

642.582 Problem Set 6 (supposedly final)

1. Say  $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$  are *cross-intersecting* if

$$A \cap B \neq \emptyset \quad \forall A \in \mathcal{A}, B \in \mathcal{B}.$$

Use the KK Theorem to show that if  $k + l \leq n$  ( $k, l \geq 1$ ) and  $\mathcal{A} \subseteq \binom{[n]}{k}$ ,  $\mathcal{B} \subseteq \binom{[n]}{l}$  are cross-intersecting, then either  $|\mathcal{A}| \leq \binom{n-1}{k-1}$  or  $|\mathcal{B}| < \binom{n-1}{l-1}$ .

2. Give an inductive proof that for  $\mathcal{A} \subseteq Q^n$  of size  $a$ ,

$$|\nabla(\mathcal{A})| \geq a(n - \log a) \tag{1}$$

(where  $\log = \log_2$ ). [Your proof should *not* give (or use) the exact edge-isoperimetric result stated in class.]

3. Say the matrix  $A = (a_{ij})_{i,j \in [n]}$  is *stochastic* if  $\sum_i a_{ij} = 1$  for each  $j$ , and notice that then, for any probability distribution  $p = (p_1, \dots, p_n)$ ,  $\bar{p} = (\bar{p}_1, \dots, \bar{p}_n)$  given by  $\bar{p}_i = \sum_j a_{ij} p_j$  is also a probability distribution. Show that if  $A$  is *doubly* stochastic and  $p, \bar{p}$  are as above, then  $H(\bar{p}) \geq H(p)$ .

4. For  $X = (X_1, \dots, X_n)$  and  $A \subseteq [n]$ , let  $f(A) = H(X_A)$ . With  $n = 5$  and sums over  $i \in [5]$  (subscripts taken mod 5), prove or disprove:

- (a)  $\sum f([5] \setminus \{i\}) + \sum f(i, i+1) \leq 2 \sum f(i-1, i, i+1)$ .  
 (b)  $\sum f([5] \setminus \{i\}) + \sum f(i-1, i+1) \leq 2 \sum f(i-1, i, i+1)$ .

5. Show that for a 3-uniform hypergraph  $\mathcal{G}$  on  $V$ ,

$$\Phi(\mathcal{G}) \leq \prod_{v \in V} d_{\mathcal{G}}(v)^{1/3}$$

(where  $\Phi(\mathcal{G})$  is the number of perfect matchings in  $\mathcal{G}$ ).

[Anti-hint: you probably *don't* want to use and/or mimic Minc for this.]

6. Let  $f(\ell)$  be the maximum number of pentagons in a graph with  $\ell$  edges. Show  $f(\ell) = O(\ell^{5/2})$ . (Note this is sharp apart from the value of the constant.)