642.582 Problem Set 2 (should be final)

1. Find (and justify) a formula for the number, say $f(n)$, of permutations $a_1, \ldots, a_n$ of $[n]$ for which there do not exist $i < j < k$ with $a_i > a_k > a_j$. (Such permutations are called 312-*avoiding*.) [You can take $f(0) = 1$.]

2. Prove that the number of partitions of $n \geq 2$ into powers of 2 is even.

[E.g. for $n = 4$, the relevant partitions are 4, 22, 211 and 1111.]

3. Suppose $p_I \in \Re$ for $I \subseteq [n]$. Show that there are A; $A_1, \ldots, A_n \subseteq A$; and a probability measure on A with

$$\Pr(A_I) = p_I \quad \forall I \subseteq [n]$$

(where $A_I = \cap_{i \in I} A_i$) if and only if

$$\sum_{K \supseteq I} (-1)^{|K \setminus I|} p_K \geq 0 \qquad \forall I \subseteq [n] \tag{1}$$

and

$$p_\emptyset = 1. \tag{2}$$

[Let's make this a 2-part problem: say necessity is (a) and sufficiency (b).]

4. Suppose $2n$ people, named $X_1, Y_1, \ldots, X_n, Y_n$, are seated (uniformly) at random around a circular table. Give (and justify) asymptotics for the probability that no X_i, Y_i are seated next to each other.

[You can think of a random cyclic ordering, instead of a random assignment to fixed seats; this is equivalent of course, but may be easier to work with.]

5. Let $V = V_1 \cup \cdots \cup V_k$ be a partition with $|V_i| = n \ \forall i$, and say $T \in \binom{V}{k}$ is a *transversal* if it meets every V_i. Show that if $h : \binom{V}{k} \to \Re$ satisfies $h(T) = 1$ for each transversal T, then there is some $S \subseteq V$ with

$$|h(S)| \geq c_k n^k,$$

where $h(S) = \sum_{T \subseteq S} h(T)$ and c_k depends only on k.

[Please use $h(X) = \sum_{E \in X} h(E)$ when $X \subseteq \binom{V}{k}$.]

6. Let $A_i = A_i^{(n)}$ be independent events with $X_i = \mathbf{1}_{A_i}$ and $\Pr(A_i) = p_i$, and set $X = \sum X_i$. Show that if $X \xrightarrow{\text{d}} \text{Po}(\mu)$ for a fixed, positive μ, then (i) $\sum p_i \to \mu$ and (ii) $\max p_i \to 0$.

[So the parameter n is "hidden": X and p_i are really $X^{(n)}$ and $p_i^{(n)}$.
You could also try the converse—not assigned but I'll read if you write.]

7.(a) Suppose we are given $A_1, \ldots, A_m \subseteq A$ and a probability measure on A, and let Y be the random variable $|\{i : A_i \text{ occurs}\}|$. As usual, let $A_I = \cap_{i \in I} A_i$ ($I \subseteq [m]$) and $S_k = \sum_{|I|=k} \Pr(A_I)$ ($0 \leq k \leq m$). Show that for any x (say $x \in \Re$, but it doesn't matter),

$$\mathsf{E}\ x^Y = \sum (x-1)^k S_k$$

(where E is expectation).

(b) We return to edge reconstruction. Assume $\mathcal{L}(G) = \mathcal{L}(G')$; suppose (just for convenience) that $V(G) = V(G') = V = [n]$; and let σ be chosen uniformly at random from the set, say A, of permutations of V. Let $E(G) = \{e_1, \ldots, e_m\}$, $E(G') = \{e'_1, \ldots, e'_m\}$,

$$A_i = \{\sigma \in A : \sigma(e_i) \in E(\overline{G})\}, \quad B_i = \{\sigma \in A : \sigma(e'_i) \in E(\overline{G})\},$$

$S_k = \sum_{|I|=k} \Pr(A_I)$ and $T_k = \sum_{|I|=k} \Pr(B_I)$ (with A_I, B_I as usual). Let

$$Y = |\{i : A_i \text{ occurs}\}|, \quad Z = |\{i : B_i \text{ occurs}\}|.$$

Show that

$$\mathsf{E}[(-1)^Y - (-1)^Z] = (-2)^m (S_m - T_m).$$

[Hint: if (as in class) $E(G) = \{e_1, \ldots, e_m\}$ and $H_I = (V, \{e_i : i \in I\})$, then for any H, $N(H, G) = |\{I : H_I \cong H\}||\text{Aut}(H)|$.]

(c) Show that if $2^{m-1} > n!$, then G is edge-reconstructible (a big improvement on Lovász except for a few small values of n).