszajn tree. We adjoin countably many elements to T_α such that 3 is satisfied.

Stage 2: Choose an increasing sequence of ordinals $\langle \alpha_n \mid n < \omega\rangle$ such that $\sup_n \alpha_n = \alpha$. Inductively construct a sequence of elements of $T \restriction \alpha$,

$$t_0 \leq_0 t_1 \leq_1 t_2 \leq_2 \ldots \leq_{n-1} t_n \leq_n t_{n+1} \leq_{n+1} \ldots$$

such that

1. $t_n \in T \restriction \beta_n$ for some $\alpha_n \leq \beta_n < \alpha$.
2. If $\exists s \in T \restriction \alpha$ and $d \in D_n$ such that $t_n \leq_n s$ and $d \leq s$, then there exists $d' \in D_n$ such that $d' \leq t_{n+1}$.

Then we put $t = \cup_n t_n \in T_\alpha$. $\square$

3 Martin's Axiom

We begin by considering an easy puzzle.

Definition. *If $f, g \in {}^\omega\omega$, then g dominates f, written $f <^* g$, if there exists $n_0 \in \omega$ such that $f(n) < g(n)$ for all $n \geq n_0$.*

Question 1: Suppose $\mathcal{F} = \{f_n \mid n < \omega\} \subseteq {}^\omega\omega$. Does there exist $g \in {}^\omega\omega$ such that $f_n <^* g$ for all $n < \omega$?

Answer: Of course. Simply define $g(n) = \max\{f_r(n) + 1 \mid r \leq n\}$.

Question 2: Suppose $\mathcal{F} = \{f_\alpha \mid \alpha < \omega_1\} \subseteq {}^\omega\omega$. Does there exist a $g \in {}^\omega\omega$ such that $f_\alpha <^* g$ for all $\alpha < \omega_1$?

Answer: Not necessarily. For example, suppose $2^{\aleph_0} = \aleph_1$ and $\mathcal{F} = {}^\omega\omega$. But what about when CH fails?

We shall translate question 2 into a question about a suitably defined poset $\mathbb{P}_{\mathcal{F}}$.

Definition. *Let $\mathbb{P}_{\mathcal{F}}$ consist of all ordered pairs $p = \langle \phi, \mathcal{F}_0 \rangle$, where $\phi \colon n \to \omega$ for some n and $\mathcal{F}_0$ is a finite subset of $\mathcal{F}$.*

Motivation: We are trying to construct $g \in {}^{\omega}\omega$ such that $f <^* g$ for all $f \in \mathcal{F}$. We regard ϕ as a candidate for $g \restriction n$. Then finite set $\mathcal{F}_0 \subseteq \mathcal{F}$ is intended as a finite set of promises, i.e. we promise that $g(m) > f(m)$ $\forall f \in \mathcal{F}_0$ and $m \in \omega \setminus \operatorname{dom}(\phi)$.

We next define a partial ordering of $\mathbb{P}_{\mathcal{F}}$ to reflect the above. As we write it, $p \leq q$ will always indicate that p is a strengthening/contains more information than q.

Definition. *$\mathbb{P}_{\mathcal{F}}$ is partially ordered by $\langle \phi, \mathcal{F}_1 \rangle \leq \langle \psi, \mathcal{F}_0 \rangle$ if*

1. *$\phi \supseteq \psi$ and $\mathcal{F}_1 \supseteq \mathcal{F}_0$, and*
2. *$\phi(m) > f(m)$ for all $m \in \operatorname{dom} \phi \setminus \operatorname{dom} \psi$ and $f \in \mathcal{F}_0$.*

Next suppose that $g \in {}^{\omega}\omega$ satisfies $f <^* g$ for all $f \in \mathcal{F}$. Then the set of "finite approximations" $G \subseteq \mathbb{P}_{\mathcal{F}}$ is defined by $\langle \phi, \mathcal{F}_0 \rangle \in G \Leftrightarrow \phi \subset g$ and $\forall f \in \mathcal{F}_0$, $f(m) < g(m)$ $\forall m \in \omega \setminus \operatorname{dom} \phi$.

Note that G has the following properties:

a) If $p, q \in G$, then there exist $r \in G$ such that $r \leq p, q$.

b) For all $p \in G$ and $q \in \mathbb{P}_{\mathcal{F}}$, if $p \leq q$ and $p \in G$, then $q \in G$.

Definition. *If $\mathbb{P}$ is any poset, then a subset $G \subseteq \mathbb{P}$ satisfying a and b is called a filter.*

Returning to our example, the following conditions hold:

c) For each $n \in \omega$, there exists $\langle \phi, \mathcal{F}_0 \rangle \in G$ such that $n \in \operatorname{dom} \phi$.

d) For each $f \in \mathcal{F}$, there exists $\langle \phi, \mathcal{F}_0 \rangle \in G$ such that $f \in \mathcal{F}_0$.

Conversely, the existence of such a subset $G \subseteq \mathbb{P}_{\mathcal{F}}$ yields a function $g \in {}^{\omega}\omega$ such that $f <^* g$ for all $f \in \mathcal{F}$.

Proposition 3.1. *Suppose that there exists a filter $G \subseteq \mathbb{P}_{\mathcal{F}}$ satisfying c and d. Then there exists $g \in {}^{\omega}\omega$ such that $f <^* g$ for all $f \in \mathcal{F}$.*

End of Lecture 8.

Lecture 9.

Definition. *If $\mathbb{P}$ is a poset, then a subset $G \subseteq \mathbb{P}$ is a filter iff*

a) For all $p, q \in \mathbb{P}$, there exists $r \in G$ such that $r \leq p, q$.

b) If $p \in G$ and $p \leq q \in \mathbb{P}$, then $q \in G$.

Definition. *For every $\mathcal{F} \subseteq {}^{\omega}\omega$, the poset $\mathbb{P}_{\mathcal{F}}$ consists of conditions $p = \langle \phi, \mathcal{F}_0 \rangle$ where $\phi \colon n \to \omega$ for some $n \in \omega$ and $\mathcal{F}_0 \subset \mathcal{F}$ is finite, ordered by $\langle \psi, \mathcal{F}_1 \rangle \leq \langle \phi, \mathcal{F}_0 \rangle$ iff*

- *$\psi \supseteq \phi$ and $\mathcal{F}_1 \supseteq \mathcal{F}_0$*
- *For all $n \in \operatorname{dom} \psi \setminus \operatorname{dom} \phi$ and $f \in \mathcal{F}_0$, $\psi(n) > f(n)$.*

Definition. *If $\mathbb{P}$ is a poset, then $D \subseteq \mathbb{P}$ is dense iff for all $p \in \mathbb{P}$, there exists $d \in D$ such that $d \leq p$.*

Example. Consider $\mathbb{P}_{\mathcal{F}}$ as above.

1. For each $n \in \omega$, $D_n = \{\langle \phi, \mathcal{F}_0 \rangle \mid n \in \operatorname{dom} \phi\}$ is dense in $\mathbb{P}_{\mathcal{F}}$.
2. For each $f \in \mathcal{F}$, $E_f = \{\langle \phi, \mathcal{F}_0 \rangle \mid f \in \mathcal{F}_0\}$ is dense.

Proposition 3.1. *Suppose there exists a filter $G \subseteq \mathbb{P}_{\mathcal{F}}$ such that*

- *$G \cap D_n \neq \emptyset$ for all $n \in \omega$,*
- *$G \cap E_f \neq \emptyset$ for all $f \in \mathcal{F}$.*

Then there exists $g \in {}^{\omega}\omega$ such that $f <^* g$ for all $f \in \mathcal{F}$.

Proof. Define $g = \cup\{\phi \mid (\exists \mathcal{F}_0)\langle \phi, \mathcal{F}_0 \rangle \in G\}$. We first check that $g \in {}^{\omega}\omega$. Let $n \in \omega$. Then there exists $\langle \phi, \mathcal{F}_0 \rangle \in G \cap D_n$. Suppose that we also have $\langle \psi, \mathcal{F}_1 \rangle \in G \cap D_n$. Since G is a filter, there exists $\langle \theta, \mathcal{F}_2 \rangle \in G$ such that $\langle \theta, \mathcal{F}_2 \rangle \leq \langle \psi, \mathcal{F}_1 \rangle, \langle \phi, \mathcal{F}_0 \rangle$. Hence $\psi(n) = \phi(n)$. Thus $g \in {}^{\omega}\omega$.

Next suppose that $f \in \mathcal{F}$. Then there exists $\langle \phi, \mathcal{F}_0 \rangle \in E_f \cap G$. We claim that $f(n) < g(n)\ \forall n \in \mathbb{N} \setminus \operatorname{dom} \phi$. As above, there exists $\langle \psi, \mathcal{F}_1 \rangle \leq \langle \phi, \mathcal{F}_0 \rangle$ such that $n \in \operatorname{dom} \psi$. By the definition of $\leq$, we have $g(n) = \psi(n) > f(n)$, since $f \in \mathcal{F}_0$. ☐

Clearly we need an axiom of the following form.

Definition (“Axiom F”)**.** *If $\mathbb{P}$ is a poset and $\{D_\alpha \mid \alpha < \omega_1\}$ is a family of dense subsets of $\mathbb{P}$, then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.*

Counterexample. Let

$$\mathbb{P}_{coll} = \{p \subseteq \omega \times \omega_1 \mid |p| < \omega \text{ and } p \text{ is the graph of a partial function}\}$$

ordered by $p \leq q$ iff $p \supseteq q$. Notice that the following sets are dense:

- $D_n = \{p \in \mathbb{P}_{coll} \mid n \in \operatorname{dom} p\}$ for each $n \in \omega$.
- $R_\alpha = \{p \in \mathbb{P}_{coll} \mid \alpha \in \operatorname{range} p\}$ for each $\alpha < \omega_1$

Suppose $g \subseteq \mathbb{P}_{coll}$ is a filter intersecting all the above dense sets. Then $g = \cup G \colon \omega \to \omega_1$ is a surjective function, contradiction.

Definition. *Let $\mathbb{P}$ be a poset.*

a) The elements $p, q \in \mathbb{P}$ are compatible iff there exists $r \in \mathbb{P}$ such that $r \leq p, q$.

b) A subset $A \subseteq \mathbb{P}$ is an antichain iff the elements of A are pairwise incompatible. (This is somewhat stronger than our old definition that we used (i.e. an antichain is a set of pairwise incomparable elements) but it works for trees at least if you view the tree upside-down *as a partial order. (If we view the tree right-side-up as a partial order, and it has a single root, then any two elements are compatible.))*

Definition. $\mathbb{P}$ *is ccc iff every antichain of* $\mathbb{P}$ *is countable.*

Example. If $\mathcal{F} \subseteq {}^{\omega}\omega$, then $\mathbb{P}_{\mathcal{F}}$ is ccc.

Proof. Suppose that $\{\langle \phi_\alpha, \mathcal{F}_\alpha \rangle \mid \alpha < \omega_1\}$ is an uncountable antichain of $\mathbb{P}_{\mathcal{F}}$. Then there exists $\alpha \neq \beta$ such that $\phi_\alpha = \phi_\beta$. But then $\langle \phi_\alpha, \mathcal{F}_\alpha \rangle, \langle \phi_\beta, \mathcal{F}_\beta \rangle$ are compatible (just union the promises), contradiction. □

Example. $\mathbb{P}_{coll}$ is not ccc.

Proof. Clearly $\{\langle 0, \alpha \rangle \mid \alpha < \omega_1\}$ is an uncountable antichain. □

Definition. $MA(\kappa)$ *is the statement:*

Whenever $\mathbb{P}$ is a nonempty ccc poset and $\mathcal{D}$ is a family of $\leq \kappa$ dense subsets, then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

Martin's Axiom (MA) is the statement $(\forall \kappa < 2^\omega) MA(\kappa)$.

Thus we have proved

Theorem 3.2 (MA)**.** *If $\mathcal{F} \subseteq {}^{\omega}\omega$ and $|\mathcal{F}| < 2^\omega$, then there exists $g \in {}^{\omega}\omega$ such that $f <^* g$ for all $f \in \mathcal{F}$.*

Remark.

1. $MA(\omega)$ is true.
2. $MA(2^\omega)$ is false.

Proof. (a) Let $\mathbb{P}$ be any poset (not necessarily ccc) and let $\{D_n \mid n \in \omega\}$ be a family of dense subsets of $\mathbb{P}$. Then we can inductively find $p_0 \geq p_1 \geq \ldots \geq p_n \geq \ldots$ such that $p_n \in D_n$. Then $G = \{q \in \mathbb{P} \mid (\exists n) p_n \leq q\}$ satisfies our requirements.

(b) Consider $\mathbb{P}_{\mathcal{F}}$, where $\mathcal{F} = {}^{\omega}\omega$. □

Theorem 3.3 ($MA + \neg CH$)**.** *There does not exist a Suslin tree.*

Proof. Suppose that $\langle T, < \rangle$ is a Suslin tree. Without loss of generality we can suppose that T is well-pruned. Let $\mathbb{P} = \langle T, > \rangle$ be the upside-down tree. Then $\mathbb{P}$ is ccc. Since T is well-pruned, for each $\alpha < \omega_1$, the set $\{x \in T \mid \mathrm{ht}_T(x) > \alpha\}$ is dense in T. By $MA + \neg CH$, there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for each $\alpha < \omega_1$. But then $\mathcal{B} = \{y \in T \mid (\exists x \in G) y < x\}$ is an uncountable branch, contradiction. ☐

3.1 Maximal almost disjoint families

Definition. *A family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is an a.d. (almost disjoint) family iff*

i. For each $A \in \mathcal{A}$, $|A| = \omega$.

ii. If $A \neq B \in \mathcal{A}$, then $|A \cap B| < \omega$.

$\mathcal{A}$ is a mad family iff $\mathcal{A}$ is a maximal a.d. family.

Example. Let $\mathbb{E} = \{2n \mid n \in \mathbb{N}\}$ and $\mathbb{O} = \{2n+1 \mid n \in \mathbb{N}\}$. Then $\{\mathbb{E}, \mathbb{O}\}$ is a mad family.

From now on, we will only be interested in infinite a.d. families.

Proposition 3.4 (ZFC)**.** *There exists an a.d. family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ with $|\mathcal{A}| = 2^\omega$.*

Proof. Label the nodes of the complete binary tree from left-to-right, then bottom-to-top, so the first few levels are $\{0\}, \{1, 2\}, \{3, 4, 5, 6\}$. Let $\mathcal{A}$ be the set of branches. ☐

Proposition 3.5 (ZFC)**.** *If $\mathcal{A} = \{A_n \mid n \in \omega\}$ is a countably infinite a.d. family, then $\mathcal{A}$ is not mad.*

Proof. For each $n \in \omega$, let $B_n = A_n \setminus \cup_{i<n} A_i$. Clearly B_n is infinite, and so we can choose $b_n \in B_n$. Clearly the B_n are distinct, so $B = \{b_n \mid n \in \mathbb{N}\}$ is infinite. It's also clear that $B \cap A_n \subseteq \{b_1, \ldots, b_n\}$ and so $\mathcal{A} \cup \{B\}$ is also a.d. ☐

Question: Suppse $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is mad. Does it follow that $|\mathcal{A}| = 2^\omega$?
We'll prove next time that MA implies a positive answer and much more.

End of Lecture 9.

Lecture 10.

Definition. *Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$ be any family. The poset $\mathbb{P}_\mathcal{A}$ consists of all conditions $\langle s, F \rangle$ where $s \subseteq \omega$ and $F \subseteq \mathcal{A}$ are finite subsets partially ordered by $\langle s', F' \rangle \leq \langle s, F \rangle$ iff*

i. $s' \supseteq s$ and $F' \supseteq F$

ii. For all $A \in F$, $s' \cap A \subseteq s$.

Lemma 3.6. *$\mathbb{P}_{\mathcal{A}}$ is ccc.*

Proof. Clear, as there are only countably many possibilities for the s in our ordered pair. ☐

Lemma 3.7. *If $A \in \mathcal{A}$, then $D_A = \{\langle s, F\rangle \in \mathbb{P}_{\mathcal{A}} \mid A \in F\}$ is dense in $\mathbb{P}_{\mathcal{A}}$.*

Proof. $\langle s, F \cup \{A\}\rangle \leq \langle s, F\rangle$ ☐

Notation: If $G \subseteq \mathbb{P}_{\mathcal{A}}$ is a filter, then $d_G = \cup\{s \mid (\exists F)\langle s, F\rangle \in G\}$.

Lemma 3.8. *If $G \subseteq \mathbb{P}_{\mathcal{A}}$ is a filter such that $G \cap D_A \neq \emptyset$, then $|d_G \cap A| < \omega$.*

Proof. Let $\langle s, F\rangle \in G$ with $A \in F$. We claim that $d_G \cap A \subseteq s$.

Suppose not, and let $n \in (d_G \cap A) \setminus s$. Then there exists $\langle s', F'\rangle \in G$ with $n \in s'$. But clearly $\langle s, F\rangle$ and $\langle s', F'\rangle$ are incompatible, as any strengthening of $\langle s, F\rangle$ cannot have as it's first element a set containing n. This is a contradiction. ☐

Theorem 3.9 ($MA(\kappa)$)**.** *Let $\mathcal{A}, \mathcal{C} \subseteq \mathcal{P}(\omega)$, where $|\mathcal{A}|, |\mathcal{C}| \leq \kappa$ and assume that*

(*) For all $C \in \mathcal{C}$ and all finite $F \subseteq \mathcal{A}$, $|C \setminus \cup F| = \omega$.

Then there exists $d \subseteq \omega$ such that

1. *$|d \cap A| < \omega$ for all $A \in \mathcal{A}$*
2. *$|d \cap C| = \omega$ for all $C \in \mathcal{C}$*

Proof. For each $C \in \mathcal{C}$ and $n \in \omega$, let $E_n^C = \{\langle s, F\rangle \in \mathbb{P}_{\mathcal{A}} \mid C \cap s \not\subseteq n\}$. By (*), E_n^C is dense in $\mathbb{P}_{\mathcal{A}}$. By $MA(\kappa)$, there exists a filter $G \subseteq \mathbb{P}_{\mathcal{A}}$ such that $G \cap D_A \neq \emptyset$ for all $A \in \mathcal{A}$ and $G \cap E_n^C \neq \emptyset$ for all $C \in \mathcal{C}$ and $n \in \omega$. Clearly d_G satisfies our requirements. ☐

Corollary (MA)**.** *Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$ be an a.d. family of size κ for some $\omega \leq \kappa < 2^\omega$. Then $\mathcal{A}$ is not maximal.*

Proof. Take $\mathcal{C} = \{\omega\}$ in the theorem. ☐

Lemma 3.10. *Let $\mathcal{B} \subseteq \mathcal{P}(\omega)$ be an a.d. family of size κ, where $\omega \leq \kappa < 2^\omega$. Let $\mathcal{A} \subseteq \mathcal{B}$ be an arbitrary subset. Then there exists $d \subseteq \omega$ such that*

1. *$|d \cap A| < \omega$ for all $A \in \mathcal{A}$*
2. *$|d \cap B| = \omega$ for all $B \in \mathcal{B} \setminus \mathcal{A}$*

Proof. Take $\mathcal{C} = \mathcal{B} \setminus \mathcal{A}$. ☐

Theorem 3.11 ($MA(\kappa)$)**.** *If $\omega \leq \kappa < 2^\omega$, then $2^\kappa = 2^\omega$.*

Proof. In ZFC, we know there exists an a.d. family of size 2^ω. Hence there exists an a.d. family $\mathcal{B} \subseteq \mathcal{P}(\omega)$ of size κ. Consider the map $\Phi\colon \mathcal{P}(\omega) \to \mathcal{P}(\mathcal{B})$ given by $d \mapsto \{B \in \mathcal{B} \mid |d \cap B| < \omega\}$. By the above lemma, Φ is onto. ☐

Corollary (MA)**.** *2^ω is regular.*

Proof. Suppose not and let $\mathrm{cf}(2^\omega) = \kappa < 2^\omega$. Since $\omega \leq \kappa < 2^\omega$, we have that $2^\kappa = 2^\omega$. Thus $\mathrm{cf}(2^\kappa) = \mathrm{cf}(2^\omega) = \kappa$, which contradict's König's Lemma. ☐

3.2 Ultrafilters over ω

Definition. *A filter over a set S is a collection $\mathcal{F} \subseteq \mathcal{P}(S)$ such that*

i. $S \in \mathcal{F}$

ii. If $X, Y \in \mathcal{F}$*, then* $X \cap Y \in \mathcal{F}$

iii. If $X \subseteq Y \subseteq S$ *and* $X \in \mathcal{F}$*, then* $Y \in \mathcal{F}$.

iv. $\emptyset \notin \mathcal{F}$.

Example. The Frechét filter on ω is $\mathcal{F}_0 = \{X \subseteq \omega \mid |\omega \setminus X| < \omega\}$.

Definition. *The filter $\mathcal{F}$ over S is an ultrafilter iff for all $X \in \mathcal{P}(S)$, either $X \in \mathcal{F}$ of $S \setminus X \in \mathcal{F}$.*

Remark. Suppose $\mathcal{F}$ is an ultrafilter on S and $A \in \mathcal{F}$. If $A = B \coprod C$, then either $B \in \mathcal{F}$ or $C \in \mathcal{F}$.

Proof. Suppose not. Then $S \setminus B, S \setminus C \in \mathcal{F}$, and so $\emptyset = A \cap (S \setminus B) \cap (S \setminus C) \in \mathcal{F}$, contradiction. $\square$

Example. Let $a \in S$. Then $\mathcal{F} = \{X \subseteq S \mid a \in X\}$ is an ultrafilter. Such an ultrafilter is said to be principal. Others are called non-principal ultrafilters.

Remark. An ultrafilter $\mathcal{U}$ on ω is nonprincipal iff $\mathcal{F}_0 \subseteq \mathcal{U}$, where $\mathcal{F}_0$ is the Frechét filter.

Lemma 3.12. *A filter $\mathcal{F}$ on S is an ultrafilter iff $\mathcal{F}$ is a maximal filter.*

Proof. ($\Rightarrow$) Suppose $\mathcal{F}$ is an ultrafilter. Suppose $X \notin \mathcal{F}$. Then $S \setminus X \in \mathcal{F}$. Thus there doesn't exist a filter containing $\mathcal{F} \cup \{X\}$. Hence $\mathcal{F}$ is maximal.

($\Leftarrow$) Suppose $\mathcal{F}$ isn't an ultrafilter. Then there exists $Y \subseteq S$ such that $Y \notin \mathcal{F}$ and $S \setminus Y \notin \mathcal{F}$. We claim that if $X \in \mathcal{F}$, then $X \cap Y \neq \emptyset$. If not, then $X \subseteq S \setminus Y$, which implies $S \setminus Y \in \mathcal{F}$, a contradiction.

It follows that if $X_1, \ldots, X_n \in \mathcal{F}$, then $X_1 \cap \ldots \cap X_n \cap Y \neq \emptyset$. Hence $\mathcal{F}^+ = \{Z \subseteq S \mid (\exists X \in \mathcal{F}) X \cap Y \subseteq Z\}$ is a filter such that $Y \in \mathcal{F}^+$. Thus $\mathcal{F}$ isn't maximal. $\square$

Theorem 3.13. *Every filter $\mathcal{F}$ over S can be extended to an ultrafilter.*

Proof. Let $\mathbb{P}$ be the set of all filters $\mathcal{F}'$ such that $\mathcal{F} \subseteq \mathcal{F}'$, ordered by inclusion. If $C \subseteq \mathbb{P}$ is a chain, then $\cup C$ is easily seen to be a filter and hence is an upper bound of C. By Zorn's Lemma, $\mathbb{P}$ contains a maximal element $\mathcal{U}$ which must be an ultrafilter. $\square$

Random question, not an exercise: How many ultrafilters contain a given filter? What does the intersection of two ultrafilters look like?

Definition. *An ultrafilter $\mathcal{U}$ on an infinite cardinal κ is uniform iff for all $x \in \mathcal{U}$, $|X| = \kappa$.*

Remark. If $\mathcal{U}$ is an ultrafilter on ω, then $\mathcal{U}$ is uniform iff $\mathcal{U}$ is nonprincipal.

Theorem 3.14. *If κ is an infinite cardinal, then there exists a uniform ultrafilter $\mathcal{U}$ on κ.*

Proof. Let $\mathcal{U}$ be an ultrafilter which extends $\mathcal{F} = \{X \subseteq \kappa \mid |\kappa \setminus X| < \kappa\}$ □

A little bit of culture: Let $\mathcal{F}_0$ be the Frechét filter. Notice that if (r_n) is a sequence of reals and $l \in \mathbb{R}$, then the following are equivalent:

- $\lim_{n\to\infty} r_n = l$
- For every $N \in \mathbb{N}^+$, $\{n \in \mathbb{N} \mid |r_n - l| < 1/N\} \in \mathcal{F}_0$.

Definition. *If $\mathcal{F}$ is any filter on ω, then we define $\lim_{\mathcal{F}} r_n = l$ iff for every $N \in \mathbb{N}^+$, $\{n \in \mathbb{N} \mid |r_n - l| < 1/N\} \in \mathcal{F}$.*

Exercise: If $\mathcal{U}$ is an ultrafilter on ω and (r_n) is a bounded sequence of reals, then there exists a unique $l \in \mathbb{R}$ such that $lim_{\mathcal{U}} r_n = l$.

Definition. *Let $\mathcal{F}$ be a filter on ω. Then $\mathcal{X} \subseteq \mathcal{F}$ generates $\mathcal{F}$ iff for all $Z \in \mathcal{F}$, there exists $X_1, \ldots, X_n \in \mathcal{X}$ such that $X_1 \cap \ldots \cap X_n \subseteq Z$. We define $d_{\mathcal{F}} = \min\{|\mathcal{X}| \mid \mathcal{X} \subseteq \mathcal{F}$ generates $\mathcal{F}\}$.*

Example. If $\mathcal{U}$ is a principal ultrafilter, then $d_{\mathcal{U}} = 1$.

Question: Suppose $\mathcal{U}$ is a nonprincipal ultrafilter on ω. What is $d_{\mathcal{U}}$?

Remark. Clearly $d_{\mathcal{U}}$ is not finite. For suppose that $\mathcal{X} = \{X_1, \ldots, X_n\}$ generates $\mathcal{U}$. Let $Z = X_1 \cap \ldots \cap X_n \in \mathcal{U}$. Then $\{Z\}$ generates $\mathcal{U}$. Express $Z = A \coprod B$ as a union of two infinite subsets. Then either $A \in \mathcal{U}$ or $B \in \mathcal{U}$, so $Z \subseteq A$ or $Z \subseteq B$, contradiction.

Definition. *If S, T are sets, then $S \subseteq^* T$ iff $|S \setminus T| < \omega$.*

Theorem 3.15 (ZFC)**.** *If $\mathcal{U}$ is a nonprincipal ultrafilter on ω, then $\omega < d_{\mathcal{U}} \leq 2^\omega$.*

Exercise: (MA) If $\mathcal{U}$ is a nonprincipal ultrafilter on ω, then $d_{\mathcal{U}} = 2^\omega$.

End of Lecture 10.

Lecture 11.

Theorem 3.15 (ZFC)**.** *If $\mathcal{U}$ is a nonprincipal ultrafilter on ω, then $\omega < d_{\mathcal{U}} \leq 2^\omega$.*

Proof. Clearly $d_{\mathcal{U}} \leq 2^\omega$. We've also seen that $d_{\mathcal{U}} \geq \omega$. So suppose that $d_{\mathcal{U}} = \omega$, and let $\{X_n \mid n \in \omega\}$ be a generating set for $\mathcal{U}$. For each $n \in \mathbb{N}$, let $Y_n = X_0 \cap \ldots \cap X_n$. Then $Y_n \in \mathcal{U}$ and so $|Y_n| = \omega$.

Thus we can inductively define an increasing sequence of natural numbers $a_0 < a_1 < \ldots < a_n < \ldots$ such that $a_n \in Y_n$. Let $A = \{a_n \mid n \in \omega\}$. Then clearly $A \subseteq^* Y_n \subseteq X_n$ for all $n \in \omega$.

First suppose that $A \notin \mathcal{U}$. Then $\omega \setminus A \in \mathcal{U}$, and hence there exists $n \in \omega$ such that $Y_n \subseteq \omega \setminus A$, and so $A \subseteq \omega \setminus Y_n$. Thus $A \subseteq^* Y_n \cap (\omega \setminus Y_n) = \emptyset$, which is a contradiction.

Thus $A \in \mathcal{U}$. Express $A = B \coprod C$, where $|B| = |C| = \omega$. Then without loss of generality $B \in \mathcal{U}$ and hence there exists $n \in \omega$ such that $Y_n \subseteq B$. Thus $C \subseteq A \subseteq^* Y_n \subseteq B$, which is a contradiction. □

Exercise: (MA) If $\mathcal{U}$ is a nonprincipal ultrafilter on ω, then $d_{\mathcal{U}} = 2^\omega$.

3.3 Other applications

Remark. Recall the following. Let μ be Lebesgue measure on $\mathbb{R}$. If $\emptyset \neq U \subseteq \mathbb{R}$ is open, then we can express $U = \coprod_{k \in K} I_k$, where each I_k is an open interval. Then $\mu(U) = \sum_{k \in K} \text{length}(I_k)$.

Definition. *A subset $N \subseteq \mathbb{R}$ is a null set iff for every $\varepsilon > 0$ there exists an open $U \subseteq \mathbb{R}$ such that $N \subseteq U$ and $\mu(U) < \varepsilon$.*

Example.

1. Every countable subset $S \subset \mathbb{R}$ is a null set.
2. The Cantor middle third set $C \subseteq \mathbb{R}$ is a null set of size 2^ω.
3. If $A_n \subseteq \mathbb{R}$ is a null set for each $n \in \omega$, then $\cup_n A_n$ is also a null set.

Theorem 3.16 (MA)**.** *Suppose $\kappa < 2^\omega$ and $A_\alpha \subseteq \mathbb{R}$ is null for all $\alpha < \kappa$. Then $A = \cup_{\alpha < \kappa} A_\alpha$ is also null.*

Proof. It is enough to show that for each $\varepsilon > 0$, there exists an open $U \subseteq \mathbb{R}$ such that $A \subseteq U$ and $\mu(U) \leq \epsilon$. From now on, fix $\varepsilon > 0$.

Definition. *Let $\mathbb{P} = \{p \subseteq \mathbb{R} \mid p$ is open and $\mu(p) < \epsilon\}$, ordered by $q \leq p$ iff $q \supseteq p$.*

Claim. For each $\alpha < \kappa$, $D_\alpha = \{p \subseteq \mathbb{R} \mid A_\alpha \subseteq p\}$ is dense.

Proof. Suppose $p \in \mathbb{P} \setminus D_\alpha$. Since A_α is null, there exists an open $q \subseteq \mathbb{R}$ such that $A_\alpha \subseteq q$ and $\mu(p \cup q) < \varepsilon$. Then $p \cup q \subseteq p$ and $p \cup q \in D_\alpha$. □

Definition. *Let $\mathcal{B}$ be the set of nonempty open intervals with rational endpoints. Let $\mathcal{C}$ be the set of finite unions of elements of $\mathcal{B}$.*

Claim. $\mathbb{P}$ is ccc.

Proof. Suppose $\{p_\alpha \mid \alpha < \omega_1\}$ is an uncountable antichain. For each $\alpha < \omega_1$, there exists $n \geq 1$ such that $\mu(p_\alpha) \leq \varepsilon - 4/n$. Since there are only countably many options for n, there is at least one n for which the inequality holds for uncountably many elements. So we may assume without loss of generality that there exists a fixed n such that $\mu(p_\alpha) \leq \varepsilon - 4/n$ for all $\alpha < \omega_1$.

For each $\alpha < \omega_1$, there exists $C \in \mathcal{C}$ such that $\mu(p_\alpha \triangle C) \leq 1/n$ for all $\alpha < \omega_1$. As above, without loss of generality there exists a fixed $C \in \mathcal{C}$ such that $\mu(p_\alpha \triangle C) \leq 1/n$ for all $\alpha < \omega_1$.

Notice that if $\alpha < \beta < \omega_1$, then $p_\alpha \cup p_\beta \subseteq C \cup (p_\alpha \setminus C) \cup (p_\beta \setminus C)$. Thus

$$\begin{aligned}\mu(p_\alpha \cup p_\beta) &\leq \mu(C) + \mu(p_\alpha \setminus C) + \mu(p_\beta \setminus C)\\ &\leq \varepsilon - 3/n + 1/n + 1/n = \varepsilon - 1/n\end{aligned}$$

□

By MA, there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Let $U = \cup G$. Then U is open and $A = \cup_{\alpha<\kappa} A_\alpha \subseteq U$.

Claim. $\mu(U) \leq \varepsilon$.

Proof. First note that $U = \cup\{p \mid p \in G \cap \mathcal{B}\}$. To see this, suppose that $x \in p \in G$. Then there exists $q \in \mathcal{B}$ such that $x \in q \subseteq p$. Since $p \leq q$ and $p \in G$, it follows that $q \in G$. Hence if $\mu(U) > \varepsilon$, there exists $p_1, \ldots, p_t \in G \cap \mathcal{B}$ such that $\mu(p_1 \cup \ldots \cup p_t) > \varepsilon$. But if $p, q \in G$, then $p \cup q \in G$, and so $p_1 \cup \ldots \cup p_t \in G$. Then $\mu(U) \leq \varepsilon$. □

□

Definition (Borel)**.** *$A \subseteq \mathbb{R}$ has strong measure 0 iff for every sequence of positive reals $a_0 \geq a_1 \geq \ldots \geq a_n \geq \ldots > 0$ there exists a sequence of open intervals I_n such that $\mu(I_n) = a_n$ and $A \subseteq \cup_n I_n$.*

Example. If $A \subseteq \mathbb{R}$ is countable, then A has strong measure 0.

Exercise: Prove that the Cantor set does not have strong measure 0.

Exercise: (MA) If $\kappa < 2^\omega$ and $A \subseteq \mathbb{R}$ has size κ, then A has strong measure 0.

Exercise: $(CH)^*$ Prove that there exists an uncountable strong measure 0 set.

Definition. *If $\mathbb{P}, \mathbb{Q}$ are posets, then $\mathbb{P} \times \mathbb{Q}$ is the poset such that $\langle p', q' \rangle \leq \langle p, q \rangle$ iff $p' \leq p$ and $q' \leq q$.*

Question: Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are ccc. Does it follow that $\mathbb{P} \times \mathbb{Q}$ is ccc?

Definition. *A poset $\mathbb{P}$ is strongly ccc iff whenever $W \subseteq \mathbb{P}$ is uncountable, there exists an uncountable $Z \subseteq W$ such that the elements of Z are pairwise compatible.*

Example.

- If $\mathcal{F} \subseteq {}^\omega\omega$, then $\mathbb{P}_{\mathcal{F}}$ is strongly ccc.
- If $\mathcal{A} \subseteq \mathcal{P}(\omega)$, then $\mathbb{P}_{\mathcal{A}}$ is strongly ccc.
- If $\varepsilon > 0$ and $\mathbb{P} = \{p \subseteq \mathbb{R} \mid p \text{ open and } \mu(p) < \varepsilon\}$, then $\mathbb{P}$ is strongly ccc.

Exercise: If $\mathbb{P}$ and $\mathbb{Q}$ are strongly ccc, then $\mathbb{P} \times \mathbb{Q}$ is strongly ccc.

Example (Not in ZFC). Suppose T is a Suslin tree and $\mathbb{P}_T = \langle T, \geq \rangle$. Then T has no uncountable subsets of pairwise compatible elements, as such a set must be a branch. Thus $\mathbb{P}$ is ccc but not strongly ccc.

Theorem 3.17 ($MA(\omega_1)$)**.** *If $\mathbb{P}$ is ccc, then $\mathbb{P}$ is strongly ccc.*

Corollary ($MA(\omega_1)$)**.** *If $\mathbb{P}$ and $\mathbb{Q}$ are ccc, then $\mathbb{P} \times \mathbb{Q}$ is strongly ccc.*

Exercise:** If T is a well-pruned Suslin tree, then $\mathbb{P}_T \times \mathbb{P}_T$ is not ccc.

End of Lecture 11.

Lecture 12.

Theorem 3.17 ($MA(\omega_1)$)**.** *If $\mathbb{P}$ is ccc, then $\mathbb{P}$ is strongly ccc.*

Proof. Suppose $W = \{w_\alpha \mid \alpha < \omega_1\} \subseteq \mathbb{P}$.

Claim. There exists $p_0 \in W$ such that every strengthening $p \leq p_0$ is compatible with uncountably many elements of W.

Proof. Suppose not. Then for every $\alpha < \omega_1$, there exists $v_\alpha \leq w_\alpha$ and $\alpha < \beta < \omega_1$ such that v_α is incompatible with w_γ for all $\gamma \geq \beta$. Then we can inductively construct a sequence $\langle v_{\alpha_i} \mid i < \omega_1 \rangle$ such that $v_{\alpha_i} \leq w_{\alpha_i}$ and v_{α_i} is incompatible with w_{α_j} for all $j > i$. But then $\{v_{\alpha_i} \mid i < \omega_1\}$ is an uncountable antichain, which is a contradiction. ☐

Consider $\mathbb{P}_0 = \{p \in \mathbb{P} \mid p \leq p_0\}$. Clearly $\mathbb{P}_0$ is also ccc. Also, for each $\alpha < \omega_1$, $D_\alpha = \{p \in \mathbb{P}_0 \mid p \leq w_\beta \text{ for some } \beta \geq \alpha\}$ is dense in $\mathbb{P}_0$. By $MA(\omega_1)$, there exists a filter $G \subseteq \mathbb{P}_0$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. If we close G upwards in $\mathbb{P}$ we get a filter $G^* \subseteq \mathbb{P}$, and then $G^* \cap W$ is an uncountable set of pairwise compatible elements. ☐

This brings us to our final application of MA.

Theorem 3.18 (MA)**.** *If T is a tree with no uncountable branches and $|T| = \kappa < 2^\omega$, then T is special.*

Corollary ($MA + \neg CH$)**.** *Every Aronszajn tree is special.*

Definition. *If T is a tree, then $f \colon T \to \omega$ is a specializing function iff for $s \neq t \in T$, if $f(s) = f(t)$ then s, t are incomparable.*

Definition. *A family of sets $\mathcal{F}$ is a Δ-system iff there exists a fixed set R, called the root, such that if $A \neq B \in \mathcal{F}$, then $A \cap B = R$.*

Remark. $R = \emptyset$ is allowed.

Theorem 3.19. *Suppose $\mathcal{F}$ is a family of finite sets such that $|\mathcal{F}| = \omega_1$. Then there exists a Δ-system $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $|\mathcal{F}_0| = \omega_1$.*

Proof. We can suppose

i. There exists a fixed $n \geq 1$ such that $|A| = n$ for all $A \in \mathcal{F}$, since there is at least one such n for which uncountably many members of $\mathcal{F}$ are size n.

ii. Inductively we may assume for all $l < n$ the result is true for all uncountable families of l-sets.

Let $S = \cup\mathcal{F}$.

Case 1: There exists $s \in S$ such that $|\{A \in \mathcal{F} \mid s \in A\}| = \omega_1$. In this case, we can suppose that $s \in A$ for all $A \in \mathcal{F}$. Consider $\mathcal{F}^* = \{A \setminus \{s\} \mid A \in \mathcal{F}\}$. By ii. there exists a Δ-system $\mathcal{F}_0^* \subseteq \mathcal{F}^*$ with root R^*. Then $\{B \cup \{s\} \mid B \in \mathcal{F}_0^*\} \subseteq \mathcal{F}$ is a Δ-system with root $R = R^* \cup \{s\}$.

Case 2: Otherwise, each $s \in S$ lies in only countably many $A \in \mathcal{F}$. Then we shall construct $\{A_\alpha \mid \alpha < \omega_1\} \subseteq \mathcal{F}$ such that $A_\alpha \cap A_\beta = \emptyset$ for all $\alpha < \beta < \omega_1$. Suppose we have defined $\{A_\gamma \mid \gamma < \alpha\}$. Let $S_\alpha = \cup\{A_\gamma \mid \gamma < \alpha\}$. Then S_α is countable and so there are only countably many $A \in \mathcal{F}$ with $A \cap S_\alpha \neq \emptyset$. Thus we can choose $A_\alpha \in \mathcal{F}$ with $A_\alpha \cap S_\alpha = \emptyset$. ☐

An important application:

Definition. *If I, J are any sets, then $\mathrm{Fn}(I, J)$ is the poset of finite partial functions $p\colon I \to J$, partially ordered by $p \leq q$ iff $p \supseteq q$.*

Theorem 3.20. *If I is arbitrary and J is countable, then $\mathrm{Fn}(I, J)$ is ccc.*

Proof. Suppose $\{p_\alpha \mid \alpha < \omega_1\} \subseteq \mathrm{Fn}(I, J)$ is an antichain. Applying the Δ-system lemma, there exists an uncountable $K \subseteq \omega_1$ and a fixed finite set $R \subseteq I$ such that $\mathrm{dom}\, p_\alpha \cap \mathrm{dom}\, p_\beta = R$ for all $\alpha \neq \beta \in K$. Since J is countable, there are only countably many possibilities for $p_\alpha \restriction R$. Hence there exist $\alpha \neq \beta \in K$ such that $p_\alpha \restriction R = p_\beta \restriction R$. But then $p_\alpha \cup p_\beta \leq p_\alpha, p_\beta$, which is a contradiction. ☐

Theorem 3.18 (MA)**.** *If T is a tree with no uncountable branches and $|T| = \kappa < 2^\omega$, then T is special.*

Proof. Let $\mathbb{P}$ consist of finite partial functions $p\colon T \to \omega$ such that

(*) if $s \neq t$ and $f(s) = f(t)$, then s, t are incompatible.

Clearly if $t \in T$, then $D_t = \{p \in \mathbb{P} | t \in \mathrm{dom}\, p\}$ is dense in $\mathbb{P}$. Suppose there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_t \neq \emptyset$ for all $t \in T$. Then $g = \cup G\colon T \to \omega$ is a specializing function. Thus it is enough to show $\mathbb{P}$ is ccc.

Suppose that $\{p_\alpha \mid \alpha < \omega_1\}$ is an antichain.

1. We can suppose that $\alpha < \beta < \omega_1$, then $|\operatorname{dom} p_\alpha| = |\operatorname{dom} p_\beta|$.
2. We can also suppose that there exists a fixed $R \subseteq T$ such that if $\alpha < \beta < \omega_1$, then $\operatorname{dom} p_\alpha \cap \operatorname{dom} p_\beta = R$.
3. We can also suppose that if $\alpha < \beta < \omega_1$, then $p_\alpha \upharpoonright R = p_\beta \upharpoonright R$.

For each $\alpha < \omega_1$, let $\operatorname{dom} p_\alpha \setminus R = \{x_{\alpha,1}, \dots, x_{\alpha,n}\}$. If $\alpha \neq \beta$, then since p_α, p_β are incomparable in $\mathbb{P}$, there exists $1 \leq k, l \leq n$ such that $p_\alpha(x_{\alpha,k}) = p_\beta(x_{\beta,l})$ while $x_{\alpha,k}$ and $x_{\beta,l}$ are comparable in T. Define

$$Y_{\alpha,k,l} = \{\beta < \omega_1 \mid \beta \neq \alpha, p_\alpha(x_{\alpha,k}) = p_\beta(x_{\beta,l}), \text{ and } x_{\alpha,k}, x_{\beta,l} \text{ are comparable}\}$$

Then $\omega_1 \setminus \{\alpha\} = \cup_{1 \leq k,l \leq n} Y_{\alpha,k,l}$. (Note that this is a finite union.)

Let $\mathcal{U}$ be a uniform ultrafilter on ω_1. Then for each $\alpha < \omega_1$, there exists $k = k(\alpha)$ and $l = l(\alpha)$ such that $Y_{\alpha,k,l} \in \mathcal{U}$. Hence there exists an uncountable $A \subseteq \omega_1$ and fixed k, l such that $k(\alpha) = k$ and $l(\alpha) = l$ for all $\alpha \in A$. Let $\alpha, \beta \in A$. Since $Y_{\alpha,k,l}, Y_{\beta,k,l} \in \mathcal{U}$, we have that $Y_{\alpha,k,l} \cap Y_{\beta,k,l} \in \mathcal{U}$ and so $|Y_{\alpha,k,l} \cap Y_{\beta,k,l}| = \omega_1$.

Notice that if $\gamma \in Y_{\alpha,k,l} \cap Y_{\beta,k,l}$, then $x_{\alpha,k}$ and $x_{\gamma,l}$ are comparable, and so are $x_{\beta,k}$ and $x_{\gamma,l}$. Also notice that if $\gamma \neq \gamma' \in Y_{\alpha,k,l} \cap Y_{\beta,k,l}$, then $x_{\gamma,l} \neq x_{\gamma',l}$, since the domains of p_γ and $p_{\gamma'}$ only overlap in R.

Since there are only countably many $t \in T$ such that $t \leq x_{\alpha,k}$ or $t \leq x_{\beta,k}$, there exists $\gamma \in Y_{\alpha,k,l} \cap Y_{\beta,k,l}$ such that $x_{\alpha,k}, x_{\beta,k} \leq x_{\gamma,l}$. Thus $x_{\alpha,k}$ and $x_{\beta,k}$ are comparable, since T is a tree. But then $\{x_{\alpha,k} \mid \alpha \in A\}$ lies in an uncountable branch, contradiction. □

Notation: $[\omega]^\omega = \{A \subseteq \omega \mid |A| = \omega\}$.

Definition. *A family $\mathcal{A} \subseteq [\omega]^\omega$ is a splitting family iff for every $B \in [\omega]^\omega$, there exists $A \in \mathcal{A}$ such that $|A \cap B| = |B \setminus A| = \omega$.*

Theorem 3.21 (*ZFC*)**.** *If $\mathcal{A} \subseteq [\omega]^\omega$ is a splitting family, then $\omega < |\mathcal{A}| \leq 2^\omega$.*

Proof. Suppose that $\mathcal{A} = \{A_n \mid n \in \omega\} \subseteq [\omega]^\omega$ is a countable splitting family. (If $\mathcal{A}$ is finite, extend it to a countably infinite splitting family.) We must show that there exists $B \in [\omega]^\omega$ such that for every $A_n \in \mathcal{A}$, either $B \subseteq^* A_n$ or $B \subseteq^* \omega \setminus A_n$.

Let $\mathcal{U}$ be a nonprincipal ultrafilter over ω. Define

$$C_n = \begin{cases} A_n & \text{if } A_n \in \mathcal{U} \\ \omega \setminus A_n & \text{if } A_n \notin \mathcal{U} \end{cases}$$

Then $C_0 \cap \ldots \cap C_n \in \mathcal{U}$ and so $|C_0 \cap \ldots \cap C_n| = \omega$. Hence we can define $b_0 < b_1 < \ldots < b_n < \ldots$ with $b_n \in C_0 \cap \ldots \cap C_n$. Then $B = \{b_n \mid n \in \mathbb{N}\} \subseteq^* C_n$ for all $n \in \mathbb{N}$. Hence $B \subseteq^* A_n$ or $B \subseteq^* \omega \setminus A_n$ for all $n \in \mathbb{N}$, as required. □

Exercise: [MA] If $\mathcal{A} \subseteq [\omega]^\omega$ is a splitting family, then $|\mathcal{A}| = 2^\omega$.

End of Lecture 12.

Lecture 13.

Exercise: If γ is a limit ordinal, then there exists a strictly increasing cofinal map $f\colon \operatorname{cf}(\gamma) \to \gamma$.
Exercise: If β, γ are limit ordinals and $f\colon \beta \to \gamma$ is a strictly increasing cofinal map, then $\operatorname{cf}(\beta) = \operatorname{cf}(\gamma)$.

4 Forcing

Our basic assumption is that ZFC is consistent. By Gödel's Completeness Theorem, this means that ZFC has a countable model, say $\langle M, E\rangle$, where E is the interpretation of the membership symbol $\in$.

We will make the following additional assumptions on $\langle M, E\rangle$.

- The membership relation E is the actual membership relation on M, i.e. $E = \in \cap M \times M$. So our model has the form $\langle M, \in\rangle$.
- M is a transitive set, i.e. if $x \in M$ and $y \in x$, then $y \in M$.

Definition. *With the above hypotheses, we say M is a countable transitive model (c.t.m.) of ZFC.*

What is the point of the second condition? It ensures that many first-order properties $\phi(x_1, \ldots, x_n)$ of elements of M are absolute, i.e. for all $a_1, \ldots, a_n \in M$, $M \models \phi[a_1, \ldots, a_n]$ iff $V \models \phi[a_1, \ldots, a_n]$.

Example.

1. If $a, b \in M$, then $M \models a \subseteq b$ iff $V \models a \subseteq b$. Note that the $\Leftarrow$ direction is always true.
2. If $a \in M$, then $M \models a$ is an ordinal iff $V \models a$ is an ordinal .

Warning: not every property is absolute.

Notice that every ordinal $\alpha \in M$ is countable. However, $M \models ZFC$, and so there exists an ordinal $\omega_1^M \in M$ such that $M \models \omega_1^M$ is the least uncountable ordinal.

Remark. We will sometimes add the assumption

- $M \models GCH$

This is justified by Gödel's Theorem that if ZFC is consistent, so is $ZFC + GCH$.

Definition. *A poset is a triple $\langle \mathbb{P}, \leq, \mathbb{1}\rangle$ such that $\leq$ partially orders $\mathbb{P}$, and $\mathbb{1}$ is the largest element of $\mathbb{P}$.*

Definition. *Suppose the triple $\langle \mathbb{P}, \leq, \mathbb{1}\rangle \in M$. Then the filter $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over M iff for all dense sets $D \subseteq \mathbb{P}$, if $D \in M$, then $G \cap D \neq \emptyset$. We sometimes write M-generic or just generic.*

Lemma 4.1. *Let M be a c.t.m. and let $\langle \mathbb{P}, \leq, \mathbb{1} \rangle \in M$. If $p \in \mathbb{P}$, then there exists a filter $G \subseteq \mathbb{P}$ such that $p \in G$ and G is $\mathbb{P}$-generic over M.*

Proof. Working in the real world (i.e. V), let $\{D_n \mid n \in \mathbb{N}\}$ be an enumeration of the dense subsets $D \subseteq \mathbb{P}$ such that $D \in M$. Then we can inductively define a sequence $p = p_0 \geq p_1 \geq \ldots \geq p_n \geq \ldots$ such that $p_{n+1} \in D_n$. Then $G = \{q \in \mathbb{P} \mid (\exists n) p_n \leq q\}$ satisfies our requirements. Notice the similarity to the proof that $MA(\omega)$ holds. □

Notation: We write $p \perp q$ iff $p, q \in \mathbb{P}$ are incompatible.

Lemma 4.2. *Suppose $\mathbb{P} \in M$ satisfies the following condition:*

(*) For all $p \in \mathbb{P}$, there exist $q, r \leq p$ such that $q \perp r$.

If $G \subseteq \mathbb{P}$ is M-generic, then $G \notin M$.

Proof. Suppose that $G \in M$. Since $M \models ZFC$, $D = \mathbb{P} \setminus G \in M$. Let $p \in \mathbb{P}$ be arbitrary. By (*), there exists $q, r \leq p$ such that $q \perp r$. Without loss of generality $q \notin G$ and $q \in D$. Thus D is dense and so $D \cap G = \emptyset$, contradiction. □

Our goal is to define a c.t.m. $M[G]$ such that $M \subseteq M[G]$ and $G \in M[G]$.

Definition. *τ is a $\mathbb{P}$-name iff*

a) τ is a set of ordered pairs.

b) for all $\langle \sigma, p \rangle \in \tau$, σ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.

Obviously this is an inductive definition based on the well-foundedness of $\in$.

Example.

1. $\emptyset$ is a $\mathbb{P}$-name.
2. If $p \in \mathbb{P}$, then $\{\langle \emptyset, p \rangle\}$ is a $\mathbb{P}$-name.

Definition. $M^{\mathbb{P}} = \{\tau \in M \mid \tau \text{ is a } \mathbb{P}\text{-name}\}$

Definition. *Suppose $G \subseteq \mathbb{P}$ is M-generic. For each $\tau \in M^{\mathbb{P}}$, the corresponding interpretation is $\tau_G = \{\sigma_G \mid (\exists p \in G)\langle \sigma, p \rangle \in \tau\}$.*

Example.

1. $\emptyset_G = \emptyset$
2. $\{\langle \emptyset, p \rangle\}_G = \begin{cases} \{\emptyset\} & p \in G \\ \emptyset & p \notin G \end{cases}$

Definition. $M[G] = \{\tau_G \mid \tau \in M^{\mathbb{P}}\}$

Theorem 4.3 (Forcing Theorem A)**.** *Let M be a c.t.m. and let $\langle \mathbb{P}, \leq, \mathbb{1}\rangle \in M$. Suppose that $G \subseteq \mathbb{P}$ is M-generic. Then*

1. *$M[G]$ is a c.t.m.*
2. *$M \subseteq M[G]$ and $G \in M[G]$*
3. *M and $M[G]$ have the same ordinals.*

sketch. (1) We will prove that the pairing axiom $(\forall x)(\forall y)(\exists z)(x \in z \wedge y \in z)$ holds in $M[G]$. Other axioms are similar (although I wonder about Choice).

Suppose $a, b \in M[G]$. Then there exist $\mathbb{P}$-names $\sigma, \tau \in M$ such that $\sigma_G = a$ and $\tau_G = b$. Consider the $\mathbb{P}$-name $\mathcal{O} \in M$ defined by $\mathcal{O} = \{\langle \sigma, \mathbb{1}\rangle, \langle \tau, \mathbb{1}\rangle\}$. Since $\mathbb{1}$ in G, it follows that $\mathcal{O}_G = \{\sigma_G, \tau_G\} = \{a, b\}$.

(2) Next we prove that $M \subseteq M[G]$. For each $x \in M$, we define the canonical $\mathbb{P}$-name $\check{x} \in M^{\mathbb{P}}$ by $\check{x} = \{\langle \check{y}, \mathbb{1}\rangle \mid y \in x\}$. By induction, we see that $\check{x}_G = \{\check{y}_G \mid y \in x\} = \{y \mid y \in x\} = x$.

Next we prove that $G \in M[G]$. Consider $\Gamma \in M^{\mathbb{P}}$ defined by $\Gamma = \{\langle \check{p}, p\rangle \mid p \in \mathbb{P}\}$. Clearly $\Gamma_G = G$.

No (3), but it seems not too bad, since being an ordinal is an absolute property. □

Definition. *Let $\phi(x_1, \ldots, x_n)$ be a first-order formula in the language of set theory with free variables $x_1, \ldots, x_n$ and let $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$. If $p \in \mathbb{P}$, then we write $p \Vdash \phi(\tau_1, \ldots, \tau_n)$ (pronounced "p forces $\phi(\ldots)$") iff for every M-generic $G \subseteq \mathbb{P}$, if $p \in G$, then $M[G] \models \phi(\tau_{1G}, \ldots, \tau_{nG})$.*

Theorem 4.4 (Forcing Theorem B)**.**

1. *With the above hypotheses on $\phi(x_1, \ldots, x_n)$ and $\tau_1, \ldots, \tau_n$, for every M-generic filter $G \subseteq \mathbb{P}$, we have $M[G] \models \phi(\tau_{1G}, \ldots, \tau_{nG})$ iff $(\exists p \in G) p \Vdash \phi(\tau_1, \ldots, \tau_n)$.*
2. *The relation $\Vdash$ is definable in M.*

Example. Let $\mathbb{P} = \mathrm{Fn}(\omega, 2)$. Then since $\omega, 2 \in M$, it follows that $\mathbb{P} \in M$. Let $G \subseteq \mathbb{P}$ be M-generic. Then $G = \Gamma_G \in M[G]$ with $\Gamma = \{\langle \check{p}, p\rangle \mid p \in \mathbb{P}\}$. Notice that for each $n \in \omega$, $D_n = \{p \in \mathbb{P} \mid n \in \operatorname{dom} p\} \in M$ is dense, so $G \cap D_n \neq \emptyset$. Thus $g = \cup G \colon \omega \to 2$. We are really more interested in $S = \{n \in \omega \mid g(n) = 1\}$.

Consider $\sigma \in M^{\mathbb{P}}$ defined by $\sigma = \{\langle \check{n}, p\rangle \mid n \in \mathbb{N}, p \in \mathbb{P}, n \in \operatorname{dom} p$ and $p(n) = 1\}$. Clearly $\sigma_G = S$. Notice that

$$
\begin{aligned}
& M[G] \models \check{3}_G \in \sigma_G \\
\Leftrightarrow\ & (\exists p \in G)(3 \in \operatorname{dom} p \wedge p(3) = 1) \\
\Leftrightarrow\ & p \Vdash \check{3} \in \sigma
\end{aligned}
$$

End of Lecture 13.

Lecture 14.

Note: As I was absent the day of this lecture, these notes were originally taken by Susan Durst.

Recall: We are dealing with M a c.t.m. and $\mathbb{P}$ a poset in M. Given a $\mathbb{P}$-generic filter G, we can construct a model $M[G]$. We have also defined the relation $p \Vdash \phi(\tau_1, \ldots, \tau_n)$ iff for every generic G containing p, $M[G] \models \phi(\tau_{1G}, \ldots, \tau_{nG})$.

Remark. If $q \leq p$ and $p \Vdash \phi(\tau_1, \ldots, \tau_n)$, then $q \Vdash \phi(\tau_1, \ldots, \tau_n)$.

Proof. Let G be generic and $q \in G$. Then $p \in G$, and thus $M[G] \models \phi(\tau_{1G}, \ldots, \tau_{nG})$. Thus $q \Vdash \phi(\tau_1, \ldots, \tau_n)$. $\square$

Definition. *If $E \subseteq \mathbb{P}$ and $p \in \mathbb{P}$, then E is dense below p iff for all $q \leq p$, there exists $r \in E$ with $r \leq q$.*

Lemma 4.5. *Let M be a c.t.m., $\mathbb{P} \in M$, and $E \subseteq \mathbb{P}$ with $E \in M$. Let G be $\mathbb{P}$-generic over M.*

a) Either $E \cap G \neq \emptyset$ or there exists $q \in G$ such that $q \perp r$ for all $r \in E$.

b) If $p \in G$ and E is dense below p, then $G \cap E \neq \emptyset$.

Proof. (a) Define

$$D = \{p \in \mathbb{P} \mid (\exists r \in E) p \leq r\} \cup \{q \in \mathbb{P} \mid q \perp r \quad \forall r \in E\}$$

Then $D \in M$. Let $q \in \mathbb{P}$ be arbitrary. If $q \notin D$, then there exists $r \in E$ such that q, r are compatible. Let $p \in \mathbb{P}$ be such that $p \leq q, r$. Then $p \in D$. Thus D is dense, so $D \cap G \neq \emptyset$. The result follows.

(b) Suppose that E is dense below p and $G \cap E = \emptyset$. By part a), there exists $q \in G$ such that $q \perp r$ for all $r \in E$. Let $q' \in G$ with $q' \leq p, q$. Since E is dense below p, there exists $r \in E$ such that $r \leq q' \leq q$, which is a contradiction. $\square$

4.1 The consistency of $\neg CH$

Recall: If I and J are sets, then $\mathrm{Fn}(I, J) = \{p \mid p \colon I \to J \text{ is a finite partial function}\}$.

Remark. If M is a c.t.m. and $I, J \in M$, then $\mathrm{Fn}(I, J) \in M$.

Lemma 4.6. *Let M be a c.t.m., $\kappa \in M$ an ordinal. If G is $\mathrm{Fn}(\kappa \times \omega, 2)$-generic over M, then $M[G] \models 2^\omega \geq |\kappa|$.*

Proof. For each $\langle \alpha, n\rangle \in \kappa \times \omega$, define the set $D_{\langle \alpha,n\rangle} = \{p \mid \langle \alpha, n\rangle \in \operatorname{dom} p\} \in M$. Clearly $D_{\langle \alpha,n\rangle}$ is dense, so $G \cap D_{\langle \alpha,n\rangle} \neq \emptyset$. Thus, letting $g = \cup G \in M[G]$ we have $g \colon \kappa \times \omega \to 2$.

For each $\alpha < \kappa$, let $f_\alpha \colon \omega \to 2$ be defined by $f_\alpha(n) = g(\alpha, n)$. Then $\langle f_\alpha \mid \alpha < \kappa\rangle \in M[G]$. Hence it is enough to show that $f_\alpha \neq f_\beta$ for all $\alpha \neq \beta$.

Let $E_{\alpha\beta} = \{p \mid (\exists n)\langle \alpha, n\rangle, \langle \beta, n\rangle \in \operatorname{dom} p \text{ and } p(\alpha, n) \neq p(\beta, n)\} \in M$. Clearly $E_{\alpha\beta}$ is dense. Thus $G \cap E_{\alpha\beta} \neq \emptyset$, and so $f_\alpha \neq f_\beta$. □

In particular, we can let $\kappa = \aleph_7^M$. Then $M[G] \models 2^\omega \geq |\aleph_7^M|^{M[G]}$. There remains the possibility that $\aleph_7^M$ is countable in $M[G]$, i.e. that we've accidentally collapsed cardinals.

For later use, we record:

Lemma 4.7. *Let M be a c.t.m. and $\kappa \in M$ be an ordinal. Then $M \models \operatorname{Fn}(\kappa \times \omega, 2)$ is ccc.*

Proof. Earlier we proved that $ZFC \vdash$"If I is arbitrary and J countable, then $\operatorname{Fn}(I, J)$ is ccc". Since $M \models ZFC$, the result follows. □

Remark. If $\mathbb{P} \in M$ is any poset, then $\mathbb{P}$ is really countable (i.e. in V it is). Hence $V \models$ "$\mathbb{P}$ is ccc". However, we might have $M \not\models$ "$\mathbb{P}$ is ccc". For example, look at the poset $\operatorname{Fn}(\omega, \omega_1^M)$.

Definition. *If $\mathbb{P} \in M$, then $\mathbb{P}$ preserves cardinals if whenever G is $\mathbb{P}$-generic over M and $\beta \in M$ is an ordinal, then $M \models$ "β is a cardinal" iff $M[G] \models$ "β is a cardinal".*

Definition. *If $\mathbb{P} \in M$, then $\mathbb{P}$ preserves cofinalities iff whenever G is $\mathbb{P}$-generic over M and $\beta \in M$ is a limit ordinal, then $\operatorname{cf}(\beta)^{M[G]} = \operatorname{cf}(\beta)^M$.*

Lemma 4.8. *If $\mathbb{P}$ preserves cofinalities, then $\mathbb{P}$ preserves cardinals.*

Proof. First notice that preservation of cardinals is only problematic for $\beta > \omega$. Assume $\mathbb{P}$ preserves cofinalities. Let $\alpha \geq \omega$ be a regular cardinal of M. Then $\operatorname{cf}(\alpha)^{M[G]} = \operatorname{cf}(\alpha)^M = \alpha$, and so α remains a cardinal in $M[G]$.

If α is a singular cardinal of M, then in M α is a limit of successor cardinals. Thus in $M[G]$ α is a limit of regular cardinals, and hence a cardinal. □

Lemma 4.9. *Suppose whenever G is $\mathbb{P}$-generic over M and M says κ is an uncountable regular cardinal, then $M[G] \models$ "κ is regular". Then $\mathbb{P}$ preserves cofinalities.*

Proof. Let $\gamma \in M$ be any limit ordinal, and let $M \models \kappa = \operatorname{cf}(\gamma)$. Clearly we can suppose that $\kappa > \omega$. By the exercises given at the beginning of the last lecture, there exists a strictly increasing $f \in M$ such that f maps κ cofinally into γ. By assumption, $M[G] \models \operatorname{cf}(\kappa) = \kappa$. Also, since $f \in M[G]$, it follows that $M[G] \models \operatorname{cf}(\kappa) = \operatorname{cf}(\gamma)$. Hence $M[G] \models \operatorname{cf}(\gamma) = \kappa$. □

We are finally ready to prove

Theorem 4.10. *If $\mathbb{P} \in M$ and $M \models$ "$\mathbb{P}$ is ccc", then $\mathbb{P}$ preserves cofinalities, and hence also cardinals.*

The main point:

Lemma 4.11. *Assume $\mathbb{P} \in M$ and $M \models$ "$\mathbb{P}$ is ccc". Let $A, B \in M$, and let G be $\mathbb{P}$-generic over M. Let $f \in M[G]$ with $f \colon A \to B$. Then there exists $F \colon A \to \mathcal{P}(B)$ such that*

1. *$F \in M$*
2. *$f(a) \in F(a)$ for all $a \in A$*
3. *for all $a \in A$, $M \models |F(a)| \leq \omega$*

Proof. Let $\tau \in M^{\mathbb{P}}$ with $\tau_G = f$. Since $M[G] \models \tau_G \colon A \to B$, there exists $p \in G$ such that $p \Vdash \tau \colon \check{A} \to \check{B}$. Define $F(a) = \{b \in B \mid (\exists q \leq p) q \Vdash \tau(\check{a}) = \check{b}\}$. Since the forcing relation is definable in M, it follows that $F \in M$.

Next let $a \in A$, and let $f(a) = b$. Then there exists $q \in G$ such that $q \Vdash \tau(\check{a}) = \check{b}$. There exists $q' \in G$ such that $q' \leq q, p$, which implies that $q' \Vdash \tau(\check{a}) = \check{b}$, and so $b \in F(a)$.

Finally, we must show that $M \models |F(a)| \leq \omega$. From now on, we work inside M. The axiom of choice tells us there exists an injection $Q \colon F(a) \to \mathbb{P}$ with $Q(b) \Vdash \tau(\check{a}) = \check{b}$ for all $b \in F(a)$. We claim that $\{Q(b) \mid b \in F(a)\}$ is an antichain, and thus countable. Suppose there exists $b \neq b' \in F(a)$, and $q \leq Q(b), Q(b')$. Then there exists a generic H with $q \in H$. Clearly $Q(b), Q(b') \in H$. But then $M[H] \models \tau_H \colon \check{A} \to \check{B}$, $\tau_H(\check{a}) = \check{b}$, and $\tau_H(\check{a}) = \check{b}'$, which is a contradiction. Thus $\{Q(b) \mid b \in F(a)\}$ is an antichain, and therefore countable. □

Proof of Theorem 4.10. If not, then by lemma 4.9, there exists $\kappa \in M$ such that $\kappa > \omega$ with $M \models$ "κ is regular" and $M[G] \models$ "κ is not regular". Hence there exists $\alpha < \kappa$ and $f \in M[G]$ such that f maps α cofinally into κ.

By lemma 4.11, there exists $F \in M$ such that

1. $F \in M$
2. $f(\xi) \in F(\xi)$ for all $\xi < \alpha$
3. $M \models |F(\xi)| \leq \omega$ for all $\xi < \alpha$.

Let $S = \cup_{\xi<\alpha} F(\xi) \in M$. Then 2 implies that S is a cofinal subset of κ. Also, computing within M we see that $M \models |S| = |\alpha| < \kappa$, contradicting $M \models$ "κ is regular". □

End of Lecture 14.

Lecture 15.

Definition. *If $\sigma \in M^{\mathbb{P}}$, a nice name for a subset of σ is an element $\tau \in M^{\mathbb{P}}$ of the form $\tau = \cup\{\{\pi\} \times A_\pi \mid \pi \in \mathrm{dom}(\sigma)\}$ where each A_π is an antichain of $\mathbb{P}$. Note that we allow the possibility that $A_\pi = \emptyset$.*

Remark. We are usually interested in cases such as $\sigma = \check{\omega}$. In this case, $\mathbb{1} \Vdash \tau \subseteq \check{\omega}$. Note that in this case the nice names are of the form $\cup\{\{\check{n}\} \times A_n \mid n \in \omega\}$, where A_n is an antichain of $\mathbb{P}$. In other words, they are functions from $\mathbb{N}$ to the set of antichains of $\mathbb{P}$. We'll see how various chain conditions (which are really antichain conditions, recall) allow us to use this observation to get a handle on these nice names.

Lemma 4.12. *If $\mathbb{P} \in M$ and $\sigma, \mu \in M^{\mathbb{P}}$, then there exists a nice name τ for a subset of σ such that $\mathbb{1} \Vdash \mu \subseteq \sigma \to \mu = \tau$.*

Proof. For each $\pi \in \mathrm{dom}(\sigma)$, let $A_\pi \subseteq \mathbb{P}$ satisfy

1. For all $p \in A_\pi$, $p \Vdash \pi \in \mu$.
2. A_π is an antichain of $\mathbb{P}$.
3. A_π is maximal subject to 1 and 2.

Using the definability of $\Vdash$ in M as well as the axiom of choice in M, we can suppose that $\{A_\pi \mid \pi \in \mathrm{dom}(\sigma)\} \in M$. Let $\tau = \cup\{\{\pi\} \times A_\pi \mid \pi \in \mathrm{dom}(\sigma)\} \in M^{\mathbb{P}}$. Clearly τ is a nice name for a subset of σ.

To show that $\mathbb{1} \Vdash \mu \subseteq \sigma \to \mu = \tau$ we must prove

(*) If $G \subseteq \mathbb{P}$ is generic over M and $\mu_G \subseteq \sigma_G$, then $\mu_G = \tau_G$.

Suppose that $\mu_G \subseteq \sigma_G$.

Claim. $\mu_G \subseteq \tau_G$.

Proof. Let $a \in \mu_G$. Since $\mu_G \subseteq \sigma_G$, there exists $\pi \in \mathrm{dom}(\sigma)$ such that $\pi_G = a$. If $A_\pi \cap G \neq \emptyset$, let $p \in A_\pi \cap G$. Then $\langle \pi, p\rangle \in \tau$ and so $a = \pi_G \in \tau_G$. Else $A_\pi \cap G = \emptyset$. Then there exists $q \in G$ such that $q \perp p$ for all $p \in A_\pi$ by lemma 4.5. Let $q' \in G$ satisfy $q' \Vdash \pi \in \mu$ and let $r \in G$ with $r \leq q', q$. Then $r \Vdash \pi \in \mu$ and $A_\pi \cup \{r\}$ is an antichain, which contradicts the maximality of A_π. □

Claim. $\tau_G \subseteq \mu_G$

Proof. Let $a \in \tau_G$. Then $a = \pi_G$ for some $\langle \pi, p\rangle \in \tau$ with $p \in G$. By definition, $p \Vdash \pi \in \mu$. Hence $a = \pi_G \in \mu_G$. □

□

Lemma 4.13. *Assume $\mathbb{P} \in M$ and that*

1. *$M \models$ "$\mathbb{P}$ is ccc and $|\mathbb{P}| = \kappa \geq \omega$"*
2. *$M \models$ "λ is an infinite cardinal and $\theta = \kappa^\lambda$".*

If $G \subseteq \mathbb{P}$ is generic over M, then $M[G] \models 2^\lambda \leq \theta$.

Proof. Working inside M, every antichain of $\mathbb{P}$ is countable. Hence there are at most κ^ω such antichains. Hence the number of nice names for subsets of $\check{\lambda}$ is at most $(\kappa^\omega)^\lambda = \kappa^\lambda = \theta$.

Let $\langle \tau_\alpha \mid \alpha < \theta \rangle \in M$ be an enumeration (possibly with repetitions) of all such nice names. Then the function $f \colon \theta \to \mathcal{P}(\lambda)^{M[G]}$ given by $\alpha \mapsto (\tau_\alpha)_G$ is an element of $M[G]$. By lemma 4.12, f is surjective. Thus $M[G] \models 2^\lambda \leq \theta$. ☐

Theorem 4.14. *Let $M \models$ "κ is an infinite cardinal such that $\kappa^\omega = \kappa$". Let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2) \in M$ and let $G \subseteq \mathbb{P}$ be generic over M. Then $M[G] \models 2^\omega = \kappa$.*

Proof. By lemma 4.13 we have that $M[G] \models 2^\omega \leq \kappa$. And we have already seen that $M[G] \models 2^\omega \geq \kappa$. ☐

Remark. Suppse that $M \models GCH$. Then $M \models \kappa^\omega = \kappa$ iff $\mathrm{cf}(\kappa) > \omega$.

4.2 The consistency of CH

Let M be any c.t.m. We look for a generic extension $M[G]$ such that $M[G] \models CH$. Our plan is to adjoin a surjection $f \colon \omega_1^M \to \mathcal{P}(\omega)^M$. Which poset $\mathbb{P}$ should we use to do this?

For our first attempt (which won't work), we will try using $\mathbb{Q} = \mathrm{Fn}(\omega_1, \mathcal{P}(\omega))^M = \mathrm{Fn}(\omega_1^M, \mathcal{P}(\omega)^M)$.

Proposition 4.15. *Let G be $\mathbb{Q}$-generic over M. Then $M[G] \models |\mathcal{P}(\omega)^M| = \omega$.*

Proof. For each $s \in \mathcal{P}(\omega)^M$, consider $D_s = \{p \in \mathbb{Q} \mid (\exists n)\langle n, s\rangle \in p\} \in M$. Clearly D_s is dense in $\mathbb{Q}$. Hence $G \cap D_s \neq \emptyset$ for all $s \in \mathcal{P}(\omega)^M$. So if $g = \cup G \in M[G]$, then $g[\omega] = \mathcal{P}(\omega)^M$. ☐

Now we go ahead and try the correct approach.

Definition. *For any infinite cardinal λ and any sets I, J we define $\mathrm{Fn}(I, J, \lambda) = \{p \mid p \colon I \to J$ is a partial function with $|p| < \lambda\}$, partially ordered by $p \leq q$ iff $p \supseteq q$.*

We will force with $\mathbb{P} = \mathrm{Fn}(\omega_1, \mathcal{P}(\omega), \omega_1)^M$, i.e. countable rather than finite approximations of the function we want to make.

Lemma 4.16. *Let $I, J, \lambda \in M$ and $M \models$ "$J \neq \emptyset$, λ is a cardinal, and $|I| \geq \lambda$". Let $\mathbb{P} = \mathrm{Fn}(I, J, \lambda)^M$ and let $G \subseteq \mathbb{P}$ be generic over M. Then $M[G] \models$ "$g = \cup G \colon I \to J$ is onto".*

Proof. Obvious. ☐

In particular, letting $\mathbb{P} = \mathrm{Fn}(\omega_1, \mathcal{P}(\omega), \omega_1)^M$, we have that $M[G] \models$ "$g = \cup G \colon \omega_1^M \to \mathcal{P}(\omega)^M$ is onto".

Problem 1: Is $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^{M[G]}$?

Problem 2: Is $\omega_1^M = \omega_1^{M[G]}$?

If the answer to both questions is yes, then $M[G] \models CH$. Notice also that a positive answer to problem 1 implies a positive answer to problem 2.

Suppose that $\omega_1^M \neq \omega_1^{M[G]}$. Then ω_1^M is countable in $M[G]$ and $g\colon \omega_1^M \to \mathcal{P}(\omega)^M$. Thus $\mathcal{P}(\omega)^M$ is countable in $M[G]$ and so $\mathcal{P}(\omega)^M \neq \mathcal{P}(\omega)^{M[G]}$.

Definition. *A partial order $\mathbb{P}$ is λ-closed iff whenever $\delta < \lambda$ and $\{p_\alpha \mid \alpha < \delta\}$ is a decreasing sequence of elements, then there exists $p \in \mathbb{P}$ such that $p \leq p_\alpha$ for all $\alpha < \delta$.*

Lemma 4.17. *If λ is regular, then $\mathrm{Fn}(I, J, \lambda)$ is λ-closed.*

Proof. Suppose $\gamma < \lambda$ and $\{p_\alpha \mid \alpha < \gamma\}$ is a descending sequence. Thus $p_0 \subseteq p_1 \subseteq \ldots \subseteq p_\alpha \subseteq \ldots$ for $\alpha < \gamma$. Then $p = \cup_{\alpha<\gamma} p_\alpha \in \mathrm{Fn}(I, J, \lambda)$ and clearly $p \leq p_\alpha$ for all $\alpha < \gamma$. ☐

End of Lecture 15.

Lecture 16.

Theorem 4.18. *Assume $\mathbb{P}, A, B, \lambda \in M$ and $M \models$ "λ is a cardinal, $\mathbb{P}$ is λ-closed, and $|A| < \lambda$". If G is $\mathbb{P}$-generic over M and $f \in M[G]$ with $f\colon A \to B$, then $f \in M$.*

Corollary. *Let $\mathbb{P} = \mathrm{Fn}(\omega_1, \mathcal{P}(\omega), \omega_1)^M \in M$. If G is $\mathbb{P}$-generic over M, then $M[G] \models CH$.*

Proof. Let $\lambda = \omega_1^M$ and $A = \omega$. Then by theorem 4.18 $({}^\omega 2)^{M[G]} = ({}^\omega 2)^M$. ☐

Corollary. *Assume $\mathbb{P} \in M$ and that $M \models$ "$\mathbb{P}$ is ω_1-closed". Then if G is $\mathbb{P}$-generic over M, we have $\omega_1^{M[G]} = \omega_1^M$.*

Proof. By theorem 4.18, $({}^\omega \omega_1)^{M[G]} = ({}^\omega \omega_1)^M$. ☐

Proof of Theorem 4.18. Let $K = ({}^A B)^M = {}^A B \cap M$. Suppose that $f \in M[G]$ with $f\colon A \to B$. We must show that $f \in K$.

Suppose not and let $\tau \in M^{\mathbb{P}}$ with $\tau_G = f$. Then there exists $p \in G$ such that

$$(*)\, p \Vdash \text{"}\tau \text{ is a function from } \check{A} \text{ to } \check{B} \text{ and } \tau \notin \check{K}\text{"}$$

We now forget about f and G and derive a contradiction from (*). From now on, we work inside M.

Let $A = \{a_\alpha \mid \alpha < \kappa\}$, where $\kappa < \lambda$. Using transfinite induction and the axiom of choice, we choose sequences $\{p_\alpha \mid \alpha \leq \kappa\} \subseteq \mathbb{P}$ and $\{z_\alpha \mid \alpha < \kappa\} \subseteq B$ such that

1. $p_0 = p$
2. If $\alpha \leq \beta \leq \kappa$, then $p_\beta \leq p_\alpha$.
3. $p_{\alpha+1} \Vdash \tau(\check{a}_\alpha) = \check{z}_\alpha$

Case 1: We are dealing with α a limit ordinal. Then, since $\mathbb{P}$ is λ-closed and $\alpha < \lambda$, there exists $p_\alpha \in \mathbb{P}$ such that $p_\alpha \leq p_\beta$ for all $\beta < \alpha$. That will satisfy our conditions.

Case 2: Suppose that p_α has been defined. Since $p_\alpha \leq p$, it follows that $p_\alpha \Vdash (\exists x \in B)\tau(\check{a}_\alpha) = x$. Hence there exists $z_\alpha \in B$ and $p_{\alpha+1} \leq p_\alpha$ such that $p_{\alpha+1} \Vdash \tau(\check{a}_\alpha) = \check{z}_\alpha$.

Let $g \in M$ be the function $g\colon A \to B$ defined by $g(a_\alpha) = z_\alpha$. Then $g \in K$. Now suppose that H is $\mathbb{P}$-generic over M with $p_\kappa \in H$. Then $p_{\alpha+1} \in H$ for all $\alpha < \kappa$ and so $\tau_H(a_\alpha) = z_\alpha$. In other words, $\tau_H = g \in K$, which contradicts (*). ☐

4.3 The consistency of $\diamondsuit$

Recall that $\diamondsuit$ is the statement:

There exists a sequence $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ such that

- $A_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$.
- For all $X \subseteq \omega_1$, the set $\{\alpha < \omega_1 \mid \alpha \cap X = A_\alpha\}$ is stationary.

Let M be any c.t.m. We seek a generic extension $M[G]$ such that $M[G] \models \diamondsuit$. So we try to generically adjoin a $\diamondsuit$-sequence, without collapsing ω_1.

Definition. *$\mathbb{P}_\diamondsuit$ is the poset of sequences $\langle A_\alpha \mid \alpha < \beta \rangle$ where $\beta < \omega_1$ and $A_\alpha \subseteq \alpha$ for each $\alpha < \beta$, partially ordered by $\langle A_\alpha \mid \alpha < \beta \rangle \leq \langle C_\gamma \mid \gamma < \delta \rangle$ iff $\beta \geq \delta$ and $C_\gamma = A_\gamma$ for all $\gamma < \delta$.*

Remark. I have written in my margin "$\mathrm{Fn}(\omega_1, 2, \omega_1)$ works 'accidentally'". Make of that what you will. (There appears to be a proof in Kunen, although I haven't read it through.)

Lemma 4.19. *$\mathbb{P}_\diamondsuit$ is ω_1-closed.*

Proof. Obvious, since everything is countable. ☐

Lemma 4.20. *Let $\mathbb{P} = \mathbb{P}_\diamondsuit^M \in M$ and let G be $\mathbb{P}$-generic over M. Then $\omega_1^{M[G]} = \omega_1^M$.*

Proof. Immediate from previous lemma and a corollary to theorem 4.18. ☐

Theorem 4.21. *Let $\mathbb{P} = \mathbb{P}_\diamondsuit^M \in M$ and let G be $\mathbb{P}$-generic over M. Then $M[G] \models \diamondsuit$.*

Proof. Let $\langle A_\alpha \mid \alpha < \omega_1^M \rangle = \cup G \in M[G]$. We shall show that $\langle A_\alpha \mid \alpha < \omega_1^M \rangle$ is a $\diamondsuit$-sequence in $M[G]$.

Let $X, C \in M[G]$ satisfy $M[G] \models$ "$X \subseteq \omega_1$ and C is a club of ω_1". Choose names $\tau, \sigma \in M^{\mathbb{P}}$ such that $\tau_G = X$ and $\sigma_G = C$. Then there exists $p \in G$ such that $p \Vdash$ "$\tau \subseteq \check{\omega}_1$ and σ is a club of $\check{\omega}_1$". We shall show that the set of conditions $\{q = \langle A_\alpha \mid \alpha < \beta \rangle$ such that for some $\alpha < \beta$, $q \Vdash \check{\alpha} \in \sigma$ and

$\tau \cap \check{\alpha} = \check{A}_\alpha\}$ is dense below p. It follows that there exists such a q with $q \in G$. Hence $M[G] \models C \cap \{\alpha < \omega_1 \mid X \cap \alpha = A_\alpha\} \neq \emptyset$. Since $C \subseteq \omega_1$ was any club in $M[G]$, we have $M[G] \models$ "$\{\alpha < \omega_1 \mid X \cap \alpha = A_\alpha\}$ is stationary".

From now on, we work inside M. Let $p^* \leq p$ be arbitrary. We shall define by induction

- ordinals $\beta_n, \delta_n < \omega_1$ for $n < \omega$
- $p_n = \langle A_\alpha \mid \alpha < \beta_n \rangle$ for $n < \omega$
- $B_n \subseteq \beta_n$ for $n < \omega$

such that the following hold:

1. $p_0 = p^*$ (which means β_0 is determined).
2. If $n < m$, then $\beta_n < \beta_m$ and $p_m < p_n$.
3. $\beta_n < \gamma_n < \beta_{n+1}$ and $p_{n+1} \Vdash \check{\gamma}_n \in \sigma$.
4. $p_{n+1} \Vdash \tau \cap \check{\beta}_n = B_n$

Suppose that $p_n, \beta_n, \gamma_{n-1}, B_{n-1}$ have been defined for some $n \geq 0$. Since $p_n \Vdash$ "σ is unbounded in $\check{\omega}_1$", there exists $t_n \leq p_n$ and $\beta_n < \gamma_n < \omega_1$ such that $t_n \Vdash \check{\gamma}_n \in \sigma$.

Let $K = \mathcal{P}(\beta_n)^M$. Since $M \models$ "$\mathbb{P}$ is ω_1-closed", and so $({}^{\beta_n}2)^{M[G]} = ({}^{\beta_n}2)^M$, we have $t_n \Vdash (\exists x \in \check{K})\tau \cap \check{\beta}_n = x$. Hence there exists $p_{n+1} \leq p_n$ and $B_n \in K$ such that $p_{n+1} \Vdash \tau \cap \check{\beta}_n = \check{B}_n$ and clearly we can choose $\beta_{n+1} = \operatorname{length}(p_{n+1}) > \gamma_n$. Thus the induction can be carried out.

Now define $\alpha = \sup_{n<\omega} \beta_n = \sup_{n<\omega} \gamma_n$. Let $p_\omega = \cup_{n<\omega} p_n = \langle A_i \mid i < \alpha \rangle \in \mathbb{P}$. By 3, $p_\omega \Vdash \check{\alpha} \in \sigma$. Let $D = \cup_{n<\omega} B_n$. By 4, $p_\omega \Vdash \tau \cap \check{\alpha} = \check{D}$. Hence letting $A_\alpha = D$ and $q = p_\omega{}^\frown A_\alpha$, we have that $q \Vdash$ "$\check{\alpha} \in \sigma$ and $\tau \cap \check{\alpha} = A_\alpha$". □

In particular, we now know that $CH + \neg SH$ is consistent. Then one may ask if $\neg SH \Rightarrow CH$. The answer is no.

Theorem 4.22. *Let $M \models$ "There exists a Suslin tree and $\kappa^\omega = \kappa$". Let $\mathbb{P} = \operatorname{Fn}(\kappa \times \omega, 2) \in M$ and let G be $\mathbb{P}$-generic over M. Then $M[G] \models$ "There exists a Suslin tree and $2^\omega = \kappa$".*

Proof. Let $M \models$ "T is an ever-branching Suslin tree". Suppose $M[G] \models$ "T isn't Suslin". Since $M[G] \models$ "T is an ever-branching ω_1-tree", we must have that $M[G] \models$ "T has an uncountable antichain". Hence $M[G] \models$ "$\exists f\colon \omega_1 \hookrightarrow T$ such that $f[\omega_1]$ is an antichain". Let $f = \tau_G$, where $\tau = M^{\mathbb{P}}$. Then there exists $p \in G$ such that $p \Vdash$ "$\tau\colon \check{\omega}_1 \hookrightarrow T$ and $\tau(\check{\alpha}) \perp \tau(\check{\beta})$ for all $\alpha < \beta < \omega_1$".

From now on we work inside M. For each $\alpha < \omega_1$, choose $p_\alpha \leq p$ such that there exists $t_\alpha \in T$ with $p_\alpha \Vdash t(\check{\alpha}) = \check{t}_\alpha$. By the Δ-system lemma, there exists $I \subseteq \omega_1$ with $|I| = \omega_1$ and a finite $R \subseteq \kappa \times \omega$ such that $\operatorname{dom} p_\alpha \cap \operatorname{dom} p_\beta = R$ for all $\alpha \neq \beta \in I$. There now exists an uncountable $J \subseteq I$ such that $p_\alpha \restriction R = p_\beta \restriction R$ for all $\alpha \neq \beta \in J$. Thus $\{p_\alpha \mid \alpha \in J\}$ are pairwise compatible in $\mathbb{P}$.

Let $\alpha \neq \beta \in J$ and choose $q \leq p_\alpha, p_\beta \leq p$. Then $q \Vdash \tau(\check{\alpha}) = \check{t}_\alpha \wedge \tau(\check{\beta}) = \check{t}_\beta$. Since $p \Vdash \tau(\check{\alpha}) \perp \tau(\check{\beta})$, we must have $t_\alpha \perp t_\beta$. Thus $\{t_\alpha \mid \alpha \in J\}$ is an uncountable antichain in T of elements from M, contradiction. $\square$

End of Lecture 16.

Lecture 17.

5 Intermediate forcing

5.1 Further results on cardinal arithmetic

Let $M \models GCH$ and we seek a generic extension $N \supseteq M$ such that $M \models 2^\omega = \omega_5 \wedge 2^{\omega_1} = \omega_7$.

Attempt 1: We proceed in two steps.

1. First $\mathbb{P} = \mathrm{Fn}(\omega_5 \times \omega, 2)^M \in M$ and let G be $\mathbb{P}$-generic over M. Then $M[G] \models 2^\omega = \omega_5 \wedge 2^{\omega_1} = \omega_5$. (You can see this by looking at the number of nice names for subsets of ω_1. Since $|\mathbb{P}| = \omega_5$, there are (ω_5^ω) antichains in $\mathbb{P}$, so there are $(\omega_5^\omega)^{\omega_1} = \omega_5$ nice names for subsets of ω_1. (We used the assumption that $M \models GCH$ here to do the cardinal arithmetic.))

2. Let $M_1 = M[G]$ and let $\mathbb{Q} = \mathrm{Fn}(\omega_7 \times \omega_1, 2, \omega_1)^{M_1} \in M_1$. Let H be $\mathbb{Q}$-generic over M_1. Then we certainly have that

   - $\mathcal{P}(\omega)^{M_1} = \mathcal{P}(\omega)^{M_1[H]}$
   - $M_1[H] \models 2^{\omega_1} \geq |\omega_7^M|$

   Unfortunately, the following result shows that we've failed.

Proposition 5.1. *Let M^* be any c.t.m. and let $\mathbb{Q} = \mathrm{Fn}(\omega_7 \times \omega_1, 2, \omega_1)^{M^*} \in M^*$. If H is $\mathbb{Q}$-generic over M^*, then $M^*[H] \models CH$.*

Proof. For each $f \in ({}^\omega 2)^{M^*}$, consider $D_f = \{q \in \mathbb{Q} \mid$ there exists a limit ordinal $\lambda < \omega_1$ such that for each $n \in \omega$, $\langle 0, \lambda+n \rangle \in \operatorname{dom} q$ and $q(0, \lambda+n) = f(n)\} \in M^*$. D_f is clearly dense in $\mathbb{Q}$. Hence if $h = \cup H \colon \omega_7 \times \omega_1 \to 2$ and $g \colon \omega_1 \to 2$ is defined by $g(\alpha) = h(0, \alpha)$, then for each $f \in ({}^\omega 2)^{M^*[H]} = ({}^\omega 2)^{M^*}$, there exists a limit ordinal $\lambda < \omega_1$ such that $g(\lambda + n) = f(n)$ for all $n < \omega$. Thus CH holds, as we may define a surjection from ω_1 onto 2^ω using g. $\square$

Attempt 2: We proceed in two steps.

1. Let $\mathbb{P} = \mathrm{Fn}(\omega_7 \times \omega_1, 2, \omega_1)^M \in M$ and let G be $\mathbb{P}$-generic over M.

2. Let $\mathbb{Q} = \mathrm{Fn}(\omega_5 \times \omega, 2)^{M_1} \in M_1$ and let H be $\mathbb{Q}$-generic over M_1. Then we'll show that $M_1[H] \models 2^\omega = \omega_5 \wedge 2^{\omega_1} = \omega_7$.

First, we need to check that cardinals are preserved. We begin by recording some easy consequences/analogues of our earlier results.

Lemma 5.2. *Let $\mathbb{P} \in M$ and suppose $M \models$ "λ is a cardinal and $\mathbb{P}$ is λ-closed". Then $\mathbb{P}$ preserves cofinalities $\leq \lambda$ and cardinalities $\leq \lambda$.*

Proof. This follows more or less immediately from theorem 4.18. ☐

Definition. *$\mathbb{P}$ has the θ-cc iff every antichain of $\mathbb{P}$ has size less than θ. For example, ccc=ω_1-cc.*

Lemma 5.3. *Suppose $\mathbb{P} \in M$ and $M \models$ "θ is a regular cardinal and $\mathbb{P}$ is θ-cc". Then $\mathbb{P}$ preserves cofinalities $\geq \theta$ and cardinalities $\geq \theta$.*

Proof. This is an analogue of theorem 4.10. ☐

So the main point of the first step of our proof is

Theorem 5.4 (*CH*)**.** *If I is any set, then $\mathrm{Fn}(I, 2, \omega_1)$ is ω_2-cc.*

Here the main point is

Lemma 5.5 (Δ-system lemma)**.** *Let $\theta \geq \omega$ be a regular cardinal and let $\kappa < \theta$ be a cardinal such that $\lambda^\kappa < \theta$ for all $\lambda < \theta$. If $\mathcal{A}$ is a family of sets such that $|\mathcal{A}| = \theta$ and $|X| = \kappa$ for all $X \in \mathcal{A}$, then there exists a Δ-system $\mathcal{B} \subseteq A$ such that $|\mathcal{B}| = \theta$.*

Remark.

- If we let $\theta = \omega_1$ and $\kappa = n \in \omega$, this is our old Δ-system lemma.
- Now suppose that $\kappa \geq \omega$. Since $\kappa^\kappa \geq \kappa^+$, we must have that $\theta \geq \kappa^{++}$. Under *GCH* we can take $\theta = \kappa^{++}$.
- (*CH*) If $\mathcal{A}$ is a family of countable sets of size ω_2, then there exists a Δ-system $\mathcal{B} \subseteq \mathcal{A}$ of size ω_2.

Proof of Δ-system lemma. Since $|\cup \mathcal{A}| \leq \theta$, we can suppose that $\cup \mathcal{A} \subseteq \theta$. Then for each $x \in \mathcal{A}$, we have that $\mathrm{otp}(x) < \kappa^+$ (otp=order type) as a subset of θ. Since $\kappa^+ < \theta$ and θ is regular, there exists an ordinal $\rho < \kappa^+$ such that $\mathcal{A}_1 = \{x \in \mathcal{A} \mid \mathrm{otp}(x) = \rho\}$ has size θ. If $\alpha < \theta$, then $|\alpha|^\kappa < \theta$ and so there are less than θ elements of $\mathcal{A}_1$ which are subsets of α. Thus $\cup \mathcal{A}_1$ is unbounded in θ.

For each $x \in \mathcal{A}_1$ and $\xi < \rho$, let $x(\xi)$ =the ξth element of x. Since θ is regular and $\rho < \theta$, there exists $\xi < \rho$ such that $\{x(\xi) \mid x \in \mathcal{A}_1\}$ is unbounded in θ. Let ξ_0 be the least such ξ. Let $\alpha_0 = \sup\{x(n) + 1 \mid x \in \mathcal{A}_1 \wedge \eta < \xi_0\}$. Then $\alpha_0 < \theta$ and $x(\eta) < \alpha_0$ for all $x \in \mathcal{A}_1$ and $\eta < \xi_0$.

Note that the number of possibilities for $\{x(\eta) \mid \eta < \xi_0\}$ is $|\alpha_0|^{|\xi_0|} \leq |\alpha_0|^\kappa < \theta$. Hence there exists a fixed function $r \colon \xi_0 \to \alpha_0$ such that $\mathcal{A}_2 = \{x \in \mathcal{A}_1 \mid x(\eta) = r(\eta) \text{ for all } \eta < \xi_0\}$ has cardinality θ.

Finally we can choose $x_\mu \in \mathcal{A}_2$ by induction on $\mu < \theta$ such that $x_\mu(\xi_0) > \max(\{\alpha_0\} \cup \sup\{x_\nu(\eta) \mid \eta < \rho, \nu < \mu\})$. Let $\mathcal{B} = \{x_\mu \mid \mu < \theta\}$. Then $\mathcal{B}$ is a Δ-system with root $\{r(\eta) \mid \eta < \xi_0\}$. ☐

Theorem 5.4 (CH)**.** *If I is any set, then $\mathrm{Fn}(I, 2, \omega_1)$ is ω_2-cc.*

Proof. Suppose that $\{p_\alpha \mid \alpha < \omega_2\} \subseteq \mathrm{Fn}(I, 2, \omega_1)$. By the Δ-system lemma, there exists $J \subseteq \omega_2$ and a fixed countable set R such that $\operatorname{dom} p_\alpha \cap \operatorname{dom} p_\beta = R$ for all $\alpha \neq \beta \in J$. The number of possibilities for $p_\alpha \restriction R$ is $2^{|R|} \leq \omega_1$. Hence there exists $\alpha \neq \beta \in J$ such that $p_\alpha \restriction R = p_\beta \restriction R$. Then p_α, p_β are compatible. $\square$

Notation: $[X]^2$ = the set of 2-subsets of X.

Ramsey's Theorem says if $\chi\colon [\mathbb{N}]^2 \to 2$ is any map, then there exists an infinite $S \subseteq \mathbb{N}$ such that $\chi \restriction [S]^2$ is constant.

Definition. *A nonprincipal ultrafilter $\mathcal{U}$ on ω is a Ramsey ultrafilter iff for every $\chi\colon [\mathbb{N}]^2 \to 2$, there exists $S \in \mathcal{U}$ such that $\chi \restriction [S]^2$ is constant.*

Exercise: (MA) There exists a Ramsey ultrafilter.

End of Lecture 17.

Lecture 18.

Theorem 5.5. *Let $M \models GCH$ and let $\mathbb{P} = \mathrm{Fn}(\omega_7 \times \omega_1, 2, \omega_1)^M \in M$. Then $\mathbb{P}$ preserves cofinalities and cardinalities. If G is $\mathbb{P}$-generic over M, then $M[G] \models \omega_5^\omega = \omega_5 \wedge 2^{\omega_1} = \omega_7 \wedge \omega_5^{\omega_1} = \omega_7$.*

Proof. Since $M \models$ "$\mathbb{P}$ is ω_1-closed and ω_2-cc", it follows from our previous results that $\mathbb{P}$ preserves cofinalities and cardinalities. Since $M \models$ "$\mathbb{P}$ is ω_1-closed", it follows that $({}^\omega\omega_5)^{M[G]} = ({}^\omega\omega_5)^M$. Hence $M[G] \models \omega_5^\omega = \omega$. Clearly $M[G] \models 2^{\omega_1} \geq \omega_7$ and hence $M[G] \models \omega_5^{\omega_1} \geq 2^{\omega_1} \geq \omega_7$.

From now on, we work inside $M \models GCH$. First, note that $|\mathbb{P}| = \omega_7^\omega \times 2^\omega = \omega_7 \times \omega_1 = \omega_7$. Hence the number of antichains of $\mathbb{P}$ is at most $|\mathbb{P}|^{\omega_1} = \omega_7^{\omega_1} = \omega_7$ and the number of nice $\mathbb{P}$-names for subsets of ω_1 is at most $\omega_7^{\omega_1} = \omega_7$. Thus $M[G] \models 2^{\omega_1} = \omega_7$. Similarly the number of nice $\mathbb{P}$-names for subsets of ω_5 is at most $\omega_7^{\omega_5} = \omega_7$ and so $M[G] \models 2^{\omega_5} = \omega_7$. It follows that $M[G] \models \omega_5^{\omega_1} \leq (2^{\omega_5})^{\omega_1} = \omega_7^{\omega_1} = \omega_7$. $\square$

Theorem 5.6. *Let $M_1 = M[G]$ be as in theorem 5.5. Let $\mathbb{Q} = \mathrm{Fn}(\omega_5 \times \omega, 2)^{M_1} \in M_1$ and let H be $\mathbb{Q}$ generic over M_1. Then $M_1[H] \models 2^\omega = \omega_5 \wedge 2^{\omega_1} = \omega_7$.*

Proof. This follows very quickly from our previous results. In particular, some of the bits of cardinal arithmetic that we showed hold in M_1 put bounds on the number of nice names we work with. $\square$

5.2 The consistency of $\neg CH + MAC$

Definition (MAC)**.** *If* $\mathbb{P}$ *is a nonempty countable poset and* $\mathcal{D}$ *is a family of* $< 2^\omega$ *dense subsets of* $\mathbb{P}$*, then there exists a filter* G *in* $\mathbb{P}$ *such that* $G \cap D \neq \emptyset$ *for all* $D \in \mathcal{D}$*.*

Remark. This is not a standard axiom.

First we present some consequences of MAC.

Theorem 5.7 (MAC)**.** *If* $X \subseteq \mathbb{R}$ *with* $|X| < 2^\omega$*, then* X *has measure* 0*.*

Proof. It is enough to show that if $\varepsilon > 0$, there exists an open subset $U \subseteq \mathbb{R}$ such that $X \subseteq U$ and $\mu(U) \leq \varepsilon$. Let $\mathcal{C}$ be the set of finite unions of open intervals with rational endpoints. Define $\mathbb{P}_\varepsilon = \{U \in \mathcal{C} \mid \mu(U) < \varepsilon\}$ ordered by $p \leq q$ iff $p \supseteq q$. If $x \in X$, then $D_x = \{p \in \mathbb{P}_\varepsilon \mid x \in p\}$ is dense in P_ε. By MAC, there exists a filter $G \subseteq \mathbb{P}_\varepsilon$ such that $G \cap D_x \neq \emptyset$ for all $x \in X$. Let $U = \cup G$. Then $X \subseteq U$ and $\mu(U) \leq \varepsilon$. □

Question: Does MAC imply that if $\lambda < 2^\omega$ and $\{X_\alpha \mid \alpha < \lambda\}$ is a collection of null subsets of $\mathbb{R}$, then $\cup_{\alpha<\lambda} X_\alpha$ is also null?

Definition. $\mathcal{A} \subseteq \mathcal{P}(\omega)$ *is an independent family if whenever* $a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathcal{A}$ *are distinct, then* $|a_1 \cap \ldots \cap a_n \cap (\omega \setminus b_1) \cap \ldots \cap (\omega \setminus b_m)| = \omega$*.*

Theorem 5.8 (MAC)**.** *If* $\mathcal{A} \subseteq \mathcal{P}(\omega)$ *is a maximal independent family, then* $|\mathcal{A}| = 2^\omega$*.*

Proof. Suppose $|\mathcal{A}| < 2^\omega$. Let $\mathbb{P} = \mathrm{Fn}(\omega, 2)$. For each finite family $a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathcal{A}$ of distinct elements and natural number $t \in \mathbb{N}$ let $D_{\underline{a},\underline{b},t}$ be the set of conditions such that there exist $t \leq k.l \in a_1 \cap \ldots \cap a_n \cap (\omega \setminus b_1) \cap \ldots \cap (\omega \setminus b_m)$ such that $p(k) = 1$ and $p(l) = 0$. Clearly each $D_{\underline{a},\underline{b},t}$ is dense in $\mathbb{P}$. By MAC, there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_{\underline{a},\underline{b},t} \neq \emptyset$ for all $\underline{a}, \underline{b}, t$. Let $g = \cup G \colon \omega \to 2$ and $C = \{n \in \omega \mid g(n) = 1\}$. Then $\mathcal{A} \cup \{C\}$ is an independent family. □

Question: Does MAC imply that every mad family has size 2^ω?

Definition. *The dominating number* $\mathfrak{d}$ *is the least size of a family* $\mathcal{D} \subseteq {}^\omega\omega$ *such that for all* $f \in {}^\omega\omega$*, there exists a* $g \in \mathcal{D}$ *such that* $f \leq^* g$*.*

Definition. *The bounding number* $\mathfrak{b}$ *is the least size of a family* $\mathcal{B} \subseteq {}^\omega\omega$ *such that there does not exist a function* $f \in {}^\omega\omega$ *such that* $g \leq^* f$ *for all* $g \in \mathcal{B}$*.*

Proposition 5.9. $\omega < \mathfrak{b} \leq \mathfrak{d} \leq 2^\omega$

□

Theorem 5.10 (MAC)**.** $\mathfrak{d} = 2^\omega$

Proof. Suppose $\mathcal{C} \subseteq {}^\omega\omega$ with $|\mathcal{C}| < 2^\omega$. Let $\mathbb{P} = \mathrm{Fn}(\omega, \omega)$. For each $f \in \mathcal{C}$ and $n \in \mathbb{N}$, let $D_{f,n}$ consist of those $p \in \mathbb{P}$ such that there exists $k > n$ with $p(k) > f(k)$. Clearly $D_{f,n}$ is dense in $\mathbb{P}$. By MAC there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_{f,n} \neq \emptyset$ for every f, n. Let $g = \cup G \colon \omega \to \omega$. Then $g \not\leq^* f$ for each $f \in \mathcal{C}$. □

We've already proved that $MA \vdash \mathfrak{b} = 2^\omega$.
Question: Does $MAC \vdash \mathfrak{b} = 2^\omega$?
Question: Does MAC imply the existence of a Ramsey ultrafilter?

Now we begin to prove the consistency of MAC. We first deal with the trivial cases.

Definition. *$p \in \mathbb{P}$ is an atom iff there does not exist $q, r \leq p$ with $q \perp r$.*

Lemma 5.11. *Suppose $\mathbb{P}$ contains an atom. Then there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for every dense subset $D \subseteq \mathbb{P}$.*

Proof. Let $p \in \mathbb{P}$ be an atom. Then $G = \{q \in \mathbb{P} \mid p \leq q \vee q \leq p\}$ works. ☐

Definition. *$\mathbb{P}$ is atomless if it contains no atoms.*

Definition. *Let $\mathbb{P}, \mathbb{Q}$ be posets. Then $i \colon \mathbb{P} \to \mathbb{Q}$ is a dense embedding if the following hold:*

a) If $p_1 \leq p_2$, then $i(p_1) \leq i(p_2)$.

b) If $p_1 \perp p_2$, then $i(p_1) \perp i(p_2)$.

c) $i(\mathbb{P})$ is dense in $\mathbb{Q}$.

Remark. No sort of injectivity is required for this definition, so the name is a little misleading.

End of Lecture 18.

Lecture 19.

Recall the definition of a dense embedding from last time.

Definition. *Let $\mathbb{P}, \mathbb{Q}$ be posets. Then $i \colon \mathbb{P} \to \mathbb{Q}$ is a dense embedding if the following hold:*

a) If $p_1 \leq p_2$, then $i(p_1) \leq i(p_2)$.

b) If $p_1 \perp p_2$, then $i(p_1) \perp i(p_2)$.

c) $i(\mathbb{P})$ is dense in $\mathbb{Q}$.

Lemma 5.12. *Suppose that $i, \mathbb{P}, \mathbb{Q} \in M$ and that $i \colon \mathbb{P} \to \mathbb{Q}$ is a dense embedding. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over M. Then $H = \{q \in \mathbb{Q} \mid (\exists p \in G) i(p) \leq q\}$ is $\mathbb{Q}$-generic over M.*

Proof. First we check that H is a filter. Clearly if $q_1 \leq q_2$ and $q_1 \in H$, then $q_2 \in H$. Next suppose that $q_1, q_2 \in H$. Then there exists $p_1, p_2 \in G$ such that $i(p_1) \leq q_1$ and $i(p_2) \leq q_2$. Let $r \in G$ with $r \leq p_1, p_2$. Then $i(r) \leq i(p_1) \leq q_1$ and $i(r) \leq i(p_2) \leq q_2$, so $i(r) \in H$ is a common strengthening of q_1, q_2.

Suppose that $D \in M$ is dense in $\mathbb{Q}$. Define $D^* = \{p \in \mathbb{P} \mid (\exists q \in D) i(p) \leq q\} \in M$. We claim that D^* is dense in $\mathbb{P}$.

Fix some $p \in \mathbb{P}$. Then there exists $q \in D$ such that $q \leq i(p)$. By the density of $i(\mathbb{P})$ in $\mathbb{Q}$, there exists p' such that $i(p') \leq q$. Since $i(p), i(p')$ are compatible, it follows that p, p' are compatible. Let $p'' \leq p, p'$. Then $p'' \in D^*$ and $p'' \leq p$. Thus D^* is dense in $\mathbb{P}$.

It follows that there exists $s \in D^* \cap G$. By definition, there exists $q \in D$ with $i(s) \leq q$. Thus $q \in D \cap H$. ☐

Lemma 5.13. *Suppose that* $i, \mathbb{P}, \mathbb{Q} \in M$ *and that* $i\colon \mathbb{P} \to \mathbb{Q}$ *is a dense embedding. Let* $H \subseteq \mathbb{Q}$ *be* $\mathbb{Q}$*-generic over* M*. Then* $G = \{p \in \mathbb{P} \mid i(p) \in H\}$ *is* $\mathbb{P}$*-generic over* M*.*

Proof. We proceed via a series of claims:

Claim. If $p \in G$ and $p \leq q \in \mathbb{P}$, then $q \in G$.

Proof. Obvious. ☐

Claim. If $p, q \in G$, then p and q are compatible.

Proof. Since $p, q \in G$, it follows that $i(p), i(q) \in H$, and so $i(p), i(q)$ are compatible. Hence p, q are compatible. ☐

Claim. If $D \in M$ is dense in $\mathbb{P}$, then $G \cap D \neq \emptyset$.

Proof. Let $D^* = i(D)$. Clearly $D^* \in M$ is dense in $\mathbb{Q}$ and so $D^* \cap H \neq \emptyset$. Hence $D \cap G \neq \emptyset$. ☐

Hence it is enough to prove

Claim. If $p, q \in G$, then there exists $r \in G$ such that $r \leq p, q$.

Proof. Let $D = \{r \in \mathbb{P} \mid r \perp p \vee r \perp q \vee r \leq p, q\}$. Clearly D is dense in $\mathbb{P}$. Hence there exists $r \in G \cap D$. By the second claim, $r \leq p, q$. ☐

☐

Lemma 5.14. *Let* $\mathbb{P} = \{p \in \mathrm{Fn}(\omega, \omega) \mid \operatorname{dom} p \in \omega\}$*. If* $\mathbb{Q}$ *is any countable nonatomic poset, then there exists a dense embedding* $i\colon \mathbb{P} \to \mathbb{Q}$*.*

The proof of this lemma is delayed. First we will derive the following consequence.

Lemma 5.15. *Let* $\mathbb{Q}, I \in M$ *with* $M \models |I| = |\mathbb{Q}| = \omega$*. If* H *is* $\mathrm{Fn}(I, 2)$*-generic over* M*, then there exists* $G \in M[H]$ *such that* $G \subset \mathbb{Q}$ *is a filter with* $G \cap D \neq \emptyset$ *for every dense* $D \subseteq \mathbb{Q}$ *with* $D \in M$*.*

Proof. By lemma 5.11, we can suppose that $\mathbb{Q}$ is atomless. By lemma 5.14, there exist $i, j \in M$ such that $i\colon \mathbb{P} \to \mathbb{Q}$ and $j\colon \mathbb{P} \to \mathrm{Fn}(I, 2)$ are dense embeddings. By lemma 5.13 applied to j, there exists $K \in M[H]$ such that $K \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter over M. By lemma 5.12 applied to i, there exists $G \in M[K] \subseteq M[H]$ such that $G \subseteq \mathbb{Q}$ is $\mathbb{Q}$-generic over M. □

Proof of Lemma 5.14. Let $\mathbb{Q} = \{q_n \mid n \in \mathbb{N}\}$. We shall define $i(p)$ for $p \in \mathbb{P}$ by induction on $n = \operatorname{dom} p$.

First let $i(\emptyset) = \mathbb{1}_{\mathbb{Q}}$. Now suppose that $i(p)$ has been defined for all $p \in \mathbb{P}$ such that $\operatorname{dom} p \leq n$. Suppose inductively that

1. For all $q \in \mathbb{Q}$, there exists $p \in \mathbb{P}$ with $\operatorname{dom} p = n$ such that $q, i(p)$ are compatible.
2. For each $l < n$, there exists $p \in \mathbb{P}$ such that $\operatorname{dom} p = l + 1$ and $i(p) \leq q_l$.

For each $p \in \mathbb{P}$ with $\operatorname{dom} p = n$, use the fact that $\mathbb{Q}$ is atomless to construct a maximal antichain $A_p = \{t_m \mid m < \omega\} \subseteq \mathbb{Q}$ of elements $q < i(p)$. If q_n is compatible with $i(p)$, we choose $t_0 \leq q_n$. Then we define $i(p^\frown\langle n, m\rangle) = t_m$. Clearly 1 and 2 still hold. So $i\colon \mathbb{P} \to \mathbb{Q}$ is a dense embedding. □

Lemma 5.16. *Suppose that $I, S \in M$ and that G is $\mathrm{Fn}(I, 2)$-generic over M. If $X \subseteq S$ with $X \in M[G]$, then there exists $I_0 \subseteq I$ such that*

1. *$I_0 \in M$ and $M \models |I_0| \leq |S|$.*
2. *$X \in M[G \cap \mathrm{Fn}(I_0, 2)]$*

Proof. We can suppose that S is infinite, since otherwise $X \in M$. Let $X = \tau_G$, where $\tau = \cup_{s \in S}\{\check{s}\} \times A_s \in M$ is a nice name for a subset of S. Let $I_0 = \cup\{\operatorname{dom}(p) \mid (\exists s \in S) p \in A_s\} \in M$. Then $M \models |I_0| \leq |S|$. Let $G_0 = G \cap \mathrm{Fn}(I_0, 2)$. Since each $A_s \subseteq \mathrm{Fn}(I_0, 2)$, τ is a $\mathrm{Fn}(I_0, 2)$-name and $\tau_G = \{s \in S \mid (\exists p \in G_0) p \in A_s\} = \tau_{G_0} \in M[G_0]$. □

Remark. Suppose that $I_0, I \in M$ with $I_0 \subset I$. Then clearly $M \models \mathrm{Fn}(I, 2) \cong \mathrm{Fn}(I_0, 2) \times \mathrm{Fn}(I \setminus I_0, 2)$, with the bijection given by $p \mapsto \langle p \restriction I_0, p \restriction I \setminus I_0\rangle$.

Next we show that forcing with $\mathrm{Fn}(I, 2)$ is the same as first forcing with $\mathrm{Fn}(I_0, 2)$ and then forcing with $\mathrm{Fn}(I \setminus I_0, 2)$.

5.3 Product forcing

Definition. $\langle \mathbb{P}_0, \leq_0, \mathbb{1}_0\rangle \times \langle \mathbb{P}_1, \leq_1, \mathbb{1}_1\rangle = \langle \mathbb{P}_0 \times \mathbb{P}_1, \leq, \mathbb{1}\rangle$, *where* $\mathbb{1} = \langle \mathbb{1}_0, \mathbb{1}_1\rangle$ *and* $\langle p_1, p_2\rangle \leq \langle q_1, q_2\rangle$ *iff* $p_1 \leq_0 q_1$ *and* $p_2 \leq_1 q_2$.

Lemma 5.17. *Suppose that $\mathbb{P}_0, \mathbb{P}_1 \in M$ and G is $\mathbb{P}_0 \times \mathbb{P}_1$-generic over M. Define $G_0 = \{p \in \mathbb{P}_0 \mid \langle p, \mathbb{1}_1\rangle \in G\}$ and $G_1 = \{q \in \mathbb{P}_1 \mid \langle \mathbb{1}_0, q\rangle \in G\}$. Then*

a) $G = G_0 \times G_1$

b) *G_0 is $\mathbb{P}_0$-generic over M.*

c) G_1 is $\mathbb{P}_1$-generic over M.

Proof. (a) If $\langle p_0, p_1\rangle \in G$, then $\langle p_0, \mathbb{1}_1\rangle, \langle \mathbb{1}_0, p_1\rangle \in G$ and so $\langle p_0, p_1\rangle \in G_0 \times G_1$. Conversely, suppose $\langle p_0, p_1\rangle \in G_0 \times G_1$. Then $\langle p_0, \mathbb{1}_1\rangle \in G$ and $\langle \mathbb{1}_0, p_1\rangle \in G$ have a common strengthening $\langle q_0, q_1\rangle \in G$. Clearly $\langle q_0, q_1\rangle \leq \langle p_0, p_1\rangle$ and so $\langle p_0, p_1\rangle \in G$.

(b) It is clear that $G_0 \subseteq \mathbb{P}_0$ is a filter. Suppose $D \in M$ is dense in $\mathbb{P}_0$. Consider $D^* = \{\langle p, q\rangle \in \mathbb{P}_0 \times \mathbb{P}_1 \mid p \in D\} \in M$. Clearly D^* is dense in $\mathbb{P}_0 \times \mathbb{P}_1$ and so $G \cap D^* \neq \emptyset$. Part (c) is proved similarly. □

End of Lecture 19.

Lecture 20.

Theorem 5.18. *Suppose $\mathbb{P}_0, \mathbb{P}_1 \in M$ and $G_0 \subseteq \mathbb{P}_0$, $G_1 \subseteq \mathbb{P}_1$ are filters. Then the following are equivalent.*

1. *$G_0 \times G_1$ is $\mathbb{P}_0 \times \mathbb{P}_1$-generic over M.*
2. *G_0 is $\mathbb{P}_0$-generic over M and G_1 is $\mathbb{P}_1$-generic over $M[G_0]$.*
3. *G_1 is $\mathbb{P}_1$-generic over M and G_0 is $\mathbb{P}_0$-generic over $M[G_1]$.*

Furthermore, if 1-3 hold, then $M[G_0 \times G_1] = (M[G_0])[G_1] = (M[G_1])[G_0]$

Proof. Assuming 1-3 are equivalent, the final sentence is obvious, as each is the smallest model containing G_0 and G_1. Clearly it's enough to prove $1 \Leftrightarrow 2$.

($1 \Rightarrow 2$) Assume 1 holds. By lemma 5.17, G_0 is $\mathbb{P}_0$ generic over M. To see that G_1 is $\mathbb{P}_1$-generic over $M[G_0]$, let $D \in M[G_0]$ be such that D is dense in $\mathbb{P}_1$. Then there exists $\tau \in M^{\mathbb{P}_0}$ and $p_0 \in G_0$ such that $D = \tau_{G_0}$ and $p_0 \Vdash$ "τ is dense in $\check{\mathbb{P}}_1$".

Define $D^* = \{\langle q_o, q_1\rangle \mid q_0 \leq p_0 \wedge q_0 \Vdash \check{q}_1 \in \tau\} \in M$. We claim that D^* is dense below $\langle p_0, \mathbb{1}\rangle$ in $\mathbb{P}_0 \times \mathbb{P}_1$. To see this, fix some $\langle r_0, r_1\rangle \leq \langle p_0, \mathbb{1}\rangle$. Then $r_0 \Vdash (\exists x \in \check{\mathbb{P}}_1)(x \in \tau \wedge x \leq \check{r}_1)$. Hence there exists $q_1 \in \mathbb{P}_1$ and $q_0 \leq r_0$ such that $q_0 \Vdash \check{q}_1 \in \tau \wedge \check{q}_1 \leq \check{r}_1$. Then $\langle q_0, q_1\rangle \leq \langle r_0, r_1\rangle$ and $\langle q_0, q_1\rangle \in D^*$.

Since D^* is dense below $\langle p_0, \mathbb{1}\rangle \in G_0 \times G_1$, it follows that there exists $\langle q_0, q_1\rangle \in D^* \cap (G_0 \times G_1)$. Then $q_0 \Vdash \check{q}_1 \in \tau$ and so $q_1 \in \tau_{G_0} = D$, and clearly $q_1 \in G_1$. Thus $q \in D \cap G_1$.

($2 \Rightarrow 1$) Assume 2 holds. It is easily seen that $G_0 \times G_1$ is a filter in $\mathbb{P}_0 \times \mathbb{P}_1$. Let $D \subseteq \mathbb{P}_0 \times \mathbb{P}_1$ be a dense subset with $D \in M$. Let $D^* = \{p_1 \in \mathbb{P}_1 \mid (\exists p_0 \in G_0)\langle p_0, p_1\rangle \in D\}$. Then $D^* \in M[G_0]$ and clearly $D^* \cap G_1 \neq \emptyset$ implies $D \cap (G_0 \times G_1) \neq \emptyset$. Hence it is enough to show that D^* is dense in $\mathbb{P}_1$.

Fix some $r_1 \in \mathbb{P}_1$. Then $D_0 = \{p_0 \in \mathbb{P}_0 \mid (\exists p_1 \leq r_1)\langle p_0, p_1\rangle \in D\} \in M$ is dense in $\mathbb{P}_0$. Hence there exists $p_0 \in D_0 \cap G_0$ and $p_1 \leq r_1$ such that $\langle p_0, p_1\rangle \in D$. But then $p_1 \in D^*$ and so D^* is dense in $\mathbb{P}_1$. □

Theorem 5.19. *Let $M \models$ "κ is a cardinal such that $\kappa^\omega = \kappa$" and let $\mathbb{P} = \mathrm{Fn}(\kappa, 2) \in M$. If G is $\mathbb{P}$-generic over M, then $M[G] \models 2^\omega = \kappa \wedge MAC$.*

Remark. This shows that $MAC \nvdash$ "2^ω is regular". To see this, let $M \models GCH$ and let $\kappa = \aleph^M_{\omega_1}$. Then $M[G] \models 2^\omega = \kappa \wedge MAC$.

Proof. Since $M \models \mathbb{P} \cong \mathrm{Fn}(\kappa \times \omega, 2)$, it follows that $M[G] \models 2^\omega = \kappa$. Next suppose that $\mathbb{Q} \in M[G]$ with $M[G] \models$ "$\mathbb{Q}$ is a countable poset" and that $\mathcal{D} = \{d_\alpha \mid \alpha < \lambda\} \in M[G]$ with $M[G] \models$ "$\lambda < \kappa = 2^\omega$ and each D_α is dense in $\mathbb{Q}$".

We can suppose that $\mathbb{Q} = \langle \omega, \prec \rangle$. By lemma 5.16, there exists $I_0 \in M$ such that

1. $M \models |I_0| \le |\omega \times \omega| = \omega$
2. $\prec \in M[G_0]$, where $G_0 = G \cap \mathrm{Fn}(I_0, 2)$.

Hence we have $\mathbb{Q} \in M[G_0]$. Let $I = \kappa \setminus I_0$. Since $M \models \mathbb{P} \cong \mathrm{Fn}(I_0, 2) \times \mathrm{Fn}(I, 2)$, we can have that $M[G] = M_1[G_1]$, where $M_1 = M[G_0]$ and G_1 is $\mathrm{Fn}(I, 2)$-generic over M_1.

Next define $X = \{\langle n, \alpha \rangle \mid n \in D_\alpha, \alpha < \lambda\} \in M_1[G_1]$. Since $X \subseteq \omega \times \lambda$, there exists $I_1 \in M_1$ such that

1. $M_1 \models |I_1| \le \max\{\omega, \lambda\} < \kappa$
2. $X \in M_1[H_0]$, where $H_0 = G_1 \cap \mathrm{Fn}(I_1, 2)$.

Let $M_2 = M_1[H_0]$. Then $\mathbb{Q} \in M_2$ and $D_\alpha \in M_2$ for each $\alpha < \lambda$. Let $J = I \setminus I_1$. Then $M_2 \models$ "$|J| = \kappa$ and $M[G] = M_2[H_1]$", where H_1 is $\mathrm{Fn}(J, 2)$-generic over M_2.

Finally, let $J = J_0 \coprod J_1$, where $M_2 \models |J_0| = \omega$. Then $M_2[H_1] = M_2[K_0][K_1]$, where K_0 is $\mathrm{Fn}(J_0, 2)$-generic over M_2 and K_1 is $\mathrm{Fn}(J_1, 2)$-generic over $M_2[K_0]$. By lemma 5.15, there exists a filter $L \in M_2[K_0]$ such that $L \subseteq \mathbb{Q}$ and $L \cap D \neq \emptyset$ for every dense subset D of $\mathbb{Q}$ with $D \in M_2$. In particular, $L \cap D_\alpha \neq \emptyset$ for each $\alpha < \lambda$. $\square$

Theorem 5.20. *Let $\kappa \in M$ be a cardinal such that $\kappa^\omega = \kappa$. Let $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$ and let G be $\mathbb{P}$-generic over M. Then the following are true in $M[G]$:*

1. *$2^\omega = \kappa$ and MAC*
2. *There exists a sequence $\langle X_\alpha \mid \alpha < \omega_1 \rangle$ of subsets of $\mathbb{R}$ such that $\mu(X_\alpha) = 0$ for all $\alpha < \omega_1$ and $\mathbb{R} = \cup_{\alpha<\omega_1} X_\alpha$.*

Proof. Our last theorem establishes 1. Since $M \models \mathbb{P} \cong \mathrm{Fn}(\omega_1, 2) \times \mathrm{Fn}(\kappa \setminus \omega_1, 2)$, it follows that $M[G] = M[G_0][H]$, where G_0 is $\mathrm{Fn}(\kappa \setminus \omega_1, 2)$-generic over M and H is $\mathrm{Fn}(\omega_1, 2)$-generic over $M[G_0]$.

From now on, we work inside $N = M[G_0]$. Express $\omega_1 = \coprod_{\alpha<\omega_1} I_\alpha$, where each $|I_\alpha| = \omega$ and let let $J_\alpha = \cup_{\beta<\alpha} I_\beta$. For each $\alpha < \omega_1$, let $H_\alpha = H \cap \mathrm{Fn}(J_\alpha, 2)$ and let $X_\alpha = \mathbb{R} \cap N[H_\alpha]$. Then $\langle X_\alpha \mid \alpha < \omega_1 \rangle \in N[H]$.

It is enough to prove $N[H] \models \mathbb{R} = \cup_{\alpha<\omega_1} X_\alpha$ and $N[H] \models \mu(X_\alpha) = 0$ for each $\alpha < \omega_1$. To show the first statement, we regard $\mathbb{R}$ as the set of Dedekind cuts of $\mathbb{Q}$. Let $r \in \mathbb{R} \cap N[H]$. Then $r \subseteq \mathbb{Q}$ and so there exists $K \subseteq \omega_1^N$ with $K \in N$ such that $N \models |K| = \omega$ and $r \in N[H \cap \mathrm{Fn}(K,2)]$. It follows that there exists $\alpha < \omega_1$ such that $K \subseteq J_\alpha$ and hence $r \in N[H_\alpha] \cap \mathbb{R} = X_\alpha$.

For the second statement, fix some $\alpha < \omega_1$. Then $N[H] = N[H_\alpha][L_\alpha]$, where L_α is $\mathrm{Fn}(\omega_1 \setminus J_\alpha, 2)$-generic over $N[H_\alpha]$. Since $\mathrm{Fn}(\omega_1 \setminus J_\alpha, 2) \cong \mathrm{Fn}(I_\alpha, 2) \times \mathrm{Fn}(\omega_1 \setminus J_{\alpha+1}, 2)$, we have that $N[H_\alpha][L_\alpha] = N[H_\alpha][L'_\alpha][L''_\alpha]$, where L'_α is $\mathrm{Fn}(I_\alpha, 2)$-generic over $N[H_\alpha]$, and L''_α is $\mathrm{Fn}(\omega_1 \setminus J_{\alpha+1}, 2)$-generic over $N[H_\alpha][L'_\alpha]$. Fix some $\varepsilon > 0$ and let $\mathbb{P}_\varepsilon = \{p \mid p$ is a finite union of open intervals with rational endpoints such that $\mu(p) < \varepsilon\}$. For each $r \in X_\alpha \subseteq N[H_\alpha]$, the set $D_r = \{p \in \mathbb{P}_\varepsilon \mid r \in p\} \in N[H_\alpha]$ is dense in $\mathbb{P}_\varepsilon$. There exists a filter $F \in N[H_\alpha][L'_\alpha]$ of $\mathbb{P}_\varepsilon$ such that $F \cap D_r \neq \emptyset$ for all $r \in X_\alpha$. Hence $X_\alpha \subseteq U = \cup F$ and $\mu(U) \leq \varepsilon$. □

End of Lecture 20.

Lecture 21.

Theorem 5.21. *Suppose that $M \models CH \wedge \omega_1 < \kappa = \kappa^\omega$. Let $\mathbb{P} = \mathrm{Fn}(\kappa, 2) \in M$ and let G be $\mathbb{P}$-generic over M. Then the following are true in $M[G]$:*

1. *$MAC + 2^\omega = \kappa$*
2. *$\omega_1 = \mathfrak{b} < \mathfrak{d} = 2^\omega$*

Lemma 5.22. *Let N be any countable transitive model and suppose that H is $\mathrm{Fn}(\omega, 2)$-generic over N. Then for each $f \in ({}^\omega\omega)^{N[H]}$, there exists $g \in ({}^\omega\omega)^N$ such that $g \not\leq^* f$.*

Proof of Theorem 5.21. Let $\mathcal{B} = ({}^\omega\omega)^M \in M[G]$. Then $M[G] \models |\mathcal{B}| = \omega_1$. We claim that $\mathcal{B}$ is unbounded in $({}^\omega\omega)^{M[G]}$. Suppose not. Then there exists $f \in ({}^\omega\omega)^{M[G]}$ such that $g \leq^* f$ for all $g \in \mathcal{B}$. Then there exists $I \in M$ such that $I \subseteq \kappa$ and $M \models |I| = \omega$ with $f \in M[G \cap \mathrm{Fn}(I,2)]$. Since $M \models \mathrm{Fn}(I,2) \cong \mathrm{Fn}(\omega,2)$, there exists $H \subseteq \mathrm{Fn}(\omega,2)$ such that $M[G \cap \mathrm{Fn}(I,2)] = M[H]$. But this contradicts lemma 5.22. □

Proof of Lemma 5.22. Let $\mathbb{P} = \mathrm{Fn}(\omega,2)$ and let $\tau \in N^{\mathbb{P}}$ satisfy $\tau_H = f$. Then there exists $p \in H$ such that $p \Vdash \tau \colon \check{\omega} \to \check{\omega}$. Let $\{q_n \mid n \in \omega\}$ enumerate the conditions $q \leq p$. Working inside N, for each $n \in \omega$, choose $r_n \leq q_n$ such that there exists $l_n \in \omega$ with $r_n \Vdash \tau(\check{n}) = \check{l}_n$. Then we can define $g \in ({}^\omega\omega)^N$ by $g(n) = l_n + 1$.

We claim that $g \not\leq^* f$. Otherwise there exists $q \leq p$ and $N \in \mathbb{N}$ such that $q \Vdash g(\check{n}) \leq \tau(\check{n})$ for all $n \geq \check{N}$. Choose $n \geq N$ such that $q_n \leq q$. Then $r_n \Vdash \check{g}(\check{n}) = l_n + 1 > \tau(\check{n}) = l_n$, which is a contradiction. □

Theorem 5.23. *Let $M \models CH \wedge \kappa^\omega = \kappa$. Let $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$ and let G be $\mathbb{P}$-generic over M. Then the following are true in $M[G]$:*

1. *$MAC + 2^\omega = \kappa$*
2. *There exists a mad family of size ω_1.*

If we argue as before, it is enough to prove

Lemma 5.24. *Let $M \models CH$ and let $\mathbb{P} = \mathrm{Fn}(\omega, 2)$. Then there exists an almost disjoint family $\mathcal{A} \in M$ such that*

a) $M \models$ "$\mathcal{A}$ is a mad family"

b) If G is $\mathbb{P}$-generic over M, then $M[G] \models$ "$\mathcal{A}$ is a mad family".

Proof. Working inside M, we shall define a suitable family $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ by induction over α. First, let $\{A_n \mid n \in \omega\}$ be a partition of ω into infinitely many infinite sets.

Using CH, we see that there are $\omega^\omega = \omega_1$ nice names τ for subsets of ω. Let $\{\langle p_\alpha, \tau_\alpha\rangle \mid \alpha < \omega_1\}$ enumerate all pairs $\langle p, \tau\rangle$ such that $p \in \mathbb{P}$ and τ is a nice name for a subset of ω.

Our basic idea is as follows. Suppose our construction fails. Then there exists a nice name τ for a subset of ω and a condition $p \in \mathbb{P}$ such that

$$(*) \quad p \Vdash |\tau| = \omega \wedge (\forall \beta < \omega_1)|\check{A}_\beta \cap \tau| < \omega$$

Let $\langle p, \tau\rangle = \langle p_\alpha, \tau_\alpha\rangle$. Then A_α is chosen to "defeat" (*). So now suppose inductively that $\omega \leq \alpha < \omega_1$ and that $\{A_\beta \mid \beta < \alpha\}$ has been defined.

Case 1: Suppose that

$$(\boxplus) \quad p_\alpha \Vdash |\tau_\alpha| = \omega \wedge (\forall \beta < \alpha)|\check{A}_\beta \cap \tau| < \omega$$

Then we shall construct A_α such that the following condition (which we will call $(+_\alpha)$) holds: For each $n < \omega$ and $q \leq p_\alpha$, there exists $r \leq q$ and $m \geq n$ such that $m \in A_\alpha$ and $r \Vdash \check{m} \in \tau_\alpha$. To accomplish this, let $\{B_i \mid i \in \omega\}$ enumerate $\{A_\beta \mid \beta < \alpha\}$ and let $\{\langle n_i, q_i\rangle \mid i \in \omega\}$ enumerate $\omega \times \{q \in \mathbb{P} \mid q \leq p_\alpha\}$. By $(\boxplus)$, for each $i \in \omega$, $q_i \Vdash |\tau_\alpha \setminus (\check{B}_0 \cup \ldots \check{B}_i)| = \omega$. Hence there exists $r_i \leq q_i$ and $m_i \geq n_i$ such that $m_i \notin B_0 \cup \ldots \cup B_i$ and $r_i \Vdash \check{m}_i \in \tau_\alpha$. Thus $A_\alpha = \{m_i \mid i \in \omega\}$ satisfies our requirements.

Case 2: Otherwise, choose any A_α such that $\{A_\beta \mid \beta < \alpha\} \cup \{A_\alpha\}$ is almost disjoint.

Now suppose that our construction fails. Then there exists $\alpha < \omega_1$ such that $p_\alpha \Vdash |\tau_\alpha| = \omega \wedge (\forall \beta < \omega_1)|\check{A}_\beta \cap \tau_\alpha| < \omega$ (we call this $(\star)$). Then $(\boxplus)$ holds. By $(\star)$, there exists $n < \omega$ and $q \leq p_\alpha$ such that $q \Vdash (\tau_\alpha \cap \check{A}_\alpha) \subseteq \check{n}$. But by our construction, there exists $r \leq q$ and $m \geq n$ such that $m \in A_\alpha$ and $r \Vdash \check{m} \in \tau_\alpha$, which is a contradiction. □

5.4 Ultrafilters over ω

Definition. *A non-principal ultrafilter $\mathcal{U}$ over ω is Ramsey iff for all $\chi\colon [\mathbb{N}]^2 \to 2$, there exists $A \in \mathcal{U}$ such that $\chi \restriction [A]^2$ is constant.*

Definition. *A non-principal ultrafilter* $\mathcal{U}$ *over* ω *is selective iff whenever* $\omega = \coprod_{n\in\omega} C_n$*, where each* $C_n \notin \mathcal{U}$*, there exists* $A \in \mathcal{U}$ *such that* $|A \cap C_n| = 1$ *for each* $n \in \omega$*.*

Proposition 5.25. *If* $\mathcal{U}$ *is Ramsey, then* $\mathcal{U}$ *is selective.*

Proof. Suppose $\mathcal{U}$ is Ramsey and that $\omega = \coprod_{n\in\omega} C_n$, where each $C_n \notin \mathcal{U}$. Define $\chi\colon [\mathbb{N}]^2 \to 2$ by $\chi(\{a,b\}) = 1$ iff $(\exists n)a, b \in C_n$. Then there exists $B \in \mathcal{U}$ such that $\chi \restriction [B]^2$ is a constant. Clearly $\chi \restriction [B]^2$ is constantly 0 (otherwise some $B \subseteq C_n$ for some n, which would imply $C_n \in \mathcal{U}$), and so $|B \cap C_n| \leq 1$ for each $n \in \omega$. Hence there exists $B \subseteq A \in \mathcal{U}$ such that $|A \cap C_n| = 1$ for each $n \in \omega$. ☐

Definition. *A non-principal ultrafilter* $\mathcal{U}$ *over* ω *is weakly selective iff whenever* $\omega = \coprod_{n\in\omega} C_n$*, where each* $C_n \notin \mathcal{U}$*, there exists* $A \in \mathcal{U}$ *such that* $|A \cap C_n| < \omega$ *for all* $n \in \omega$*.*

Remark. Note that if there is some $k \in \omega$ such that $|A \cap C_n| < k$ for all $n \in \omega$, there exists some $B \in \mathcal{U}$ such that $|B \cap C_n| = 1$ for all $n \in \omega$.

Definition. *A non-principal ultrafilter* $\mathcal{U}$ *over* ω *is a P-point iff whenver* $\{A_n \mid n \in \omega\} \subseteq \mathcal{U}$*, there exists* $B \in \mathcal{U}$ *such that* $B \subseteq^* A_n$ *for each* $n \in \omega$*.*

Theorem 5.26. *If* $\mathcal{U}$ *is a non-principal ultrafilter over* ω*, the following are equivalent:*

1. $\mathcal{U}$ *is weakly selective.*
2. $\mathcal{U}$ *is a P-point.*

Definition. *A subset* $D \subseteq \mathbb{R}$ *is discrete iff for all* $d \in D$*, there exists* $\varepsilon > 0$ *such that* $B_\varepsilon(d) \cap D = \{d\}$*.*

Definition. *A non-principal ultrafilter* $\mathcal{U}$ *over* ω *is discrete iff for every* $f\colon \mathbb{N} \to \mathbb{R}$ *there exists* $A \in \mathcal{U}$ *such that* $f[A]$ *is discrete.*

Theorem 5.27. *If* $\mathcal{U}$ *is a P-point, then* $\mathcal{U}$ *is discrete.*

End of Lecture 21.

Lecture 22.

Theorem 5.26. *If* $\mathcal{U}$ *is a non-principal ultrafilter over* ω*, the following are equivalent:*

1. $\mathcal{U}$ *is weakly selective.*
2. $\mathcal{U}$ *is a P-point.*

Proof. (2⇒1) Suppose $\omega = \coprod_{n\in\omega} C_n$, where each $C_n \notin \mathcal{U}$. Then each $A_n = \omega \setminus C_n \in \mathcal{U}$. Since $\mathcal{U}$ is a P-point, there exists $B \in \mathcal{U}$ such that $B \subseteq^* A_n$ for each $n \in \omega$. Clearly $|B \cap C_n| < \omega$ for each $n \in \omega$.

(1⇒2) Suppose that $\{A_n \mid n \in \omega\} \subseteq \mathcal{U}$. Replacing each A_n by $\cap_{l<n} A_l$ if necessary, we can suppose $A_0 \supseteq A_1 \supseteq \dots \supseteq A_n \supseteq \dots$ After deleting finitely many elements from A_n, we can also suppose that $A_n \supsetneq A_{n+1}$ for each $n \in \omega$ and $A_n \subseteq \omega \setminus n$ for each $n \in \omega$. In particular, $\cap_n A_n = \emptyset$. For each $n \in \omega$, let $C_n = A_n \setminus A_{n+1} \notin \mathcal{U}$ and consider the partition $\omega = \coprod_n C_n$. Since $\mathcal{U}$ is weakly selective, there exists $B \in \mathcal{U}$ such that $|B \cap C_n| < \omega$ for all $n \in \omega$. Clearly $B \subseteq^* A_n$ for each $n \in \omega$. □

Theorem 5.27. *If $\mathcal{U}$ is a P-point, then $\mathcal{U}$ is discrete.*

Proof. Suppose that $f\colon \mathbb{N} \to \mathbb{R}$ and that $R = f[\mathbb{N}]$.

Case 1: If there exists $r \in R$ such that $f^{-1}(r) \in \mathcal{U}$, then $A = f^{-1}(r)$ satisfies our requirement.

Case 2: Suppose there exists $r \in R$ such that for all $n \geq 1$, $f^{-1}((r - 1/n, r + 1/n)) \in \mathcal{U}$. Clearly there is a unique such point. Hence working with $\mathbb{N} \setminus f^{-1}(r) \in \mathcal{U}$, we can suppose that we are in the next case.

Case 3: Otherwise, for each $r \in R$, there exists $n_r \geq 1$ such that $B_r = \{l \in \omega \mid |f(l) - r| \geq 1/n_r\} \in \mathcal{U}$. Since $\mathcal{U}$ is a P-point, there exists $A \in \mathcal{U}$ such that $A \subseteq^* B_r$ for all $r \in R$. We claim that $f[A]$ is discrete. To see this, let $a \in A$ and $r = f(a)$. Since $A \subseteq^* B_r$, there are only finitely many $l \in A$ such that $|f(l) - r| < 1/n_r$. Hence there exists $\varepsilon > 0$ such that $B_\varepsilon(r) \cap f[A] = \{r\}$. □

So the summary so far:

Ramsey ⇒ selective ⇒ weakly selective/P-point ⇒ discrete ⇒ nonprincipal

We will show the first arrow reverses.

Theorem 5.28. *If $\mathcal{U}$ is non-principal, then the following are equivalent:*

a) $\mathcal{U}$ is Ramsey.

b) $\mathcal{U}$ is selective.

c) If $\{A_n \mid n \in \omega\} \subseteq \mathcal{U}$, then there exists a strictly increasing function $g\colon \omega \to \omega$ such that $g(n+1) \in A_{g(n)}$ for each $n \in \omega$, and $\operatorname{range} g \in \mathcal{U}$.

Proof. (a⇒ b) We have already seen this.

(c⇒ a) Let $\chi\colon [\omega]^2 \to 2$ be any function. For each $n \in \omega$, there exists $\varepsilon_n \in 2$ such that $B_n = \{m \in \omega \mid m > n \wedge \chi(\{n, m\}) = \varepsilon_n\} \in \mathcal{U}$. Let $A_n = \cap_{l<n} B_n \in \mathcal{U}$. Note that $A_0 \supseteq A_1 \supseteq \dots \supseteq A_n \supseteq \dots$ and if $m \in A_n$, then $\chi(\{l, m\}) = \varepsilon_l$ for $l \leq n$.

Let $g\colon \omega \to \omega$ be a strictly increasing function such that $g(n+1) \in A_{g(n)}$ for all $n \in \omega$ and $\operatorname{range} g \in \mathcal{U}$. If $n < m$, then $g(m) \in A_{g(m-1)} \subseteq A_{g(n)}$ and so

$\chi(\{g(n), g(m)\}) = \varepsilon_{g(n)}$. For each $\varepsilon \in 2$, let $D_\varepsilon = \{g(n) \mid n < \omega \wedge \varepsilon_{g(n)} = \varepsilon\}$. Then $\operatorname{range} g = D_0 \coprod D_1 \in \mathcal{U}$ and so there exists $\varepsilon \in 2$ such that $D_\varepsilon \in \mathcal{U}$. Clearly $\chi \restriction [D_\varepsilon]^2$ is constantly ε.

(b$\Rightarrow$ c) Suppose that $\{A_n \mid n \in \omega\} \subseteq \mathcal{U}$. Since $\mathcal{U}$ is selective, $\mathcal{U}$ is weakly selective and a P-point. Hence there exists $B \in \mathcal{U}$ such that $B \subseteq^* A_n$ for each $n \in \omega$. Let $B = \{b_n \mid n \in \omega\}$ be the strictly increasing enumeration. Unfortunately, there is no reason to suppose that $b_{n+1} \in A_{b_n}$. To accomplish this, we must pass to a "suitably sparse" subset of B.

Define $f\colon \omega \to \omega$ by $f(n) =$ the least b_k such that $B \setminus A_n \subseteq b_k$. Suppose we can find a subset $D \subseteq B$ such that $D \in \mathcal{U}$ and if $D = \{d_n \mid n \in \omega\}$ is the increasing enumeration, then $d_{n+1} > f(d_n)$ for all $n \in \omega$. Then since $B \setminus A_{d_n} \subseteq f(d_n)$, it follows that $d_{n+1} \in A_{d_n}$. Thus $g(n) = d_n$ satisfies our requirements.

To see that D exists, we define a partition $B = \coprod_n F_n$ into finite sets as follows. Let $F_0 = \{b_0\}$. Suppose that F_n has been defined and that $b_{t_n} = \max(F_n)$. Then $F_{n+1} = \{b_k \mid t_n + 1 \leq k \leq t_{n+1}\}$, where $b_{t_{n+1}} > \max\{f(b_r) \mid r \leq t_n\}$. By considering the partition $\omega = (\omega \setminus B) \coprod \coprod_{n\in\omega} F_n$, the selectivity of $\mathcal{U}$ gives $E \subseteq B$ with $E \in \mathcal{U}$ such that $|E \cap F_n| = 1$ for each $n \in \omega$. Let $E_0 = E \cap \coprod_n F_{2n}$ and $E_1 = E \cap \coprod_n F_{2n+1}$. Then there exists $i \in 2$ such that $E_i \in \mathcal{U}$. Clearly $D = E_i$ satisfies our requirements. □

Theorem 5.29 (*MAC*)**.** *There exists a Ramsey ultrafilter of* ω*.*

Proof. We shall construct a selective ultrafilter $\mathcal{U}$. Let $\{\mathcal{A}_\alpha \mid \alpha < 2^\omega\}$ be the set of all partitions of ω. We construct by induction on $\alpha < 2^\omega$ an increasing sequence of subsets $S_\alpha \subseteq [\omega]^\omega$ such that

1. $S_0 = \mathcal{F}_0$ is the Fréchet filter.
2. S_α is closed under finite intersections and $|S_\alpha| \leq |\alpha| + \omega$.
3. If $\beta < \alpha$, then either there exists $A \in \mathcal{A}_\beta$ such that $A \in S_\alpha$ or $\exists X \in S_\alpha$ such that $|X \cap A| \leq 1$ for all $A \in \mathcal{A}_\beta$.

If we can complete the construction, let $\mathcal{U} \supseteq \cup_{\alpha<2^\omega} S_\alpha$ be any ultrafilter. Then $\mathcal{U}$ is clearly selective.

Suppose S_β has been defined for all $\beta < \alpha$. If $\lim \alpha$, we can set $S_\alpha = \cup_{\beta<\alpha} S_\beta$. Hence we can suppose $\alpha = \beta + 1$.

Case 1: There exists $A \in \mathcal{A}_\beta$ such that $X \cap A \in [\omega]^\omega$ (i.e. $X \cap A$ is infinite) for all $X \in S_\beta$. Then we can let $S_{\beta+1} = S_\beta \cup \{A \cap X \mid X \in S_\beta\}$.

Case 2: Otherwise we claim that

$$(*) \text{ For all } X \in S_\beta,\ |\{A \in \mathcal{A}_\beta \mid X \cap A \neq \emptyset\}| = \omega.$$

Suppose not. Then there exists $X \in S_\beta$ such that $X \subseteq A_1 \coprod \ldots \coprod A_n$ for some $A_1, \ldots, A_n \in \mathcal{A}_\beta$. Let $\mathcal{U}$ be any non-principal ultrafilter containing S_β. Since $X \in \mathcal{U}$, there exists $1 \leq i \leq n$ such that $A_i \in \mathcal{U}$, which means that we are in case 1, which is a contradiction.

Let $\mathbb{P}$ consist of finite subsets p of ω such that $|p \cap A| \leq 1$ for each $A \in \mathcal{A}_\beta$, ordered by $p \leq q$ iff $p \supseteq q$. Obviously $|\mathbb{P}| = \omega$. For each $X \in S_\beta$ and $n \in \omega$, let $D_{X,n} = \{p \in \mathbb{P} \mid (\exists m > n) m \in p \cap X\}$. By (*), $D_{X,n}$ is dense in $\mathbb{P}$. By MAC, there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_{X,n} \neq \emptyset$ for each X, n. Let $Z = \cup G$. Then $S_{\beta+1} = S_\beta \cup \{Z \cap X \mid X \in S_\beta\}$ satisfies our requirements. $\square$

End of Lecture 22.

Lecture 23.

Theorem 5.30. *There exists a weakly selective ultrafilter $\mathcal{U}$ which is not selective.*

Proof. We first ensure that $\mathcal{U}$ is not selective. Choose $\omega = A_0 \supset A_1 \supset \ldots \supset A_n \supset \ldots$ such that $\cap_n A_n = \emptyset$ and $|A_n \setminus A_{n+1}| = \omega$ for all $n \in \omega$. Let $\mathcal{S} = \{T \subseteq \omega \mid |T \cap (A_n \setminus A_{n+1})| \leq 1 \text{ for all } n \in \omega\}$. Let $\mathcal{F}$ be the set of finite intersections of elements of $\{\omega \setminus T \mid T \in \mathcal{S}\}$. Then $\mathcal{F}$ consists of all sets of the form $\omega \setminus C$, where there exists $k < \omega$ such that $|C \cap (A_n \setminus A_{n+1})| \leq k$ for all $n \in \omega$. In particular, each $Z \in \mathcal{F}$ is infinite and $\mathcal{F}$ contains the Fréchet filter.

We intend that $\{A_n \mid n \in \omega\} \cup \mathcal{F} \subseteq \mathcal{U}$ some ultrafilter. Unfortunately $|\mathcal{F}| = 2^\omega$, so we can't deal with it directly using MAC. Instead we do the following. Let $\{\mathcal{B}_\alpha \mid \alpha < 2^\omega\}$ be the set of all partitions of ω. We construct by induction on $\alpha < 2^\omega$ an increasing chain of subsets $S_\alpha \subseteq [\omega]^\omega$ such that

1. $S_0 = \{A_n \mid n \in \omega\}$
2. S_α is closed under finite intersections and $|S_\alpha| \leq |\alpha| + \omega$
3. If $X \in S_\alpha$ and $Z \in \mathcal{F}$, then $X \cap Z \neq \emptyset$.
4. If $\beta < \alpha$, then either there exists $B \in \mathcal{B}_\alpha$ with $B \in S_\alpha$ or there exists $X \in S_\alpha$ with $|B \cap X| < \omega$ for all $B \in \mathcal{B}_\beta$. (In other words, S_α will work as a weakly selective filter for all the partitions with index $< \alpha$.)

Suppose we can complete the induction. Then there exists an ultrafilter $\mathcal{U} \supseteq \mathcal{F} \cup \cup_{\alpha < 2^\omega} S_\alpha$. Clearly $\mathcal{U}$ is weakly selective but not selective.

So suppose $\alpha > 0$ and S_β has been defined for each $\beta < \alpha$. If $\lim \alpha$, then we let $S_\alpha = \cup_{\beta < \alpha} S_\beta$. So suppose that $\alpha = \beta + 1$.

Case 1: Suppose that there exists $B \in \mathcal{B}_\beta$ such that $X \cap B \cap Z \neq \emptyset$ for each $X \in S_\beta$ and $Z \in \mathcal{F}$. Then we let $S_{\beta+1} = S_\beta \cup \{X \cap B \mid X \in S_\beta\}$.

Case 2: Otherwise, let $\mathbb{P}$ consist of all finite subsets $p \subset \omega$ with partial ordering $p \leq q$ iff $p \supseteq q$ and for all $B \in \mathcal{B}_\beta$, if $q \cap B \neq \emptyset$, then $p \cap B = q \cap B$. For each $X \in S_\beta$ and $k < \omega$, let $D_{X,k} = \{p \in \mathbb{P} \mid (\exists n < \omega) |p \cap X \cap (A_n \setminus A_{n+1})| > k\}$.

Claim. $D_{X,k}$ is dense in $\mathbb{P}$.

Before proving the claim, we complete the proof of the theorem. By MAC, there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D_{X,k} \neq \emptyset$ for all X, k. Let $Y = \cup G$. Clearly $|Y \cap B| < \omega$ for all $B \in \mathcal{B}_\beta$. Hence it is enough to prove that $Y \cap X \cap (\omega \setminus C) \neq \emptyset$ for all $X \in S_\beta$ and C such that there exists $k \in \omega$ such that $|C \cap (A_n \setminus A_{n+1})| \leq k$ for all $n \in \omega$. But this follows from the fact that $G \cap D_{X,k} \neq \emptyset$.

Proof of claim. Suppose $D_{X,k}$ is not dense. Then there exists $p \in \mathbb{P}$ such that $q \notin D_{X,k}$ for all $q \leq p$. Let $\{B_i \mid i \in F\} = \{B \in \mathcal{B}_\beta \mid B \cap p \neq \emptyset\}$. Clearly $|F| < \omega$. Let $\mathcal{B}^*_\beta = \mathcal{B}_\beta \setminus \{B_i \mid i \in F\}$. Then for each $n \in \omega$, $|X \cap (\cup \mathcal{B}^*_\beta) \cap (A_n \setminus A_{n+1})| \leq k$, and so $\omega \setminus (X \cap (\cup \mathcal{B}^*_\beta)) \in \mathcal{F}$.

Let $\mathcal{U}'$ be any nonprincipal ultrafilter such that $\mathcal{F} \cup S_\beta \subseteq \mathcal{U}'$. Then $X \cap (\cup \mathcal{B}^*_\beta) \notin \mathcal{U}'$ and so $(\cup_{i \in F} B_i \cap X) \in \mathcal{U}'$. Hence there exists $i \in F$ such that $B_i \in \mathcal{U}'$, which means we are in Case 1, contradiction. □

□

Theorem 5.31 (MA). *There exists a discrete ultrafilter which is not a P-point.*

Proof. Our first step is to ensure that the ultrafilter is not weakly selective, which is equivalent to being a P-point. Choose $\omega = A_0 \supset A_1 \supset \ldots \supset A_n \supset \ldots$ such that $\cap_n A_n = \emptyset$ and $|A_n \setminus A_{n+1}| = \omega$ for all $n \in \omega$. Let $\mathcal{S} = \{T \subseteq \omega \mid |T \cap (A_n \setminus A_{n+1})| < \omega \text{ for all } n \in \omega\}$. Let $\mathcal{F} = \{\omega \setminus T \mid T \in \mathcal{S}\}$. Then $\mathcal{F}$ is closed under finite intersections and contains the Fréchet filter.

Next, let $\{f_\alpha \mid \alpha < 2^\omega\}$ be the set of all functions $f \colon \mathbb{N} \to \mathbb{R}$. We shall construct by induction on $\alpha < 2^\omega$ an increasing sequence of subsets $S_\alpha \subseteq [\omega]^\omega$ such that

1. $S_0 = \{A_n \mid n \in \omega\}$
2. S_α is closed under finite intersections and $|S_\alpha| \leq |\alpha| + \omega$
3. $X \cap Z \neq \emptyset$ for each $X \in S_\alpha$ and $Z \in \mathcal{F}$.
4. If $\beta < \alpha$, then there exists $X \in S_\alpha$ such that $f_\beta[X]$ is discrete.

Assuming the induction can be completed, any ultrafilter $\mathcal{U} \supseteq \mathcal{F} \cup \cup_{\alpha < 2^\omega} S_\alpha$ satisfies our requirements.

Suppose $\alpha > 0$ and S_β has been defined for each $\beta < \alpha$. If $\lim \alpha$, then we let $S_\alpha = \cup_{\beta < \alpha} S_\beta$. So we can suppose $\alpha = \beta + 1$. For simplicity's sake, let $f = f_\beta$. Let $R = f[\mathbb{N}]$. Let $\mathcal{U}$ be a nonprincipal ultrafilter such that $\mathcal{F} \cup S_\beta \subseteq \mathcal{U}$.

Case 1: There exists $d \in R$ such that $f^{-1}(d) \in \mathcal{U}$. Then we can let $S_\alpha = S_\beta \cup \{X \cap f^{-1}(d) | X \in S_\beta\}$. Hence we can suppose that no such point $d \in R$ exists; in particular, R is infinite.

Case 2: There exists $d \in R$ such that for all $n \geq 1$, $f^{-1}((d - 1/n, d + 1/n)) \in \mathcal{U}$. Clearly there exists a unique such point $d \in R$. Note that $f^{-1}(d) \notin \mathcal{U}$. Hence if $R^* = R \setminus \{d\}$, then $f^{-1}(R^*) \in \mathcal{U}$. So working with R^* instead of R, we may assume that we are in the following final case.

Case 3: For each $d \in R$, there exists $n_d \geq 1$ such that $f^{-1}((d - 1/n_d, d + 1/n_d)) \notin \mathcal{U}$. We will treat this case next time. □

End of Lecture 23.

Lecture 24.

We are continuing our proof from last time, assuming that we are in the third case presented there.

Definition. *$\mathbb{P}$ consists of conditions of the form $p = \langle \phi, \mathcal{P} \rangle$, where*

- *$\phi\colon R \to \omega$ is a finite function such that $\phi(d) \geq n_d$ for all $d \in \operatorname{dom}\phi$, and the intervals $(d - 1/\phi(d), d + 1/\phi(d))$ are all disjoint. (We will write $J_p = \cup_{d \in \operatorname{dom}\phi}(d - 1/\phi(d), d + 1/\phi(d))$.)*
- *$\mathcal{P}$ consists of finitely many pairs $\langle X, n \rangle$ such that $X \in S_\beta$ and if $W = (X \setminus f^{-1}(J_p)) \cap (A_n \setminus A_{n+1})$ then $f[W]$ is infinite.*

We order $\mathbb{P}$ by $\langle \phi', \mathcal{P}' \rangle \leq \langle \phi, \mathcal{P} \rangle$ iff $\phi' \supseteq \phi$ and $\mathcal{P}' \supseteq \mathcal{P}$.

Definition. *If $G \subseteq \mathbb{P}$ is a filter, then $D_G = \cup\{\operatorname{dom}\phi \mid (\exists \mathcal{P})\langle \phi, \mathcal{P} \rangle \in G\}$*

Claim. If $G \subseteq \mathbb{P}$ is a filter, then D_G is discrete. (Obvious.)

Claim. $\mathbb{P}$ is ccc.

Proof. Suppose that $\{\langle \phi_i, \mathcal{P}_i \rangle \mid i \in \omega_1\} \subseteq \mathbb{P}$. Then there exist $i \neq j$ such that $\phi_i = \phi_j = \phi$. Clearly $\langle \phi, \mathcal{P}_i \cup \mathcal{P}_j \rangle \leq \langle \phi_i, \mathcal{P}_i \rangle, \langle \phi_j, \mathcal{P}_j \rangle$. $\square$

Claim. If $p = \langle \phi, \mathcal{P} \rangle \in \mathbb{P}$, then there exists a cofinite subset $C \subseteq R \setminus J_p$ such that

(*) For all $d \in C$, there exists $m \in \mathbb{N}^+$ such that $\langle \phi \cup \{\langle d, m \rangle\}, \mathcal{P} \rangle \in \mathbb{P}$

Proof. Suppose that $d \in R \setminus J_p$ and that for all $m \in \mathbb{N}^+$, $\langle \phi \cup \{\langle d, m \rangle\}, \mathcal{P} \rangle \notin \mathbb{P}$. Then there exists $\langle X, n \rangle \in \mathcal{P}$ such that for all $m \in \mathbb{N}^+$, $(d - 1/m, d + 1/m)$ contains a cofinite set of $f[W]$, where $W = (X \setminus f^{-1}(J_p)) \cap (A_n \setminus A_{n+1})$. For a given $\langle X, n \rangle \in \mathcal{P}$, there is at most one such $d \in R \setminus J_p$, as disjoint intervals can't both contain a cofinite subset of $f[W]$. The result follows. $\square$

Definition. *For each $X \in S_\beta$, $\mathcal{D}_X$ consists of those $p = \langle \phi, \mathcal{P} \rangle$ such that one of the following holds:*

1. *There exists $n \in \omega$ and $d \in \operatorname{dom}\phi$ such that $X \cap (A_n \setminus A_{n+1}) \cap f^{-1}(d)$ is infinite.*
2. *There exists $n \in \omega$ such that $\langle X, n \rangle \in \mathcal{P}$.*

Claim. For each $X \in S_\beta$, $\mathcal{D}_X$ is dense.

Proof. Let $p = \langle \phi, \mathcal{P} \rangle \in \mathbb{P}$. For each $n \in \omega$, let $W_n = (X \setminus f^{-1}(J_p)) \cap (A_n \setminus A_{n+1})$. Let $I = \{n \in \omega \mid |W_n| = \omega\}$. Then $\cup_{n \in I} W_n \in \mathcal{U}$. If there exists $n \in I$ such that $f[W_n]$ is infinite, then $\langle \phi, \mathcal{P} \cup \{\langle X, n \rangle\}\rangle \in \mathbb{P}$. Hence we can suppose that $f[W_n]$ is finite for each $n \in I$.

If there exists $n \in I$ and $d \in f[W_n]$ such that

1. $f(t) = d$ for infinitely many $t \in W_n$, and
2. $\langle \phi \cup \{\langle d, m\rangle\}, \mathcal{P}\rangle = q \in \mathbb{P}$ for some $m \in \mathbb{N}^+$

then q satisfies condition i. and so $q \in \mathcal{D}_X$.

So suppose this fails. Then by our third claim, there exists finitely many elements $d_1, \ldots, d_s \in R \setminus J_p$ such that $W_n \setminus \cup_{1\le i\le s} f^{-1}(d_i)$ is finite for all $n \in I$. But this means that $\cup_{k\in I} W_k \setminus \cup_{1\le i\le s} f^{-1}(d_i) \notin \mathcal{U}$ and so $\cup_{1\le i\le s} f^{-1}(d_i) \in \mathcal{U}$. But then $f^{-1}(d_i) \in \mathcal{U}$ for some i, which is a contradiction. ☐

Definition. *For each $X \in S_\beta$ and $n, l \in \omega$, $\mathcal{E}_{X,n,l}$ consists of those $p = \langle \phi, \mathcal{P}\rangle \in \mathbb{P}$ such that one of the following holds:*

1. *$p \perp p' = \langle \phi', \mathcal{P}'\rangle$ whenever $\langle X, n\rangle \in \mathcal{P}'$.*
2. *There exists $t > l$ with $f(t) \in \operatorname{dom}\phi$ and $t \in X \cap (A_n \setminus A_{n+1})$.*

Claim. $\mathcal{E}_{X,n,l}$ is dense.

Proof. Let $p = \langle \phi, \mathcal{P}\rangle \in \mathbb{P}$. Then without loss of generality we can suppose that $\langle X, n\rangle \in \mathcal{P}$. In particular, $f[W]$ is infinite, where $W = (X \setminus f^{-1}(J_p)) \cap (A_n \setminus A_{n+1})$. Hence the result follows by the third claim. ☐

Concluding the proof of Theorem 5.31. By MA, there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap \mathcal{D}_X \neq \emptyset$ and $G \cap \mathcal{E}_{X,n,l} \neq \emptyset$ for every $X \in S_\beta$ and $n, l \in \omega$. Let D_G be the associated discrete subset of $\mathbb{R}$ and let $Y = f^{-1}(D_G)$. We claim that for all $X \in S_\beta$ and $Y \in \mathcal{F}$, we have $X \cap Y \cap Z \neq \emptyset$, and so we can define $S_{\beta+1} = S_\beta \cup \{X \cap Y \mid X \in S_\beta\}$.

Let $p = \langle \phi, \mathcal{P}\rangle \in G \cap \mathcal{D}_X$. If there exists $d \in \operatorname{dom}\phi$ such that $X \cap (A_n \setminus A_{n+1}) \cap f^{-1}(d)$ is infinite, then we are done. Hence we can suppose that $\langle X, n\rangle \in \mathcal{P}$.

Notice that Z contains a cofinite subset of $A_n \setminus A_{n+1}$. Hence there exists an $l \in \omega$ such that for all $t > l$, if $t \in (A_n \setminus A_{n+1}) \cap X$, then $t \in Z$. Choose $q \le p$ with $q \in \mathcal{E}_{X,n,l}$ and let $q = \langle \phi', \mathcal{P}'\rangle$. Then there exists $l < t \in X \cap (A_n \setminus A_{n+1})$ with $f(t) \in \operatorname{dom}\phi'$. Hence $t \in X \cap (A_n \setminus A_{n+1}) \cap f^{-1}(D_G) \cap Z \subseteq X \cap Y \cap Z$. ☐

We ended the class with some discussion of results on cardinal arithmetic, starting from Easton forcing (which concerns only regular cardinals) and discussing the progress made in analyzing cardinal arithmetic with singular cardinals, specifically results of Silver, Magidor, and Shelah. We begin our discussion of Easton forcing next time.

End of Lecture 24.

Lecture 25.

5.5 Easton forcing

Definition. *An Easton index function is a function E such that*

a) $\operatorname{dom}(E)$ *is a set of regular cardinals.*

b) For each $\kappa \in \operatorname{dom}(E)$, $E(\kappa)$ is a cardinal such that $\operatorname{cf}(E(\kappa)) > \kappa$.

c) If $\kappa, \theta \in \operatorname{dom} E$ with $\kappa < \theta$, then $E(\kappa) \leq E(\theta)$.

The idea here is that $E(\kappa)$ is a plausible value for 2^κ for every κ in $\operatorname{dom} E$. The fact that $\operatorname{dom} E$ could only contain regular cardinals turned out to be a rather deep fact.

Definition. *If E is an Easton index function, then $\mathbb{P}(E)$ is the set of functions p such that*

a) $\operatorname{dom} p = \operatorname{dom} E$

b) For all $\kappa \in \operatorname{dom} p$, $p(\kappa) \in \operatorname{Fn}(E(\kappa), 2, \kappa)$

c) For every regular λ, $|\{\kappa \in (\lambda \cap \operatorname{dom} E) \mid p(\kappa) \neq \emptyset\}| < \lambda$.

Remark.

i. When we write every regular λ, we mean every regular λ, even those regular $\lambda \notin \operatorname{dom} E$.

ii. Clause c) has content for those regular λ such that $\aleph_\lambda = \lambda$, i.e. for weakly inaccessible λ.

Definition. *If E is an Easton index function an λ is any cardinal, then $E^+_\lambda = E \restriction \{\kappa \in \operatorname{dom} E \mid \kappa > \lambda\}$, and $E^-_\lambda = E \restriction \{\kappa \in \operatorname{dom} E \mid \kappa \leq \lambda\}$.*

Remark. Obviously E^+_λ and E^-_λ are also Easton index functions.

Lemma 5.32. *If E is an Easton index function and λ is any cardinal, then $\mathbb{P}(E) \cong \mathbb{P}(E^-_\lambda) \times \mathbb{P}(E^+_\lambda)$.*

Lemma 5.33. *If E is an Easton index function such that* $\operatorname{dom} E \cap \lambda^+ = \emptyset$, *then $\mathbb{P}(E)$ is λ^+-closed.*

Corollary. *If E is any Easton index function, then $\mathbb{P}(E^+_\lambda)$ is λ^+-closed.*

Lemma 5.34. *Suppose that λ is a regular cardinal such that $2^{<\lambda} = \lambda$. If E is an Easton index function such that* $\operatorname{dom} E \subseteq \lambda^+$, *then $\mathbb{P}(E)$ is λ^+-cc.*

Proof. For each $p \in \mathbb{P}(E)$, let $d(p) = \cup\{\{\kappa\} \times \operatorname{dom} p(\kappa) \mid \kappa \in \operatorname{dom} E\}$. Then $|d(p)| < \lambda$ by part c) of the definition of $\mathbb{P}(E)$. Now suppose that $\{p_\alpha \mid \alpha < \lambda^+\} \subseteq \mathbb{P}(E)$. By the Δ-system lemma, there exists $X \subseteq \lambda^+$ such that $|X| = \lambda^+$ and $\{d(p_\alpha) \mid \alpha \in X\}$ forms a Δ-system with root r. Since $|r| < \lambda$ and $2^{|r|} \leq \lambda$, there exist distinct $\alpha \neq \beta \in X$ such that $p_\alpha(\kappa)(i) = p_\beta(\kappa)(i)$ for all $\langle \kappa, i\rangle \in r$. Hence p_α, p_β are compatible. $\square$

Corollary. *Suppose λ is a regular cardinal such that $2^{<\lambda} = \lambda$. If E is any Easton index function, then $\mathbb{P}(E^-_\lambda)$ is λ^+-cc.*

Lemma 5.35. *Suppose that $M \models GCH$ and that $\mathbb{P} = \mathbb{P}(E) \in M$ for some Easton index function E. Then $\mathbb{P}$ preserves cofinalities and hence also preserves cardinals.*

Proof. Suppose not. Then there exists a $\mathbb{P}$-generic filter G and $\theta, \lambda \in M$ such that $M \models$ "θ is uncountable and regular" and $M[G] \models \lambda = \mathrm{cf}(\theta) < \theta$. Then λ is regular in $M[G]$ and hence is also regular in M. Working inside M, let $\mathbb{P}_0 = \mathbb{P}(E^-_\lambda)$ and $\mathbb{P}_1 = \mathbb{P}(E^+_\lambda)$. Then $\mathbb{P} \cong \mathbb{P}_0 \times \mathbb{P}_1$, and so $M[G] = M[G_1][G_0]$, where G_1 is $\mathbb{P}_1$-generic over M and G_0 is $\mathbb{P}_0$-generic over $M[G_1]$.

Let $f \in M[G]$ be a cofinal map from λ into θ. Since $M \models$ "$\mathbb{P}_1$ is λ^+-closed", it follows that $M[G_1] \models 2^{<\lambda} = \lambda$. Hence $\mathbb{P}(E^-_\lambda)^M = \mathbb{P}(E^-_\lambda)^{M[G_1]}$ and $M[G_1] \models$ "$\mathbb{P}_0$ is λ^+-cc". Hence there exists $F \in M[G_1]$ such that the following are true in $M[G_1]$:

a) $F \colon \lambda \to \mathcal{P}(\theta)$ and $f(\alpha) \in F(\alpha)$ for all $\alpha \in \lambda$

b) $|F(\alpha)| \leq \lambda$ for all $\alpha \in \lambda$.

Since $M \models$ "$\mathbb{P}_1$ is λ^+-closed", it follows that $F \in M$.

Clearly a) also holds in M. To see that b) is also true in M, note that for each $\alpha \in \lambda$, there exists $g_\alpha \in M[G_1]$ such that $g_\alpha \colon \lambda \to F(\alpha)$ is surjective. Since $M \models$ "$\mathbb{P}_1$ is λ^+-closed", it follows that $g_\alpha \in M$. Thus $M \models |F(\alpha)| \leq \lambda$. But now working inside M, we see that $\cup_{\alpha<\lambda} F(\alpha)$ is a cofinal subset of θ of size at most λ, which contradicts the regularity of θ in M. $\square$

End of Lecture 25.

Lecture 26.

Theorem 5.36. *Let $M \models GCH$ and let $\mathbb{P} = \mathbb{P}(E)$ for some Easton index function E. If $G \subseteq \mathbb{P}$ is M-generic, then $M[G] \models 2^\kappa = E(\kappa)$ for all $\kappa \in \operatorname{dom} E$.*

Proof. Let $\kappa \in \operatorname{dom} E$. First we shall check that $M[G] \models 2^\kappa \geq E(\kappa)$. Let $f \in M$ be a bijection $f \colon E(\kappa) \times \kappa \to E(\kappa)$. Then for any $\alpha < \beta < E(\kappa)$, the set $D_{\alpha\beta} = \{p \in \mathbb{P} \mid \exists \xi < \kappa \text{ such that } p(\kappa)(f(\alpha, \xi)) \neq p(\kappa)(f(\beta, \xi))\}$ is dense. Working in $M[G]$, define $A_\alpha = \{\xi < \kappa \mid (\exists p \in G) p(\kappa)(f(\alpha, \xi)) = 1\}$. Then $A_\alpha \in \mathcal{P}(\kappa)$ and $A_\alpha \neq A_\beta$ for all $\alpha < \beta < E(\kappa)$. Thus $M[G] \models 2^\kappa \geq E(\kappa)$.

Finally we show that $M[G] \models 2^\kappa \leq E(\kappa)$. Working inside M, let $\mathbb{P}_0 = \mathbb{P}(E^-_\kappa)$ and $\mathbb{P}_1 = \mathbb{P}(E^+_\kappa)$. Then $M \models \mathbb{P} \cong \mathbb{P}_1 \times \mathbb{P}_0$ and so $M[G] = M[G_1][G_0]$, where G_1 is $\mathbb{P}_1$-generic over M and G_0 is $\mathbb{P}_0$-generic over $M[G_1]$. Still working inside M, for each $\lambda \in E^-_\kappa$, we have

$$|\operatorname{Fn}(E(\lambda), 2, \lambda)| \leq |\operatorname{Fn}(E(\kappa), 2, \kappa)| = E(\kappa)^{<\kappa} \times 2^{<\kappa} = E(\kappa)$$

Note that we used GCH for the last equality. Hence we have that $E(\kappa) \leq |\mathbb{P}_0| \leq E(\kappa)^\kappa = E(\kappa)$. Since $M \models$ "$\mathbb{P}_1$ is κ^+-closed", the following are true in $M[G_1]$:

a) $E(\kappa)^\kappa = E(\kappa)$

b) $2^{<\kappa} = \kappa$ and so $\mathbb{P}_0$ is κ^+-cc.

Hence working inside $M[G_1]$, there are at most $|\mathbb{P}_0|^\kappa = E(\kappa)^\kappa = E(\kappa)$ antichains in $\mathbb{P}_0$ and at most $E(\kappa)^\kappa = E(\kappa)$ nice $\mathbb{P}_0$-names for subsets of $\check{\kappa}$. Hence $M[G] = M[G_1][G_0] \models 2^\kappa \leq E(\kappa)$. $\square$

Question: What is the value of 2^θ for a cardinal θ such that $\theta \notin \mathrm{dom}(E)$? Answer: Intuitively speaking, the least possible value.

We define $E^*(\theta) = \sup(\{E(\kappa) \mid \kappa \in \mathrm{dom}\, E \wedge \kappa \leq \theta\} \cup \{\theta^+\})$. Then, let
$$E'(\theta) = \begin{cases} E^*(\theta) & \text{if } \mathrm{cf}(E^*(\theta)) > \theta \\ E^*(\theta)^+ & \text{otherwise} \end{cases}$$

Exercise: With the above hypotheses, if θ is any cardinal, then $M[G] \models 2^\theta = E'(\theta)$.

End of Lecture 26.
