MATH 336: Review homework

Spring 2016

Due: Tuesday, January 26, 2016

Solve the below differential equation and linear algebra review problems. Please review any concepts that are difficult and/or unfamiliar; I will also be happy to meet and discuss any questions you may have either in office hours or via appointment. There is also one introductory MATLAB problem. A (possibly improper) subset of them will be graded. All calculations should be done analytically, unless marked with an (M). (M) problems require the use of MATLAB.

1. (20 points) Consider the first-order ordinary differential equation (ODE)

$$\dot{y} = y^4 - y^2.$$

   (a) Find the steady states (also known as equilibria, fixed points, etc.).

   (b) Use the graph of $f(y) = y^4 - y^2$ to determine the stability of the steady states found in (a).

   (c) Suppose that $y(0) = 0.2$. Find $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$.

   (d) Suppose that $y(0) = 1.2$. Find $\lim_{t\to\infty} y(t)$.

2. (20 points) Solve the following second-order linear ODEs:

   (a) $2y'' - 5y' - 3y = 0$

   (b) $y'' - 10y' + 25y = 0$

   (c) $y'' + 4y' + 7y = 0, \quad y(0) = 1,\ y'(0) = 2.$

   Note that (c) is an initial-value problem, while in (a) and (b) you should find the general solution.

3. (10 points) Recall that second-order equations can be recast in an equivalent manner as **second-order systems** (in general this is true in higher dimensions as well). As an example, consider the second-order linear initial value problem (IVP)

$$\ddot{y} - 6\dot{y} + 25y = 0, \quad y(0) = 1, \quad \dot{y}(0) = \frac{1}{3}.$$

Convert the above into a first-order system of equations, with initial condition. That is, find a 2×2 matrix A and vectors $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{x}_0$ such that the IVP takes the equivalent form

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0.$$

Hint: $x_1 := y$, $x_2 := \dot{y}$.

4. (20 points) Compute the eigenvalues λ of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}.$$

Furthermore, for each λ, find a corresponding eigenvector $\mathbf{v}$.

5. (10 points) The eigenvalues and corresponding eigenvectors "essentially" give us the general solution to a linear system of ODEs. For example, if the 2×2 matrix A has *distinct* eigenpairs $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$, then the general solution of

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1}$$

takes the form

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2,$$

for arbitrary $c_1, c_2 \in \mathbb{R}$ (generally determined by initial conditions). **Use this "fact" to find the general solution of equation** (1) **with the matrix appearing in problem 4.** How do you expect the solution to behave as $t \to \infty$ and $t \to -\infty$?
Note: Exceptions to this "fact" occur when the matrix does not have an eigenbasis, i.e. a spanning set of eigenvectors. From your ODE course, this essentially corresponds to the cases of repeated and complex conjugate eigenvalues. **Please review these cases as well, and know how to construct their general solutions**. Please see me if you have questions.

6. (20 points) (M) This problem is designed to get you familiar MATLAB; you will not need to write your own code, but should instead get familiar with the software and its basic functionality.

Suppose I have an experiment that measures the concentration of some bacteria over 5 days. The results of this experiment are in the form of ordered pairs (t, y), where y measures the bacterial concentration. The complete data set is below:

$$(t, y) = \{(0, 0.04), (0.6, 0.35), (0.7, 0.5), (2, 0.95), (2.25, 1.1), (4, 1.45), (4.5, 1.7)\}.$$

We may guess (don't worry about why yet) that this growth should be logarithmic, say of the form $y = \log(t + 1)$. As a first step to check the accuracy of this prediction, we might want to plot both the data and the "guess" on the same set of axes. The following (complete) MATLAB code does this:

```
%Clear all previous plots and variables
clear all; close all; clc


%Enter the data as vectors
TData=[0,0.6,0.7,2,2.25,4,4.5];
YData=[0.04,0.35,0.5,0.95,1.1,1.45,1.7];

%Define the function you'd like to compare the data to
f=@(t) log(t+1);  %Functional form
TFun=0:0.01:5;    %Make a list of t values (from 0 to 5, spaced equally...
                  %... with parition 0.01
F=f(TFun);        %Function as a list of numbers (i.e. a vector)

%Perform all of the plotting
figure(1)
plot(TData,YData,'xk','LineWidth',2);
hold on;  %Lets you view multiple plots on same figure
plot(TFun,F,'-b','LineWidth',2);
xlabel('t')
ylabel('concentration of bacteria');
legend('Data','y=log(t+1)')
title('Bacterial growth over 5 days');
```

Note the text preceded by a % (green text) is ignored by the computer, and is there to help clarify (known as a *comment*). **Enter this code as a script (an *m-file*, say *hw1No6.m*) in the MATLAB editor (do NOT type it directly into the command window; this is not good practice for writing code), run it, and attach the resulting plot.** Please try to understand all of the commands and their basic syntax. See the Resources section on the webpage for more tutorials, and see me if you have further questions. Also, Google can be a good friend here.