Math 250:08, Workshop 1 Solutions

1. (a)(i)

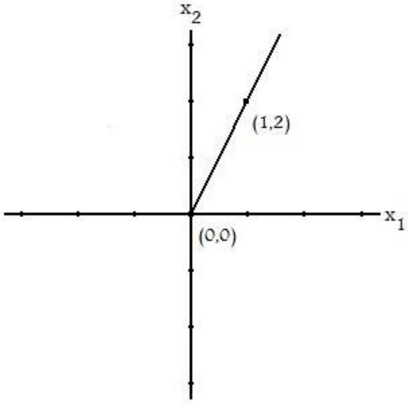
$$\begin{aligned}
\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & -5 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}\right) &= \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & -5 \end{bmatrix}\begin{bmatrix} -1 \\ -7 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 2*(-1)+3*(-7)+(-1)*(-1) \\ 1*(-1)+0*(-7)+(-5)*(-1) \end{bmatrix} \\
&= \begin{bmatrix} -22 \\ 4 \end{bmatrix}
\end{aligned}$$

(ii)

$$\begin{aligned}
\left(\begin{bmatrix} -3 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}^T\right)\begin{bmatrix} -5 \\ 2 \end{bmatrix} &= \left(\begin{bmatrix} -3 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 2 \\ 2 & 0 \end{bmatrix}\right)\begin{bmatrix} -5 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} -3 & 0 \\ 2 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -5 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} (-3)*(-5)+0*2 \\ 2*(-5)+4*2 \\ 2*(-5)+3*2 \end{bmatrix} \\
&= \begin{bmatrix} 15 \\ -2 \\ -4 \end{bmatrix}
\end{aligned}$$

(b)(i) $(A+B^T)^T((I_m)^T v) = (A^T+B)(I_m v) = (A^T+B)v = A^T v + Bv$ (either of the last two forms is fine)
(ii) This expression does not exist. This is because $(A^T+B)^T = A+B^T$ is an $m \times n$ matrix while $(I_m)^T v = I_m v = v$ is a vector in $\mathbb{R}^m$; for a matrix-vector product to be defined, the number of columns in the matrix must equal the number of components in the vector.

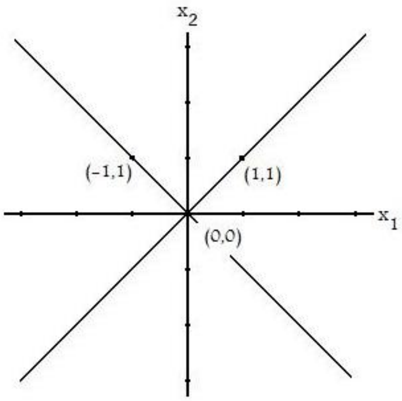
2. (a) The set $\{(x_1,x_2) \in \mathbb{R}^2 \,|\, 2x_1 - x_2 = 0 \text{ and } x_1 \geq 0\}$ is the intersection of the line satisfying $x_2 = 2x_1$ and the right half-plane satisfying $x_1 \geq 0$. It's a ray starting at the origin:



This is not a subspace, because it's not closed under scalar multiplication. For example, the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is in it, but the scalar multiple $(-1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ is not, since its x_1-coordinate is negative.

The smallest subspace of $\mathbb{R}^2$ containing this set is the line defined by $x_2 = 2x_1$, i.e. the set $\{(x_1, x_2) \in \mathbb{R}^2 \,|\, 2x_1 - x_2 = 0\}$.

(b) The set $\{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 - x_2 = 0 \text{ or } x_1 + x_2 = 0\}$ is the union of two lines, one defined by $x_2 = x_1$ and the other by $x_2 = -x_1$:



This is not a subspace, because it's not closed under vector addition. For example, the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are both in it, but their sum $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is not, since its coordinates satisfy neither equation.

The smallest subspace of $\mathbb{R}^2$ containing this set is $\mathbb{R}^2$ itself.

(c) The set $\{(x_1,x_2,x_3) \in \mathbb{R}^3 \,|\, x_1 + x_2 - x_3 = 0 \text{ and } x_1 - 2x_2 = 0\}$ is the intersection of two planes, one defined by $x_1 + x_2 - x_3 = 0$ and the other by $x_1 - 2x_2 = 0$. That intersection is the solution set to the following system of linear equations:

$$\begin{cases} x_1 + x_2 - x_3 & = & 0 \\ x_1 - 2x_2 & = & 0 \end{cases}$$

Because row equivalent augmented matrices correspond to equivalent systems of linear equations, one can find the general solution to this system by forming an augmented matrix and row reducing:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 1 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \end{array}\right]$$

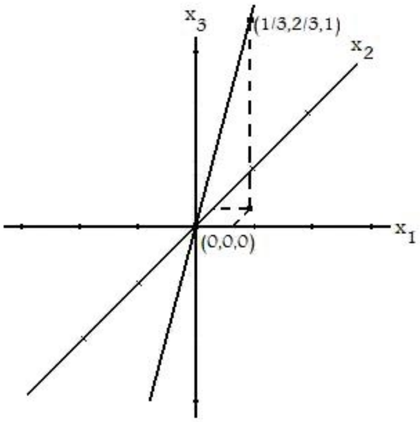
That last augmented matrix corresponds to a simpler system:

$$\begin{cases} x_1 \qquad - \frac{2}{3}x_3 & = & 0 \\ \qquad x_2 - \frac{1}{3}x_3 & = & 0 \end{cases}$$

By writing the basic variables x_1 and x_2 in terms of the free variable x_3, one obtains the general solution:

$$\begin{cases} x_1 & = & \frac{2}{3}x_3 \\ x_2 & = & \frac{1}{3}x_3 \\ x_3 & \text{free} \end{cases}$$

The vector form of the general solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x_3 \\ \frac{2}{3}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix}$. Thus the intersection of the two planes is the set of all scalar multiples of the vector $\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix}$, or in set notation $\{(\frac{1}{3}x_3, \frac{2}{3}x_3, x_3) \in \mathbb{R}^3 \,|\, x_3 \in \mathbb{R}\}$. This is the line containing the origin and the point $(\frac{1}{3}, \frac{2}{3}, 1)$:



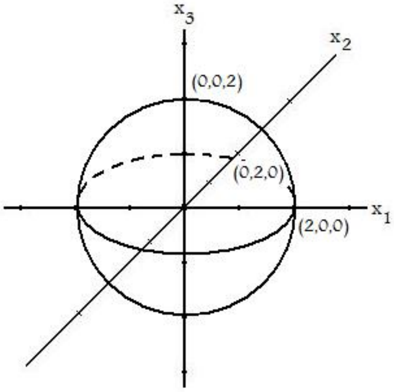
This line is a subspace of $\mathbb{R}^3$. To see that, one must first note that it is non-empty, since for example it contains the point $(0,0,0)$. It now remains check that it is (1) closed under vector addition and (2) closed under scalar multiplication:

(1) Let $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$ be points on this line. Then $u_1+u_2-u_3=0$ and $v_1+v_2-v_3=0$, which means $(u_1+u_2-u_3)+(v_1+v_2-v_3)=(u_1+v_1)+(u_2+v_2)-(u_3+v_3)=0$. Thus the coordinates of $u+v=(u_1+v_1,u_2+v_2,u_3+v_3)$ satisfy the first equation used to define the line. Similarly, $u_1-2u_2=0$ and $v_1-2v_2=0$, so $(u_1-2u_2)+(v_1-2v_2)=(u_1+v_1)-2(u_2+v_2)=0$, and the coordinates of $u+v$ also satisfy the second equation used to define the line. Thus $u+v$ is on the line.

(2) Let $u=(u_1,u_2,u_3)$ be a point on the line, and let $s\in\mathbb{R}$. Then $u_1+u_2-u_3=0$, so $s(u_1+u_2-u_3)=(su_1)+(su_2)-(su_3)=0$. Thus the coordinates of $s\times u=(su_1,su_2,su_3)$ satisfy the first equation used to define the line. Similarly, $u_1-2u_2=0$, so $s(u_1-2u_2)=(su_1)-2(su_2)=0$, and the coordinates of $s\times u$ also satisfy the second equation used to define the line. Thus $s\times u$ is on the line.

That completes the proof that the line is a subspace.

(d) The set $\{(x_1,x_2,x_3)\in\mathbb{R}^3\,|\,x_1^2+x_2^2+x_3^2=4\}$ is a sphere centered at the origin with radius 2:



This is not a subspace. The easiest way to see this is, probably, to note that it does not contain $(0,0,0)$; it was proved in class that every subspace must contain the zero vector. One could just as well show that it's not closed under vector addition or that it's not closed under scalar multiplication.

The smallest subspace of $\mathbb{R}^3$ containing this set is $\mathbb{R}^3$ itself.

3. In this game, each of the über-knight's moves can be modeled as a vector in $\mathbb{R}^3$, e.g. the moves

(i) 1 right, 1 backward, and 3 down
(ii) 2 right, 1 forward, and 2 down

can be represented by the vectors $\begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$, respectively. Under this identification, making a move corresponds to adding the appropriate movement vector to the über-knight's location. Thus the set of points that can be realized by the über-knight, starting from the origin, is equal to the set of all linear combinations of the four movement vectors with non-negative integer coefficients. (The über-knight cannot, for example, make half of a move, which is why the coefficients must be integers.) Because moves (i)' and (ii)' are the negatives of moves (i) and (ii), respectively, any linear combination of all four with non-negative integer coefficients can be written as a linear combination of (i) and (ii) with integer coefficients. Thus asking whether the point $(22,29,2)$ can be reached is the same as asking whether the system

$$x_1 \begin{bmatrix} -1 \\ -1 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 22 \\ 29 \\ 2 \end{bmatrix}$$

has a solution with both x_1 and x_2 integers. Because row equivalent augmented matrices correspond to equivalent systems of linear equations, one can determine this by forming the corresponding augmented matrix and row reducing:

$$\left[\begin{array}{cc|c} -1 & -2 & 22 \\ -1 & 1 & 29 \\ -3 & -2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & -22 \\ -1 & 1 & 29 \\ -3 & -2 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & -22 \\ 0 & 3 & 7 \\ 0 & 4 & -64 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & -22 \\ 0 & 3 & 7 \\ 0 & 0 & -\frac{220}{3} \end{array}\right]$$

The last row of the final augmented matrix corresponds to the equation $0 = -\frac{220}{3}$, so this system is inconsistent and it's impossible for the über-knight to reach the point $(22, 29, 2)$.

NOTE: In the statement of the problem, the directions "right" and "left" are reversed from what was intended. If "right" corresponds to positive movement on the x_1-axis, "left" to negative movement on the x_1-axis, and everything else remains the same, then it is possible for the über-knight to reach $(22, 29, 2)$. One sequence of moves that will land it there is to make move (i)' 12 times then move (ii) 17 times. It's left as an exercise to show this. Both interpretations were given full credit.